# Chapter 1

# Introduction

One of the objectives of a researcher in mathematics is to introduce/catch a function and study its various properties. The influence of such properties pervades various branches of Mathematics in particular and those of Science in general. The theory of functions were studied and enriched by many eminent mathematicians right from the ancinet time to the recent era; the names of L. Euler, C. F. Gauss, G. Szegö, A. Erdélyi, H. W. Gould, L. Carlitz, W. A. Al-Salam, G. Gasper, M. E. H. Ismail, M. Rahman, A. Verma, W. N. Bailey, T. M. Koorwinder, H. L. Manocha, H. Exton are among the major contributors. The study of the Function theory in the light of the Differential equations or Group theory or Generating function or recurrence relation or Series/Intrgral Transforms influenced many branches of Science such as Physical Science, Chemical Science, Life Science, Earth Science, Space Science, Economics, etc.

The functions whose nomenclature become inevitable for the further study or towards the application, constitute the class of functions known as the *Special Functions.* Many of the functions belonging to this class are arising from a natural phenomenon or a physical phenomenon; most of such functions are found to be the solution(s) of (Ordinary or Partial) Differential equations. The investigation of many research problems in Special Functions remains incomplete without two friendly functions : the Gamma function and the Beta function. They are extensively used along with their various properties, and with this reason, they are also

believed to be the member of class of Special Functions.
It goes without saying that the Special Functions are playing vital role in various applied branches such as Heat conduction, solutions of Wave equations, Acoustics, Electrical current, Fluid dynamics, Moments of inertia and Quantum mechanics, study of Hydrogen atom, Potential theory, Orbital mechanics, Vibration phenomenon. The displacement and rotations of the plate and a stiffener are approximated by separate sets of the Jacobi polynomials.

Among various generalization of the well known Special Functions; in particular the polynomial functions, the family of Bessel function etc., an interesting development took place during last decades of the $20^{th}$ Century. The scalar function theory began to extend in matrix forms. The variable(s) treated as usual to be real or complex, but the parameters were replaced by the square matrices. This development received a good support from the literature like [24, 49]. The work carried out in this direction also found the place in Statistics, Group representation theory [49], Scattering theory [31], Differential equations [50, 58, 79], Fourier series expansions [19], Interpolation and quadrature [57, 99], Splines [22], medical imaging [18] etc. There is a large number of eminent researchers who contributed to the development includes L. Jodar, E. Defez, J. Cortes and J. Sastre [20, 21, 51–54, 57–61, 86, 87], M. A. Pathan [76], Ayman Shehata [92–94, 96] and many others [1, 25–28, 105, 108]. Beginning with the matrix analogues of Pochhammer symbol and Gamma and Beta functions, the orthogonal polynomials such as those of Laguerre, Hermite, Legendre, Gegenbauer, Jacobi etc., and also the Bessel function and its associated functions were extended to matrix forms and various properties have been studied. Moreover, the classical polynomials were provided two variables matrix forms and their properties were investigated (see [76, 105]).

It will be worth mentioning here that the theory of Infinite product and Weierstrass definition of the Gamma function in [80, Ch-1] have been extended to the matrix product and Gamma matrix function along with their various properties (see [54]).
We list out the symbols, formulas, definitions in the subsequent section of 'Matrix Forms'.

## 1.1 Definitions, Notations and Formulae

In this section we provide the basic definition, formulas and notations which will be appering in this thesis. We first glance through the preliminary of the scalar forms.

## 1.2 Scalar Form:

It will be worth while to state the scalar forms of the formulas, definitions etc. prior to their matrix forms. We begin with the symbol: $(*)_m$.
For $\alpha \in \mathbb{C}\backslash\{0\}$, $m \in \mathbb{N}$, the product

$$\alpha(\alpha+1)(\alpha+2)...(\alpha+m-1)$$

is denoted by $(\alpha)_m$. This symbol is called the Pochhammer symbol, named after L. A. Pochhammer (1890). In fact, A. L. Crelle (1831) used this notation earlier but P. E. Appell (1880) nomenclatures $(\alpha)_n$ as Pochhammer symbol. It is defined as follows [80].

$$(\alpha)_m = \begin{cases} 1, & \text{if } m = 0 \\ \alpha(\alpha+1)\ldots(\alpha+(m-1)), & \text{if } m \geq 1. \end{cases} \tag{1.2.1}$$

The Gamma function is defined as [80]:

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt, \;\; \Re(\alpha) > 0. \tag{1.2.2}$$

The Beta function denoted by $\mathfrak{B}(\alpha,\beta)$, is defined as [80]:

$$\mathfrak{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1} dt, \quad \Re(\alpha) > 0, \;\; \Re(\beta) > 0. \tag{1.2.3}$$

The relation between the Gamma function and Beta function is given by

$$\mathfrak{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\quad \Re(\alpha)>0,\ \ \Re(\beta)>0. \tag{1.2.4}$$

The connection between the Pochhammer symbol and Gamma function is

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{1.2.5}$$

This may be regarded as the definition of the Pochhammer symbol $(\alpha)_n$ for $n \in \mathbb{C}$. The Gamma function is also expressible as follows [80].

$$\Gamma(\alpha)=\lim_{n\to\infty}\frac{(n-1)!\ n^{\alpha}}{(\alpha)_n}. \tag{1.2.6}$$

The following formulas are useful; their matrix analogues which appear in the next section,

$$(\alpha)_{n+k}=(\alpha)_n(\alpha+n)_k. \tag{1.2.7}$$

$$(\alpha)_{-n}=\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-a)_n}. \tag{1.2.8}$$

$$(\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}. \tag{1.2.9}$$

$$(\alpha)_{kn}=k^{kn}(\alpha/k)_n((\alpha+1)/k)_n((\alpha+2)/k)_n...((\alpha+k-1)/k)_n. \tag{1.2.10}$$

As a generalization of the infinite geometric series, C. F. Gauss in 1813, introduced the series which is denoted and defined by [80, 102]

$${}_2F_1[a,b;c;x]=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m\ z^m}{(c)_m\ \ m!},\quad |z|<1. \tag{1.2.11}$$

If $|z|=1$, then the series converges absolutely for $\Re(c-a-b)>0$. This is well known as the Hypergeometric series or the Gauss series. The generalized hypergeometric series is denoted and expressed as [80, 102]

$${}_pF_q[a_1\cdots a_p;b_1\cdots b_q:z]=\sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_p)_m\ z^m}{(b_1)_m\cdots(b_q)_m\ m!}. \tag{1.2.12}$$

1. If $p \leq q$, the series converges absolutely for all finite $|z|$.
2. If $p = q + 1$, series converges for $|z| < 1$ and diverges for $|z| > 1$.
3. If $|z| = 1$, the series converges absolutely for $\Re(b_1) + \cdots + \Re(b_q) > \Re(a_1) + \cdots + \Re(a_p)$.

### 1.2.1 Other Generalized Functions

The generalized hypergeometric function ${}_pF_q[z]$ has many generalized function forms to which it would reduce when the parameters involved are specialized appropriately. Some of such functions are stated below.

