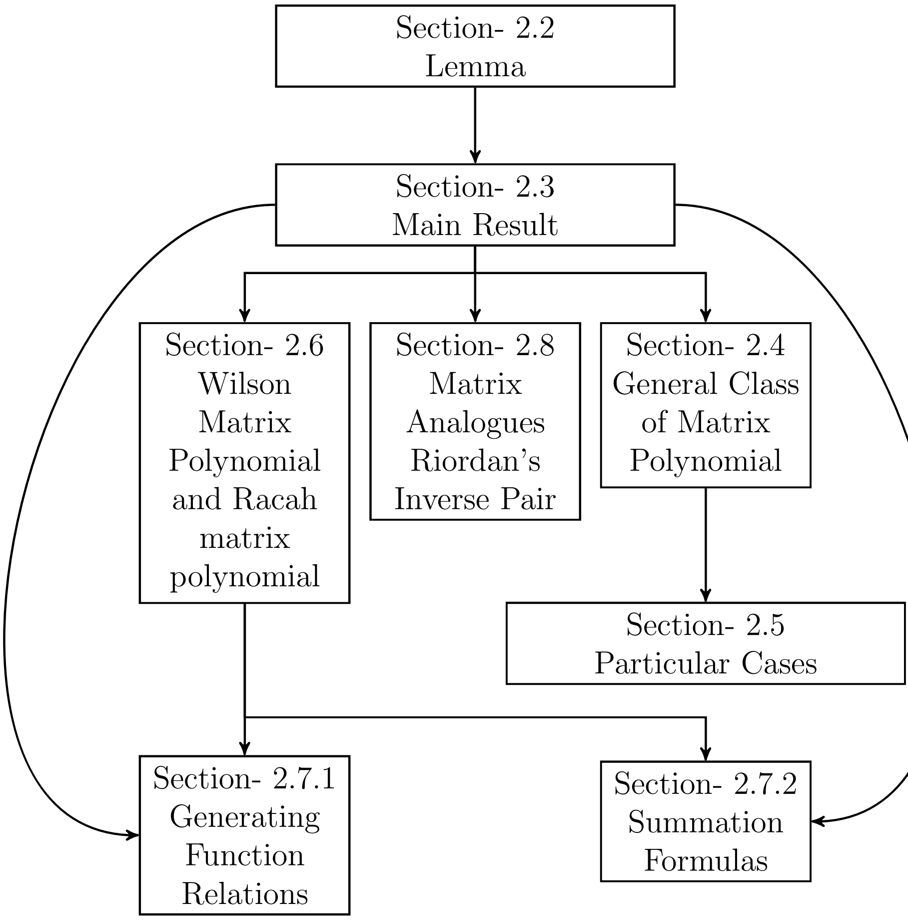
Chapter 2

A General Inverse Matrix Series Relation and Associated Polynomials-I



2.1 Introduction

In this chapter, we establish a general inverse matrix series relation and thereby deduce a matrix polynomial of general nature. For this polynomial, we obtain the inverse series relation, generating function relations and summations formulas. This polynomial turns out to be a matrix analogue of the general class of polynomials [75]:

$$\{g_n^c(x,r,s)|n=0,1,2,3..\}$$

where $s\in\mathbb{N}, r\in\mathbb{Z}$ and c is arbitrary complex parameter.
This polynomial is generated by the relation [75]:

$$(1-t)^{-c}G\left(\frac{xt^s}{(1-t)^r}\right)=\sum_{k=0}^{\infty}g_n^c(x,r,s)t^n. \tag{2.1.1}$$

From this, the explicit representation occurs in the form [75]:

$$g_n^c(x,r,s;x)=\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(c+rk)_{n-sk}}{(n-sk)!}\ \gamma_k\ x^k. \tag{2.1.2}$$

This polynomial contains the special polynomials such as the extended Jacobi polynomial, Classical Jacobi polynomial, Lagurre polynomial etc. The inverse series of the polynomial (2.1.2) is given by [101]

$$\gamma_n\ x^n\ =\ \sum_{k=0}^{sn}\frac{(-1)^{sn-k}(c+(rk/s))(c)_{rn}}{(c)_{rn-sn+k+1}(sn-k)!}g_k^c(x,r,s). \tag{2.1.3}$$

In order to obtain the inverse series of the proposed matrix polynomial, we establish a general matrix inversion theorem as a special case of which we deduce the inverse series relation of matrix analogue of (2.1.2).

2.2 Lemma

The proof of inversion theorem uses the particular inverse matrix series relation which we state and prove here as

Lemma 2.2.1. *The following matrix series relations hold true.*

$$A_N = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma^{-1}(A + kB - NI + I)\ B_k \tag{2.2.1}$$

$$\Leftrightarrow$$

$$B_N = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma(A + NB - kI)\ (A + kB - kI)\ A_k. \tag{2.2.2}$$

where A and B are positive stable matrices in $\mathbb{C}^{p\times p}$.