- ${}_pR_q(\alpha, \beta; z)$ function: This function is defined as [23]

$$
\begin{aligned}
{}_pR_q(\alpha, \beta; z) &= {}_pR_q\left[\begin{matrix}\gamma_1, \gamma_2 \ldots, \gamma_p \\ \delta_1, \delta_2 \ldots, \delta_q\end{matrix} \mid \alpha, \beta; z\right] \\
&= \sum_{n \geq 0} \frac{1}{\Gamma(n\alpha + \beta)} \frac{(\gamma_1)_n (\gamma_2)_n \ldots (\gamma_p)_n}{(\delta_1)_n (\delta_2)_n \ldots (\delta_q)_n} \frac{z^n}{n!}, \qquad (1.2.13)
\end{aligned}
$$

  where $\alpha, \beta \in \mathbb{C}, \Re(\alpha), \ \Re(\beta), \ \Re(\gamma_i), \ \Re(\delta_j) > 0$ for all $i = 1, 2, ...p$ and $j = 1, 2, ...q$.

- The generalised M-series:

  It is defined as [91]

$$
{}_pM_q^{\alpha,\beta}(a_1, ..., a_p; b_1, ...b_q; z) = \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n z^n}{\Gamma(n\alpha + \beta)(b_1)_n (b_2)_n \ldots (b_q)_n}, \qquad (1.2.14)
$$

  where $z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0$.

- The Wright function

  This function is defined by [106]

$$
{}_p\psi_q\left[\begin{array}{c}(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_p,\beta_p);\\(\eta_1,\mu_1),(\eta_2,\mu_2),\ldots,(\eta_q,\mu_q);\end{array} z\right]
= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n\beta_1)\Gamma(\alpha_2+n\beta_2)...\Gamma(\alpha_p+n\beta_p)z^n}{\Gamma(\eta_1+n\mu_1)\Gamma(\eta_2+n\mu_2)...\Gamma(\eta_q+n\mu_q)n!}. \tag{1.2.15}
$$

- Mittag-Leffler function and its generalization:

  The function

$$
E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.2.16}
$$

where $z,\alpha\in\mathbb{C},\ \Re(\alpha)>0$, is due to Gosta Mittag-Leffler [71] which is well known as the Mittag-Leffler function.

Wiman [110] generalized this in the form:

$$
E_{\alpha,\,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{1.2.17}
$$

This was further extended in different forms by T. R. Prabhakar [77]:

$$
E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \qquad Re(\alpha,\ \beta,\ \gamma)>0 \tag{1.2.18}
$$

and subsequently by Shukla and Prajapati [98]:

$$
E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.2.19}
$$

where $\alpha,\beta,\gamma\in\mathbb{C};\ \Re(\alpha,\ \beta,\ \gamma)>0$ and $0<q<1$.

Recently, there is introduction of a more general form [73]:

$$
E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r}\frac{z^n}{n!}, \tag{1.2.20}
$$

where $\alpha,\ \beta,\ \gamma,\ \lambda\ \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \ \delta,\ \mu > 0,\ \ r \in \mathbb{N} \cup \{-1, 0\}$ and $s \in \mathbb{N} \cup \{0\}$. It is noteworthy that this function besides containing above three functions, also includes some other functions namely,

(i) Bessel-Maitland function [69, Eq.(1.7.8), p.19]:

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu + n\mu + 1)} \frac{z^n}{n!}, \tag{1.2.21}$$

(ii) the Dotsenko function [69, Eq.(1.8.9), p.24]:

$${}_2R_1(a, b; c, \omega; \nu; z) \quad = \quad \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\ \Gamma(b + n\frac{\omega}{\nu})}{\Gamma(c + n\frac{\omega}{\nu})} \frac{z^n}{n!}, \tag{1.2.22}$$

(iii) a particular form $(m = 2)$ of extension of Mittag-Leffler function due to Saxena and Nishimoto given by [89]:

$$E_{\gamma,K}[(\alpha_j, \beta_j)_{1,2}; z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{Kn}}{\Gamma(\alpha_1 n + \beta_1)\ \Gamma(\alpha_2 n + \beta_2)} \frac{z^n}{n!}, \tag{1.2.23}$$

and (iv) the Elliptic function [68, Eq.(1), p.211]:

$$K(k) = \frac{\pi}{2}\ {}_2F_1\left(\begin{array}{lll} \frac{1}{2}, & \frac{1}{2}; & k^2 \\ 1; & & \end{array}\right). \tag{1.2.24}$$

All these functions are tabulated below as particular cases of (1.2.19).

Table-1

| **Function** | **r** | **s** | $\boldsymbol{\alpha}$ | $\boldsymbol{\beta}$ | $\boldsymbol{\gamma}$ | $\boldsymbol{\delta}$ | $\boldsymbol{\lambda}$ | $\boldsymbol{\mu}$ |
|---|---|---|---|---|---|---|---|---|
| Mittag-Leffler | 0 | 1 | $\alpha$ | 1 | 1 | 1 | - | - |
| Wiman | 0 | 1 | $\alpha$ | $\beta$ | 1 | 1 | - | - |
| Prabhakar | 0 | 1 | $\alpha$ | $\beta$ | $\gamma$ | 1 | - | - |
| Shukla and Prajapati | 0 | 1 | $\alpha$ | $\beta$ | $\gamma$ | q | - | - |
| Bessel-Maitland | 0 | 0 | $\mu$ | $\nu+1$ | - | - | - | - |
| Dotsenko | -1 | 1 | $\omega/\nu$ | c | $a$ | 1 | b | $\omega/\nu$ |
| Saxena-Nishimoto | 1 | 1 | $\alpha_1$ | $\beta_1$ | $\gamma$ | K | $\beta_2$ | $\alpha_2$ |
| Elliptic | -1 | 1 | 1 | 1 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 |

### 1.2.2 Inverse Series Relation

Invers seires relations have been found useful in the study of the Combinatorial Identities. The significance of an inverse pair is that

- each pair implies an orthogonal relation which itself would generate one or more identities
- each identity in the inverse pair may be regarded as a companion of the other
- in the form of the inverse series, a new identity will be found
- particular choices of the variable/parameters in the inverse pair would yield a new pair
- in proving the inverse series, a prover has a choice.

A series is said to be the inverse series of a given series if one of the series when substituted into the other, simplifies to the expression involving the Kronecker delta :

$$\delta_{ij} = \begin{cases} 0, & if \quad j \neq i \\ 1, & if \quad j = i \end{cases} .$$

To illustrate this, consider the inverse pair

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \quad b_n = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} a_k. \tag{1.2.25}$$

Here, if the second series is substituted into the first series then the inner sum simplifies to the form

$$\sum_{k=j}^{n} (-1)^{k+j} \binom{n}{k} \binom{k}{j} = \delta_{nj},$$

thus proving one side of inverse relation. The poof of the converse part is similar. It is interesting to see that the coefficient matrix of each series in (1.2.25) gives rise to the square matrices which are inverses of each other. This can be illustrated as follows by taking $n = 0, 1, 2, 3$ in turn in (1.2.25).

<u>Coefficient matrix of first series</u>

$$\begin{matrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{matrix}$$

<u>Coefficient matrix of second series</u>

$$\begin{matrix} 1 & -1 & 1 & -1 \\ & 1 & -2 & 3 \\ & & 1 & -3 \\ & & & 1 \end{matrix}$$

The product of matrices of these arrays in either order, turns out to be the identity matrix. This eventually proves the inverse series relations of the series in (1.2.25). The application of inverse series relations can be seen in Coding theory [103], Partition theory, Approximation theory, Distribution theory and Probability theory [29].

### 1.2.3 Polynomials and Their Inverse Series Relations

In this section, we state the pairs of inverse series relations involving certain orthogonal polynomials such as the Laguerre polynomial, the Legendre polynomial, the Hermite polynomial, the Chebyshev polynomial, the Gegenbauer polynomial, the Jacobi polynomial, Wilson polynomial, Racah polynomial etc. occurring in the literature hitherto on the Special Functions (see [4, 12, 14, 65, 80]).

- Laguerre polynomial and its inverse series relation:

$$\left.\begin{aligned} L_m^{(\alpha)}(x) &= \sum_{k=0}^{m} \frac{(-1)^k\,(1+\alpha)_m\,x^k}{(1+\alpha)_k (m-k)!\;k!} \\ &\Leftrightarrow \\ x^m &= \sum_{k=0}^{n} \frac{(-1)^k\,m!\,(1+\alpha)_m}{(1+\alpha)_k\,(m-k)!}\,L_k^{(\alpha)}(x) \end{aligned}\right\}$$

- Konhauser polynomial and its inverse series relation:

$$\left.\begin{aligned} Z_m^{(\alpha)}(x,k) &= \sum_{j=0}^{m} \frac{\Gamma(m+\alpha+1)(-1)^j\,x^{jk}}{\Gamma(kj+\alpha+1)(m-j)!\;j!} \\ &\Leftrightarrow \\ x^{mk} &= \sum_{j=0}^{m} \frac{(-1)^j\,m!\Gamma(km+\alpha+1)}{\Gamma(j+\alpha+1)(m-j)!\;j!}\,Z_j^{(\alpha)}(x,k) \end{aligned}\right\}$$

- Hermite polynomial and its inverse series relation:

$$\left.\begin{aligned} H_m(x) &= \sum_{k=0}^{[m/2]} \frac{(-1)^k\,m!(2x)^{m-2k}}{(m-2k)!\;k!} \\ &\Leftrightarrow \\ \frac{x^m}{m!} &= \sum_{k=0}^{[m/2]} \frac{m!H_{m-2k}(x)}{2^m(m-2k)!k!}. \end{aligned}\right\}$$

- Legendre polynomial and its inverse series relation:

$$\left.\begin{aligned}
P_m(x) &= \sum_{k=0}^{[m/2]} \frac{(-1)^k (1/2)_{m-k} (2x)^{m-2k}}{(m-2k)!\ k!} \\
&\Leftrightarrow \\
(2x)^m &= \sum_{k=0}^{[m/2]} \frac{(2m-4k+1) m! P_{m-2k}(x)}{(3/2)_{m-k} k!}.
\end{aligned}\right\}$$

- Gegenbauer polynomial and its inverse series relation:

$$\left.\begin{aligned}
C_m^{\alpha}(x) &= \sum_{k=0}^{[m/2]} \frac{(-1)^k (\alpha)_{m-k} (2x)^{n-2k}}{(m-2k)!\ k!} \\
&\Leftrightarrow \\
\frac{(2x)^m}{m!} &= \sum_{k=0}^{[m/2]} \frac{(\alpha+m-2k)}{(\alpha)_{m-k+1} k!} C_{m-2k}^{\alpha}(x).
\end{aligned}\right\}$$

- Jacobi polynomial and its inverse series relation:

$$\left.\begin{aligned}
P_m^{(\alpha,\beta)}(x) &= \frac{(1+\alpha)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k\ (1+\alpha+\beta+m)_k}{(1+\alpha)_k k!} \left(\frac{1-x}{2}\right)^k \\
&\Leftrightarrow \\
\frac{(1-x)^m}{2^m (1+\alpha)_m} &= \sum_{k=0}^{m} \frac{(-m)_k (1+\alpha+\beta)_k (1+\alpha+\beta+2k)}{(1+\alpha+\beta)_{m+k+1} (1+\alpha)_k} P_k^{(\alpha,\beta)}(x)
\end{aligned}\right\}$$

- Racah polynomial and its inverse series relation:

$$\left.\begin{aligned}
R_m(x(x+\delta+\gamma+1);\alpha,\beta,\gamma,\delta) &= \sum_{k=0}^{m} \frac{(-m)_k (1+\alpha+\beta+m)_k (-x)_k}{(1+\alpha)_k (\beta+\delta+1)_k (\gamma+1)_k k!} \\
&\quad \times (x+\gamma+\delta+1)_k \\
&\Leftrightarrow \\
\frac{(-x^m)(x+\gamma+\delta+1)_m}{(1+\alpha)_m (\beta+\delta+1)_m (\gamma+1)_m} &= \sum_{k=0}^{m} \frac{(-m)_k (1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{m+1} k!} \\
&\quad \times R_k(x(x+\delta+\gamma+1);\alpha,\beta,\gamma,\delta).
\end{aligned}\right\}$$

- Wilson polynomial and its inverse series relation:

$$\left.\begin{aligned}
\frac{P_m(x^2)}{(a+b)_m(a+c)_m(a+d)_m} &= \sum_{k=0}^{m} \frac{(-m)_k(a+b+c+d+m-1)_k}{(a+b)_k(a+c)_k(a+d)_k} \\
&\quad \times \frac{(a+ix)_k(a-ix)_k}{(a+b+c+d+k-1)_{m+1}k!} \\
\Leftrightarrow \\
\frac{(a+ix)_k(a-ix)_k}{(a+b)_m(a+c)_m(a+d)_m} &= \sum_{k=0}^{m} \frac{(-m)_k(a+b+c+d+2k-1)_k}{(a+b)_k(a+c)_k(a+d)_k k!} \\
&\quad \times \frac{R_k(x(x+\delta+\gamma+1);\alpha,\beta,\gamma,\delta)}{(a+b+c+d+k-1)_{m+1}}.
\end{aligned}\right\}$$

- Hahn polynomial and its inverse series relation:

$$\left.\begin{aligned}
Q_m(x;\alpha,\beta,N) &= \sum_{k=0}^{m} \frac{(-m)_k(1+\alpha+\beta+m)_k(-x)_k}{(1+\alpha)_k(-N)_k k!} \\
\Leftrightarrow \\
\frac{(-x)_m}{(1+\alpha)_m(-N)_m} &= \sum_{k=0}^{m} \frac{(-m)_k(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{m+1}k!} Q_k(x;\alpha,\beta,N).
\end{aligned}\right\}$$