Proof. The proof runs as follows. We begin with substituting the values $N = 0$ and 1 and obtain the following inverse matrix relation.

For $N = 0$, the trivial relation holds. That is,

$$A_0 = \Gamma^{-1}(A + I)\ B_0 \Leftrightarrow B_0 = \Gamma(A)A\ A_0.$$

For $N = 1$, we have from (2.2.1),

$$\begin{aligned}
A_1 &= \sum_{k=0}^{1} (-1)^k \binom{1}{k} \Gamma^{-1}(A + kB - I + I)\ B_k \\
&= \Gamma^{-1}(A)\ B_0 - \Gamma^{-1}(A + B)B_1 \\
\Leftrightarrow \Gamma^{-1}(A + B)B_1 &= \Gamma^{-1}(A)\ B_0 - A_1. \\
\Leftrightarrow \qquad B_1 &= \Gamma(A + B)\ \Gamma^{-1}(A)\ B_0 - \Gamma(A + B)\ A_1. \\
\Leftrightarrow \qquad B_1 &= \Gamma(A + B)\Gamma^{-1}(A)\Gamma(A)A\ A_0 - \Gamma(A + B - I + I)\ A_1. \\
\Leftrightarrow \qquad B_1 &= \Gamma(A + B)\ A\ A_0 - \Gamma(A + B - I)(A + B - I)\ A_1.
\end{aligned} \tag{2.2.3}$$

Thus for $N = 0, 1$, the series (2.2.1) holds if only if series (2.2.2) holds.

The lemma is now proved for any $N \in \mathbb{N}$.

Part(I): The series (2.2.2) $\Rightarrow$ the series (2.2.1). For that we assume that (2.2.2) holds true. If $\overline{\eta}$ stands for the right hand side of (2.2.1), then on substituting the series for B_k from (2.2.2), we get

$$\begin{aligned}\overline{\eta} &= \sum_{k=0}^{N}(-1)^k\binom{N}{k}\Gamma^{-1}(A+kB-NI+I)B_k\\ &= \sum_{k=0}^{N}(-1)^k\binom{N}{k}\Gamma^{-1}(A+kB-NI+I)\\ &\quad\times\sum_{i=0}^{k}(-1)^i\binom{k}{i}\Gamma(A+kB-iI)\ (A+iB-iI)A_i.\end{aligned}$$

Now in view of the double series identity (1.3.27), we have

$$\begin{aligned}\overline{\eta} &= \sum_{i=0}^{N}\sum_{k=0}^{N-i}(-1)^k\binom{N}{k+i}\binom{k+i}{i}\Gamma^{-1}(A+kB+iB-NI+I)\\ &\quad\times\ \Gamma(A+kB+iB-iI)\ (A+iB-iI)\ A_i\\ &= A_N+\sum_{i=0}^{N-1}\sum_{k=0}^{N-i}(-1)^k\binom{N}{k+i}\binom{k+i}{i}\Gamma^{-1}(A+kB+iB-NI+I)\\ &\quad\times\Gamma(A+kB+iB-iI)\ (A+iB-iI)\ A_i\\ &= A_N+\sum_{i=0}^{N-1}\binom{N}{i}\sum_{k=0}^{N-i}(-1)^k\binom{N-i}{k}\Gamma^{-1}(A+kB+iB-NI+I)\\ &\quad\times\Gamma(A+kB+iB-iI)(A+iB-iI)\ A_i,\end{aligned}$$

in which we used the identity:

$$\binom{N}{k+i}\binom{k+i}{i}=\binom{N}{i}\binom{N-i}{k}.$$

We note that the simplification of the product of the Gamma matrix function and inverse Gamma matrix function simplifies to the matrix polynomial in k.

For illustration, let us take $N = 5, i = 2$ and denote $A + iB - iI$ by C, then

$$\begin{aligned}
&\Gamma(C + kB)\Gamma^{-1}(C + kB - 2I)\\
&= \Gamma(C + kB - I + I)\Gamma^{-1}(C + kB - 2I)\\
&= (C + kB - I)\Gamma(C + kB - I)\Gamma^{-1}(C + kB - 2I)\\
&= (C + kB - I)(C + kB - 2I)\Gamma(C + kB - 2I)\Gamma^{-1}(C + kB - 2I)\\
&= (C + kB)^2 - 3I(C + kB) + 2I\\
&= \sum_{r=0}^{2} S_r\, k^r,
\end{aligned}$$

say, where $S_2 = B^2 \neq O$. Thus,

$$\begin{aligned}
\overline{\eta} &= A_N + \sum_{i=0}^{N-1}\binom{N}{i}\left[\sum_{k=0}^{N-i}(-1)^k\binom{N-i}{k}\sum_{r=0}^{N-i-1} T_r\, k^r\right](A + iB - iI)A_i\\
&= A_N + \sum_{i=0}^{N-1}\binom{N}{i}\left[\sum_{r=0}^{N-i-1} T_r \sum_{k=0}^{N-i}(-1)^k\binom{N-i}{k} k^r\right](A + iB - iI)A_i.
\end{aligned}$$