- Dual Hahn polynomial and its inverse series relation:

$$\left.\begin{aligned}
R_m(x(x+\alpha+\beta+1);\alpha,\beta,N) &= \sum_{k=0}^{m} \frac{(-m)_k(1+\alpha+\beta-x)_k(-x)_k}{(1+\alpha)_k(-N)_k k!} \\
\Leftrightarrow \\
\frac{(1+\alpha+\beta+x)_m(-x)_m}{(1+\alpha)_m(-N)_m m!} &= \sum_{k=0}^{m} \frac{(-m)_k}{m!} R_k(x(x+\alpha+\beta+1);\alpha,\beta,N)
\end{aligned}\right\}$$

- Extended Jacobi polynomial and its inverse series relation:

$$\left.\begin{aligned}
\mathcal{F}^c_{m,l,s}[\alpha_1\cdots\alpha_p;\beta_1\cdots\beta_q:x] &= \sum_{k=0}^{\lfloor m/s\rfloor} \frac{(-m)_{sk}(c+m)_{lk}\,(\alpha_1)_k\cdots(\alpha_p)_k x^k}{(\beta_1)_k\cdots(\beta_q)_k k!}. \\
\Leftrightarrow \\
\frac{(\alpha_1)_m\cdots(\alpha_p)_m}{(\beta_1)_m\cdots(\beta_q)_m}\frac{x^m}{m!} &= \sum_{k=0}^{sm} \frac{(-sm)_k}{(sm)!\;k!}\frac{(c+k+(lk/s))}{(c+k)_{lk+1}} \\
&\quad \times \mathcal{F}^c_{k,l,s}[\alpha_1\cdots\alpha_p;\beta_1\cdots\beta_q:x]
\end{aligned}\right\}$$

- Generalized Humbert polynomials and its inverse series relation:

$$\left.\begin{aligned} P_n(m,x,\eta,\alpha,c) &= \sum_{k=0}^{[n/m]} \tbinom{\alpha-n+mk}{k}\tbinom{\alpha}{n-mk}\eta^k c^{\alpha-n-k+mk}(-mx)^{n-mk} \\ &\Leftrightarrow \\ \frac{(-mx)^n}{n!} &= \sum_{k=0}^{[n/m]}(-\eta)^k \frac{(\alpha-n+mk)c^{n-k-\alpha}}{(\alpha-n+k)k!}\Gamma(\alpha+1+k-n) \\ &\quad \times P_{n-mk}^{\alpha}(m,x,\eta,c). \end{aligned}\right\}$$

- Humbert polynomial and its inverse series relation:

$$\left.\begin{aligned} \Pi_{n,m}^{\gamma}(x) &= \sum_{k=0}^{[n/m]}(-1)^k \frac{(mx)^{n-mk}}{\Gamma(-\gamma-n+mk-k+1)(n-mk)!\ k!} \\ &\Leftrightarrow \\ \frac{(-mx)^n}{n!} &= \sum_{k=0}^{[n/m]}(-1)^k \frac{-\gamma-n+mk}{-\gamma-n+k}\frac{\Gamma(-\gamma-n+k+1)}{k!}\Pi_{n-mk,m}^{\gamma}(x). \end{aligned}\right\}$$

- The Kinney polynomial and its inverse series relation:

$$\left.\begin{aligned} k_n(m,x) &= \sum_{k=0}^{[n/m]}(-1)^k \frac{(mx)^{n-mk}}{\Gamma(-(1/m)-n+mk-k+1)(n-mk)!\ k!} \\ &\Leftrightarrow \\ \frac{(-mx)^n}{n!} &= \sum_{k=0}^{[n/m]}(-1)^k \frac{-(1/m)-n+mk}{-(1/m)-n+k}\frac{\Gamma((-1/m)-n+k+1)}{k!} \\ &\quad \times k_{n-mk}(m,x). \end{aligned}\right\}$$

- The Pincherle polynomial and its inverse series relation:

$$\left.\begin{aligned} p_n(x) &= \sum_{k=0}^{[n/3]}(-1)^k \frac{(1/2)_{n-2k}}{(n-3k)!\ k!}(3x)^{n-3k} \\ &\Leftrightarrow \\ \frac{(3x)^n}{n!} &= \sum_{k=0}^{[n/3]} \frac{((1/2)+n-3k)}{((1/2)+n-k)}\frac{1}{(1/2)_{n-k}k!}\ p_{n-3k}(x). \end{aligned}\right\}$$

- Meixner polynomial and its inverse series relation:

$$\left.\begin{aligned} &M_n(x;\beta,c) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{(\beta)_k\ k!} \left(1-\frac{1}{c}\right)^k \\ &\Leftrightarrow \\ &\frac{(-x)_n}{(\beta)_n n!}\left(1-\frac{1}{c}\right)^n = \sum_{k=0}^{n}(-n)_k\ M_k(x;\beta,c). \end{aligned}\right\}$$

- Krawtchouk polynomial and its inverse series relation:

$$\left.\begin{aligned} &K_n(x;p,N) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{(-N)_k\ k!} \left(\frac{1}{p}\right)^k \\ &\Leftrightarrow \\ &\frac{(-x)_n}{(-N)_n n!}\left(\frac{1}{p}\right)^n = \sum_{k=0}^{n}(-n)_k\ K_k(x;p,N). \end{aligned}\right\}$$

General class of polynomials due to R.Panda [75] and its inverse series relation[101]:

$$\left.\begin{aligned} &g_m^c(x;r,s) = \sum_{k=0}^{[m/s]} \frac{(c+rk)_{m-sk}}{(m-sk)!}\gamma_k\ x^k \\ &\Leftrightarrow \\ &\gamma_m x^m = \sum_{k=0}^{sm} \frac{(-1)^{sm-k}(c+(rk/s))(c)_{rm}}{(c)_{rm-sm+k+1}(sm-k)!}\ g_k^c(x;r,s). \end{aligned}\right\}$$

Another general class of polynomial studied by Singal and S.Kumari [100] and its inverse series relation are given by

$$\begin{aligned} &f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k\ \gamma_{n-mk}x^{n-mk} \\ &\Leftrightarrow \\ &= \sum_{n=0}^{n} \frac{c+nr-mrk}{c+nr-k} \binom{-c-nr+k}{k}\ f_{n-mk}^c(x,y,r,m). \end{aligned}$$

### 1.2.4 Riodan's Inversion Pairs

John Riordan [82] studied and classified a number of inverse series relations. For example, he classified certian inverse series relations into Simplest classes, Gould classes, Simpler Legendre classes, Chebyshev classes, Simpler Chebyshev classes and Legendre-Chebyshev classes. Many of these inverse pairs are reducible to certain polynomials; for example, the Laguerre polynomial, Legendre polynomial etc.

Here we list out (Table 1.1) those classes of inverse series relations which can be extended to the matrix forms.

The following tables enlist them.

TABLE 1.1: Gould classes of inverse series relations

$$F(n) = \sum C_{n,k}\, G(k); \quad G(n) = \sum (-1)^{n-k}\, D_{n,k}\, F(k)$$

| Sr.no | $A_{n,k}$ | $B_{n,k}$ |
|---|---|---|
| (1) | $\binom{p+qk-k}{n-k}$ | $\frac{p+qk-k}{p+qn-k}\binom{p+qn-k}{n-k}$ |
| (2) | $\frac{p+qn-n+1}{p+qk-n+1}\binom{p+qk-k}{n-k}$ | $\binom{p+qn-k}{n-k}$ |
| (3) | $\binom{p+qn-n}{k-n}$ | $\frac{p+qn-n}{p+qk-n}\binom{p+qk-n}{k-n}$ |
| (4) | $\frac{p+qk-k+1}{p+qn-k+1}\binom{p+qn-n}{k-n}$ | $\binom{p+qk-n}{k-n}$ |

TABLE 1.2: Simpler Legendre classes of inverse series relations

$$F(n) = \sum C_{n,k}\ G(k); \quad G(n) = \sum (-1)^{n-k}\ D_{n,k}\ F(k)$$

| Sr. no | $C_{n,k}$ | $D_{n,k}$ |
|---|---|---|
| (1) | $\binom{p+n+k}{n-k}$ | $\frac{p+2k+1}{p+n+k+1}\binom{p+2n}{n-k}$ |
| (2) | $\binom{p+2n}{n-k}$ | $\frac{p+2n}{p+n+k}\binom{p+n+k}{n-k}$ |
| (3) | $\binom{p+n+k}{k-n}$ | $\frac{p+2n+1}{p+n+k+1}\binom{p+2k}{k-n}$ |
| (4) | $\binom{p+2k}{k-n}$ | $\frac{p+2k}{p+n+k}\binom{p+n+k}{k-n}$ |
| (5) | $\binom{p+2n}{k}$ | $\frac{p+2n}{p+2n-3k}\binom{p+2n-3k}{k}$ |
| (6) | $\frac{p+2n-4k+1}{p+2n-k+1}\binom{p+2n}{k}$ | $\binom{p+2n-3k}{k}$ |

TABLE 1.3: Legendre-Chebyshev classes of inverse series relations

$$F(n) = \sum C_{n,k} G(k); \quad G(n) = \sum (-1)^{n-k} D_{n,k} F(k)$$

| Sr. no | $C_{n,k}$ | $D_{n,k}$ |
|---|---|---|
| (1) | $\binom{p+cn}{n-k}$ | $\frac{p+cn}{p+ck}\binom{p+n+ck-k-1}{n-k}$ |
| (2) | $\binom{p+cn}{k-n}$ | $\frac{p+cn}{p+ck}\binom{p+ck+k-n-1}{k-n}$ |
| (3) | $\binom{p+ck}{n-k}$ | $\frac{p+ck}{p+cn}\binom{p+cn+n-k-1}{n-k}$ |
| (4) | $\binom{p+ck}{k-n}$ | $\frac{p+ck}{p+cn}\binom{p+cn-n+k-1}{k-n}$ |

*Continue*

| Sr. no | $C_{n,k}$ | $D_{n,k}$ |
|---|---|---|
| (5) | $\frac{p+ck+1}{p+cn-n+k+1}\binom{p+cn}{n-k}$ | $\binom{p+n+ck-k}{n-k}$ |
| (6) | $\frac{p+ck+1}{p+cn+n-k+1}\binom{p+cn}{k-n}$ | $\binom{p+ck+k-n}{k-n}$ |
| (7) | $\frac{p+cn+1}{p+ck-n+k+1}\binom{p+ck}{n-k}$ | $\binom{p+cn+n-k}{n-k}$ |
| (8) | $\frac{p+cn+1}{p+ck-n+k+1}\binom{p+ck}{n-k}$ | $\binom{p+cn+n-k}{n-k}$ |

## 1.3 Matrix Form:

### 1.3.1 Preliminaries

Let $\mathbb{C}^{p\times p}$ be a family of square matrices of order $p$ having in general, complex entries. For a matrix $A$ in $\mathbb{C}^{p\times p}$, let $\sigma(A)$ be the set of all eigenvalues of $A$. The matrix $A$ is said to be positive stable matrix if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$.

A matrix polynomial of degree $n$ in $x$ is a polynomial of the form:

$$F(x) = P_n x^n + P_{n-1}x^{n-1} + P_{n-2}x^{n-2} + ... + P_1 x + P_0, \tag{1.3.1}$$

where the matrices $P_0, P_1, P_2, \ldots, P_n$ are in $\mathbb{C}^{p\times p}$ and $P_n$ is a non-zero matrix. Let $A, B$ be matrices in $\mathbb{C}^{p\times p}$ and $r \in \mathbb{C}$. Then the 2- norm has the following properties:

1.$\| rA \| = |r| \, \| A \|$

2.$\| A \| = 0$ if and only if $A = 0$

3.$\| A + B \| \leq \| A \| + \| B \|$

4.$\| AB \| \leq \| A \| \| B \|$

5. $\| I_{p\times p} \| = 1$

We illustrate the following types of the norm of a square matrix [88]. The Column norm of a matrix $A$ denoted by $\| A \|_1$ is defined by

$$\| A \|_1 = \max_j \sum_i |a_{ij}|. \tag{1.3.2}$$

The Row norm of a matrix $A$ denoted by $\|A\|_\infty$ is defined by

$$\| A \|_\infty = \max_i \sum_j |a_{ij}|. \tag{1.3.3}$$

The 2- norm or the Euclidean norm of a matrix $A$ is denoted and defined by

$$\| A \|_2 = \sup_{x=0} \frac{\| Ax \|_2}{\| x \|_2} = max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\}, \tag{1.3.4}$$

where $A^*$ denotes the transposed conjugate of $A$ and for any vector u in the p-dimensional complex space, $\| u \|_2 = (u^*u)^{\frac{1}{2}}$ is the Euclidean norm of u.

**Example 1.** Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We find that

$$\| A \|_1 = max[1+4+7, 2+5+8, 3+6+9] = max[12, 15, 18] = 18,$$

$$\| A \|_2 = (1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2)^{1/2} = 16.88,$$

$$\| A \|_\infty = max[1+2+3, 4+5+6, 7+8+9] = max[6, 15, 24] = 24.$$

**Example 2.** If we consider the matrix

$$A = \begin{bmatrix} 1+2i \\ 2i \\ 1 \end{bmatrix},$$

then we have

$$\| A \|_2 = ((1-2i)(1+2i) + (-2i)(2i) + 1)^{1/2} \ = \ \sqrt{10}.$$

Now, if $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z$ which are defined on an open set $\Omega$ of the complex plane, and if $\sigma(A) \subset \Omega$ then from the properties of the matrix functional calculus [24], it follows that

$$f(A)g(A) = g(A)f(A). \tag{1.3.5}$$

Moreover, if $B$ in $\mathbf{C}^{p\times p}$ is a matrix for which $\sigma(B) \subset \Omega$, and $AB = BA$ , then

$$f(A)g(B) = g(B)f(A). \tag{1.3.6}$$

Exponential Matrix function is defined as follows [11, Eq.(6), p.2]

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + ... \tag{1.3.7}$$

where $A$ is positive stable matrices in $C^{p\times p}$.

We note that

$$\begin{aligned} x^I &= e^{I \log x} \\ &= \sum_{n=0}^{\infty} \frac{I^n (logx)^n}{n!} = I \sum_{n=0}^{\infty} \frac{(logx)^n}{n!} \\ &= e^{logx} I = xI. \end{aligned}$$

Hence, $x^A = x(IA) = xA$ for $A \neq O$.

Also, we have [13, p.379]

$$\|e^A\| \leq e^{\|A\|}.$$

Using the Schur decomposition of $A$, it follows that [107]

$$\|e^{tA}\| \leq e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{1/2} t)^k}{k!}, \quad t \geq 0. \tag{1.3.8}$$

If $A$ is matrix in $\mathbb{C}^{p\times p}$ such that $\Re(\lambda) > 0$ for all eigenvalues $\lambda$ of $A$, then $\Gamma(A)$ is well defined as

$$\Gamma(A) = \int_0^\infty e^x x^{A-I}\, dx, \qquad x^{A-I} = exp((A-I)\ln x). \tag{1.3.9}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = (\Gamma(z))^{-1} = \frac{1}{\Gamma(z)}$ is an entire function of complex variable $z$ [45, p. 253] and thus for any matrix $A$ in $C^{p\times p}$, the functional calculus [24] shows that $\Gamma^{-1}(A)$ is a well defined matrix.

If $I$ denotes the identity matrix of order $p$ and $A+nI$ is invertible for every integer $n \geq 0$ then ([54])

$$\Gamma^{-1}(A) = A(A+I)\cdots(A+(n-1)I)\Gamma^{-1}(A+nI). \tag{1.3.10}$$

From this, the functional equation of the gamma matrix function occurs in the form

$$A\ \Gamma(A) = \Gamma(A+I) \tag{1.3.11}$$

which readily follows for $n = 1$. For a matrix $A$ in $\mathbf{C}^{p\times p}$, the Pochhammer matrix symbol is defined by [54]

$$(A)_n = \begin{cases} I, & if \quad n = 0 \\ A(A+I)\cdots(A+(n-1)I), & if \ n \geq 1. \end{cases} \tag{1.3.12}$$

If we denote by $\mu(A)$ the logarithmic norm of A, and by $\tilde{\mu}(A)$ the number $-\mu(-A)$, then

$$\mu(A) = max\{z; z \in \sigma[(A + A^H)/2]\}, \tag{1.3.13}$$

and

$$\tilde{\mu}(A) = min\{z; z \in \sigma[(A + A^H)/2]\}. \tag{1.3.14}$$

If $A$ and $B$ are positive stable matrices in $\mathbf{C}^{p\times p}$, then the Beta function is defined as

$$\mathbf{B}(A, B) = \int_0^1 t^{A-I}(1-t)^{B-I}\, dt. \tag{1.3.15}$$

Let $A$ and $B$ be commutative matrices in $\mathbf{C}^{p\times p}$ such that $A$, $B$ and $A + B$ are positive stable matrices, then

$$\mathbf{B}(A, B) = \Gamma(A)\Gamma(B)\Gamma^{-1}(A + B). \tag{1.3.16}$$

For an arbitrary matrix A in $\mathbb{C}^{p\times p}$,

$$(A)_{m+k} = (A)_m(A + mI)_k. \tag{1.3.17}$$

Also, if $I - A - nI$ is invertible for all $n \geq 0$, then

$$(A)_{n-k} = (-1)^k\; n!\; (A)_n\; (I - A - nI)_k^{-1}. \tag{1.3.18}$$

If $A - nI$ is invertible for all $n \geq 1$, then in view of the product

$$(-A + I)_n(A - I)^{-1}(A - 2I)^{-1} \cdots (A - nI)^{-1} = (-1)^n\; I,$$

we define

$$(A)_{-n} = (A - I)^{-1}(A - 2I)^{-1} \cdots (A - nI)^{-1} = (-1)^n(-A + I)_n^{-1}. \tag{1.3.19}$$

Hence,

$$\Gamma(A-nI)\Gamma^{-1}(A)=(A)_{-n}=(-1)^n(-A+I)_n^{-1}. \tag{1.3.20}$$

For any matrix A in $C^{p\times p}$ and for $|x|<1$, the following series expansion holds [53].

$$(1-x)^{-A}=\sum_{n=0}^{\infty}\frac{(A)_n}{n!}x^n. \tag{1.3.21}$$

If $A\in\mathbb{C}^{r\times r}$ is a positive stable matrix and $n\geq 1$ is an integer, then the gamma matrix function can also be defined in the form of a limit as [54]

$$\Gamma(A)=\lim_{n\to\infty}(n-1)!\,(A)_n^{-1}\,n^A. \tag{1.3.22}$$

It is to be mentioned here that the notation $\Delta$ is used to denote the forward difference operator, hence we adopt the notation $\prec m;A\succ$ for

$$(A)_{mk}=m^{mk}\prod_{i=1}^{m}\left(\frac{A+(i-1)I}{m}\right)_k. \tag{1.3.23}$$

In particular, for non negative integer $n$,

$$\prec m;-nI\succ=(-nI)_{mk}=\frac{(-1)^{mk}\;n!}{(n-mk)!}I=m^{mk}\prod_{i=1}^{m}\left(\frac{-n+i-1}{m}\;I\right)_k. \tag{1.3.24}$$

The generalized hypergeometric matrix function is defined as follows [94, Eq. (2.2), p. 608].

**Definition 1.3.1.** *If $\{A_i;i=1,2,\ldots,\ell\}$, and $\{B_j;j=1,2,\ldots,m\}$, are sequences of matrices in $C^{p\times p}$ such that $B_j+nI$ are invertible for all $n\geq 0$, then the generalized hypergeometric matrix function is defined as*

$$\begin{aligned}&{}_\ell F_m\left(A_1,A_2,\ldots,A_\ell;B_1,B_2,\ldots,B_m;x\right)\\ =&\sum_{n=0}^{\infty}(A_1)_n(A_2)_n\ldots(A_\ell)_n\;[(B_1)_n]^{-1}[(B_2)_n]^{-1}\ldots[(B_m)_n]^{-1}\;\frac{x^n}{n!}.\end{aligned} \tag{1.3.25}$$

*Here, the series converges for all $x$ if $\ell \leq m$. If $\ell = m+1$, then the series converges for $|x| < 1$. If $\ell > m+1$, then the series diverges for all $x \neq 0$.*

The notation $\binom{r}{s}$ will stand only for the binomial coefficient $\dfrac{r!}{(r-s)!s!}$, but will not be used to denote the column matrix.

For $Q(k,n) \in \mathbb{C}^{p\times p}$, $n, k \geq 0$ and $m, s \in \mathbb{N}$, there hold the double series identities [76, 80]:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{[k/m]} Q(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} Q(j,k+mj), \tag{1.3.26}$$

$$\sum_{k=0}^{N}\sum_{j=0}^{N-k} Q(k,j) = \sum_{j=0}^{N}\sum_{k=0}^{j} Q(k,j-k), \tag{1.3.27}$$

$$\sum_{k=0}^{mn}\sum_{j=0}^{[k/m]} Q(k,j) = \sum_{j=0}^{n}\sum_{k=0}^{mn-mj} Q(k+mj,j), \tag{1.3.28}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} Q(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} Q(j,j+k), \tag{1.3.29}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} Q(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} Q(j,k-j), \tag{1.3.30}$$

$$\sum_{k=1}^{n}\sum_{j=1}^{k} Q(k,j) = \sum_{j=1}^{n}\sum_{k=0}^{n-j} Q(k+j,j), \tag{1.3.31}$$

$$\sum_{k=s}^{sn}\sum_{j=1}^{[k/s]} Q(k,j) = \sum_{j=1}^{n}\sum_{k=sj}^{sn} Q(k,j). \tag{1.3.32}$$

### 1.3.2 Examples

**Example 1.3.1.** *Let us evaluate* $\Gamma(A)$*, where*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

*The eigen values of* $A$ *are* $1, 2$.

*Now, we have [1, Eq.(8), p. 64]*

$$\Gamma(A) = \int_0^\infty e^{-t} t^{A-I} dt.$$

*We first find* $t^{A-I}$*, where*

$$A - I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

*Since for* $n \geq 1$,

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

*and*

$$e^{B \log t} = \sum_{n=0}^{\infty} \frac{B^n (\log t)^n}{n!},$$

*we find that*

$$t^{A-I} = t^{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix}.$$

*Hence,*

$$e^{-t} t^{A-I} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t}(t-1) & te^{-t} \end{bmatrix}.$$

*Consequently,*

$$\Gamma\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Alternatively, following [83], if $f(s)$ is a scalar function which is analytic in some region $R$ of complex plane, then

$$f(s) = \sum_{k=0}^{\infty} \beta_k \, s^k.$$

Now if $P$ be an $n \times n$ matrix with characteristic polynomial $\Delta(s)$ and eigenvalues $\lambda_i$ then f(s) may be written as $f(s) = \Delta(s)Q(s) + R(s)$, where $R(s)$ is of degree $\leq n-1$. Now, from Cayley-Hamilton theorem,

$$f(\lambda_i) = R(\lambda_i) = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k. \tag{1.3.33}$$

This yields the system of simultaneous equations in $\alpha_k' s$. Thus for matrix function $f(A)$, we have

$$f(A) = R(A) = \sum_{k=0}^{n-1} \alpha_k A^k.$$

The $\alpha_k' s$ are determined from (1.3.33). Thus, taking

$$P = A - I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

we find the eigen values to be $\lambda_1 = 0, \lambda_2 = 1$. The corresponding system of equations is

$$1 = \alpha_0 + 0;$$

$$t = \alpha_0 + \alpha_1.$$

This provides us $\alpha_0 = 1, \alpha_1 = t - 1$. Now,

$$t^A = e^{P \log t} = \alpha_0 I + \alpha_1 P = \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix},$$

Consequently,

$$\Gamma\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right) = \int_0^{\infty} e^{-t} \begin{bmatrix} 1 & 0 \\ t-1 & t \end{bmatrix} dt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Note 1.3.1.** *The above example illustrates that if the eigen value $\lambda$ of a matrix $A$ has real part positive, then $\Gamma(A)$ is defined; otherwise, $\Gamma(A)$ will not be defined. The following example illustrates this. Thus, whenever we encounter the Gamma matrix function, we must have the associated matrix to be positive stable.*

**Example 1.3.2.** *Consider the matrix* $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$.

*Its eigen values are* $0, 5$. *Here* $A - I = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} = P$, *say. Hence, we have to find matrix representation of* $t^P$. *The eigen values to be* $\lambda_1 = 4, \lambda_2 = -1$. *The corresponding system of equations is*

$$t^4 = \alpha_0 + 4\alpha_1;$$

$$t^{-1} = \alpha_0 - \alpha_1.$$

*Solving this system, we find* $\alpha_0 = \dfrac{t^4 + 4t^{-1}}{5}$ *and* $\alpha_1 = \dfrac{t^4 - t^{-1}}{5}$.

*Now,*

$$t^A = e^{P \log t} = \alpha_0 I + \alpha_1 P = \frac{1}{5}\begin{bmatrix} 3t^4 + 2t^{-1} & t^4 - t^{-1} \\ 6t^4 - 6t^{-1} & 2t^4 + 3t^{-1} \end{bmatrix},$$

*Hence,*

$$\Gamma\left(\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\right) = \frac{1}{5}\int_0^\infty e^{-t}\begin{bmatrix} 3t^4 + 2t^{-1} & t^4 - t^{-1} \\ 6t^4 - 6t^{-1} & 2t^4 + 3t^{-1} \end{bmatrix} dt. \tag{1.3.34}$$

*Since*

$$\int_0^\infty e^{-t} t^{z-1} dt$$

*is divergent for* $\Re(z) \le 0$, *hence one of the integrals occurring in (1.3.34)*

$$\int_0^\infty e^{-t} t^{0-1} dt$$

*will be divergent because* $\Re(z) = 0$.

*Thus, for the given matrix* $A$ *whose eigen value are* $0$ *and* $5$, $\Gamma(A)$ *is not defined.*

## 1.4 Matrix Polynomials

As mentioned in earlier section, we enlist the explicit representations of the matrix polynomials below.

- Laguerre matrix polynomial [52, Eq.(3.7), p.58]:

$$L_m^{(A,\lambda)}(x) = \sum_{n=0}^{m} \frac{(-1)^n}{n!(m-n)!} \,(A+I)_m \,[(A+I)_n]^{-1}\, (\lambda x)^n. \tag{1.4.1}$$

- Konhauser matrix polynomial [108, Eq.(27), p.197]:

$$Z_m^{(A,\lambda)}(x;k) = \Gamma(kmI + A + I) \sum_{n=0}^{m} \frac{(-1)^n (\lambda x)^{nk}}{(m-n)!n!}\, \Gamma^{-1}(knI + A + I). \tag{1.4.2}$$

- Hermite matrix polynomial [55, Eq.(12), p.14]:

$$H_m(x, A) = m! \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k!(m-2k)!}\, (x\sqrt{2A})^{m-2k}. \tag{1.4.3}$$

- Jacobi matrix polynomial [21, Eq.(16), p.793]:

$$\begin{aligned} P_m^{(A,B)}(x) \;&=\; (-1)^m \frac{[(B+I)_m]}{m!} \sum_{k=0}^{m} \frac{(-mI)_k}{k!} (A+B+mI+I)_k \\ &\times [(B+I)_k]^{-1} \left(\frac{1+x}{2}\right)^k. \end{aligned} \tag{1.4.4}$$

- The Gegenbauer matrix polynomials [90, Eq.(15), p.104]:

$$C_m^A(x) = \sum_{k=0}^{[m/2]} (-1)^k \frac{(A)_{m-k}}{(m-2k)!\; k!}\, (2x)^{m-2k}. \tag{1.4.5}$$

- Legendre matrix polynomial [97, p.355]:

$$P_m(x, A) \;=\; \sum_{k=0}^{m} \frac{(-1)^k (m+k)! (A)_k^{-1}}{k!(m-k)!} \left(\frac{1-x}{2}\right)^k. \tag{1.4.6}$$

- Chebychev matrix polynomial of first kind [20, Eq.(36), p.115]:

$$T_m(x, A) = \sum_{k=0}^{m} \frac{(-1)^k (m+k-1)!}{(m-k)!\ 2^k\ k!} \Gamma(A)\Gamma(A+kI)(1-x)^k. \quad (1.4.7)$$

- Chebychev matrix polynomial of second kind [70, Eq.(2.3), p.1040]:

$$U_m(x, A) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (m-k)!}{(m-2k)!\ k!} (x\sqrt{2A})^{m-2k}. \quad (1.4.8)$$

The Generalized Humbert matrix polynomial in two variables has been studied by M.A.Pathan [76, Eq.(2.3), p.210] which coincides with Generalized Humbert matrix polynomial in one variable given below, when $a = y = 1$.

- Generalized Humbert matrix polynomial:

$$P_n^A(m, x, \eta, c) = \sum_{k=0}^{[n/m]} \eta^k \frac{c^{A-(n-(m-1)k)I}}{(n-mk)!\ k!} \Gamma^{-1}(A+(1-n+mk-k)I)$$
$$\times \Gamma(A+I)(-mx)^{n-mk}. \quad (1.4.9)$$

- Humbert matrix polynomial [76, Eq.(2.8), p.211]:

$$\Pi_{n,m}^A(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(-A)_{n-mk+k}}{(n-mk)!\ k!} (mx)^{n-mk}. \quad (1.4.10)$$

- Pincherle matrix polynomials [76, Eq.(2.10),p.212]:

$$P_n^A(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k (A)_{n-2k} (3x)^{n-3k}}{(n-3k)!\ k!}. \quad (1.4.11)$$

- Kinny matrix polynomials (cf. [14, Eq.(1.2.27), p.12]):

$$K_n^A(m, x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{((-1/m)I)_{n-mk+k}}{(n-mk)!\ k!} (mx)^{n-mk}. \quad (1.4.12)$$

## 1.5 Operators $E$ and $\Delta$

It is well known that the forward difference operator $\Delta$ on a real function $f(x)$ is defined as [44, Eq.(5.2.1), p.175]

$$\Delta f(x) = f(x+h) - f(x),$$

where $h$ is the length of sub intervals of a given interval $[a, b]$, say. Similarly. the shift oprator $E$ is defined as [44, Eq.(5.2.3), p.175]

$$Ef(x) = f(x+h),$$

wherein the number $h$ is same as specified above.

We now consider the action of these two operators on a matrix polynomial function (1.3.1). They are defined as follows.

**Definition 1.5.1.** *Let $P(x)$ be a matrix polynomial of degree $n$ in $x$ of the form (1.3.1), then*

$$E(P(x)) = P(x+h) \tag{1.5.1}$$

*and*

$$\Delta P(x) = P(x+h) - P(x), \tag{1.5.2}$$

*where the constant $h > 0$ is difference of the equidistant values of $x$.*

*Also, for $r \in \mathbb{N}, E^r P(x) = E^{r-1}(EP(x))$, and $\Delta^r P(x) = \Delta^{r-1}(\Delta P(x))$.*

*For $r = 0, E^0 := \mathcal{I}, \Delta^0 := \mathcal{I}$, the identity operator.*

### 1.5.1 Auxiliary result

We prove the following Lemma which will be used in the derivation of inverse matrix series relation in the next chapters.

**Lemma 1.5.1.** *Let $P(x)$ be a matrix polynomial in $x$ of degree less than $n$, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(a+kh)=O, \tag{1.5.3}$$

*where $n \geq 1$ and $a, h$ are constants.*

*Proof.* If the $deg[P(x)] < n$, then in view of Definition 1.5.1, $\Delta^n$ annihilates $P(x)$ (cf. [44], p. 179). To see this, take $P(x) = A_1x + A_0$, a linear matrix polynomial, then

$$\Delta^2 P(x) = \Delta^2(A_1x + A_0) = O$$

as $\Delta^2 x = 0$. This may be generalized easily. Thus,

$$\Delta^n P(x) = O.$$

Now, $\Delta = E - \mathcal{I}$ [44, Eq.(5.2.11), p.178], hence using the operator binomial expansion,

$$\begin{aligned}
\Delta^n P(x) &= (E-\mathcal{I})^n P(x) \\
&= \left[E^n - \binom{n}{1}E^{n-1} + \binom{n}{2}E^{n-2} + ... + (-1)^n \mathcal{I}\right]P(x) \\
&= E^nP(x) - \binom{n}{1}E^{n-1}P(x) + ... + (-1)^n P(x) \\
&= P(x+nh) - \binom{n}{1}P(x+(n-1)h) + ... + (-1)^n P(x) \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{n-k}P(x+kh).
\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(x+kh) = O.$$

which leads to the lemma for $x = a$. □

## 1.6 Fractional Order Integral and Derivatives

Let $x > 0$ and $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$. Then the Riemann-Liouville type fractional order integral and derivatives of order $\mu$ are given by ([6], [85])

$$(\mathbf{I}_a^\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt \tag{1.6.1}$$

and

$$\mathbf{D}_a^\mu f(x) = (\mathbf{I}_a^{n-\mu} \mathbf{D}^n f(x)), \quad \mathbf{D} = \frac{d}{dx}. \tag{1.6.2}$$

Using these operators, Bakhet and et.al.[6], studied the fractional order integrals and derivatives of Wright hypergeometric matrix function and incomplete Wright hypergeometric matrix function.

We define the fractional order integral of $x^A$ as follows [6].

**Definition 1.6.1.** *Let $A$ be a positive stable matrix in $\mathbb{C}^{p\times p}$ and $\mu \in C$ such that $\Re(\mu) > 0$. Then, the Riemann-Liouville fractional integrals of order $\mu$ may be defined as follows*

$$\mathbf{I}^\mu(x^A) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^A dt. \tag{1.6.3}$$

**Lemma 1.6.1.** *Let $A$ be a positive stable matrix in $\mathbb{C}^{p\times p}$ and $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$. Then from (1.6.3),*

$$\mathbf{I}^\mu(x^{A-I}) = \Gamma(A)\Gamma^{-1}(A+\mu I)x^{A+(\mu-1)I}. \tag{1.6.4}$$

*Proof.* Using the definition, we have

$$\begin{aligned} \mathbf{I}^\mu(x^{A-I}) &= \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^{A-I} dt. \\ &= \frac{1}{\Gamma(\mu)} \int_0^x x^{\mu-1} \left(1 - \frac{t}{x}\right)^{\mu-1} t^{A-I} dt. \end{aligned}$$

Now, putting $u = \dfrac{t}{x}$, this gives

$$\begin{aligned}
\mathbf{I}^{\mu}(x^{A-I}) &= \frac{1}{\Gamma(\mu)} \int_0^x x^{\mu}(1-u)^{\mu-1}(ux)^{A-I} du. \\
&= \frac{x^{A+(\mu-1)I}}{\Gamma(\mu)} \int_0^x (1-u)^{\mu-1} u^{A-I} du. \\
&= \frac{x^{A+(\mu-1)I}}{\Gamma(\mu)} B(\mu I, A) \\
&= \Gamma(A)\Gamma^{-1}(A+\mu I) x^{A+(\mu-1)I}. \qquad (1.6.5)
\end{aligned}$$

□