Here the inner two series are $(N - i)^{th}$ difference of the polynomial of degree $N - i - 1$, hence it follows from Lemma 1.5.1 that $\overline{\eta} = A_N + O$.
Also, it may be seen that the diagonal elements $(-1)^N\Gamma^{-1}(A+NB-NI+I)$ of the series (2.2.1) and those given by $(-1)^N\Gamma(A+NB-NI+I)$ of the series (2.2.2) are non singular matrices for every matrix $A \neq -NB+NI-jI, j = 0, 1, 2, \ldots,$ hence the inverse of the block matrix corresponding to the each series in the statement of the lemma, will be unique. Since (2.2.2)$\Rightarrow$ (2.2.1), it follows that (2.2.1) $\Leftrightarrow$ (2.2.2). □

It is interesting to note that if either the sequence $\{A_i : i = 0, 1, 2, \ldots\}$ or $\{B_i : i = 0, 1, 2, \ldots\}$ is chosen to be $\{\binom{0}{i}I\}$, then the matrix series orthogonality relation is obtained. This is stated as

Corollary 2.2.1. *There holds the matrix series orthogonality relation:*

$$\binom{0}{N}I = \sum_{k=0}^{N}(-1)^k\binom{N}{k}\,\Gamma^{-1}(A + kB - NI + I)\;\Gamma(A + kB)\;A.$$

Proof. In Lemma 2.2.1 Putting $A_k = \binom{0}{k} I$, we get

$$\binom{0}{N} I = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma^{-1}(A + kB - NI + I)\ B_k. \tag{2.2.4}$$

Here B_k is evaluated from (2.2.2) as follows.

$$\begin{aligned} B_N &= \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma(A + NB - kI)\ (A + kB - kI)\ A_k \\ &= \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma(A + NB - kI)\ (A + kB - kI) \binom{0}{k} I \\ &= \Gamma(A + nB)A. \end{aligned} \tag{2.2.5}$$

Using this in (2.2.4), proves the corollary. □

Now using the lemma, we prove as a main result, the general inversion theorem in the next section.

2.3 Main Result

A general inverse matrix series relation is stated as

Theorem 2.3.1. *Let A and B be positive stable matrices in $\mathbb{C}^{p \times p}$ such that $A + \jmath B - \ell I \neq O$ for every non negative integers $\jmath, \ell$, and $n \neq sm, m \in \boldsymbol{N} \cup \{0\}, s \in \boldsymbol{N} \backslash \{1\}$, then*

$$\boldsymbol{F}(n) = \sum_{k=0}^{\lfloor n/s \rfloor} (-1)^{n-sk} \frac{\Gamma^{-1}(A + skB - nI + I)}{(n - sk)!} \boldsymbol{G}(k) \tag{2.3.1}$$

if and only if

$$\boldsymbol{G}(n) = \sum_{k=0}^{sn} \frac{(A + kB - kI)\Gamma(A + snB - kI)}{(sn - k)!} \boldsymbol{F}(k) \tag{2.3.2}$$

and

$$\sum_{k=0}^{n} \frac{(A+kB-kI)\Gamma(A+nB-kI)}{(n-k)!}\, \boldsymbol{F}(k) = O, \tag{2.3.3}$$

in which the floor function $\lfloor r \rfloor$ = *floor* r, *represents the greatest integer* $\leq r$.

Proof. (I) The series (2.3.1) $\Rightarrow$ the series (2.3.2). Denoting the right hand side of the series (2.3.2) by $\overline{\phi}(n)$ and substituting for $\mathbf{F}(k)$ from (2.3.1), we get

$$\begin{aligned}
\overline{\phi}(n) &= \sum_{k=0}^{sn} \frac{(A+kB-kI)\Gamma(A+snB-kI)}{(sn-k)!}\mathbf{F}(k) \\
&= \sum_{k=0}^{sn} \frac{(A+kB-kI)\Gamma(A+snB-kI)}{(sn-k)!} \\
&\quad \times \sum_{i=0}^{\lfloor k/s \rfloor} (-1)^{k-si}\, \frac{\Gamma^{-1}(A+siB-kI+I)}{(k-si)!}\, \mathbf{G}(i).
\end{aligned}$$

Here using the double series relation (1.3.28), we further get

$$\begin{aligned}
\overline{\phi}(n) &= \sum_{i=0}^{n} \sum_{k=0}^{sn-si} (-1)^k\, \frac{(A+kB+siB-kI-siI)}{(sn-si-k)!\, k!}\Gamma(A+snB-kI-siI) \\
&\quad \times\Gamma^{-1}(A+siB-siI-kI+I)\, \mathbf{G}(i) \\
&= \mathbf{G}(n) + \sum_{i=0}^{n-1} \sum_{k=0}^{sn-si} (-1)^k \frac{(A+kB+siB-kI-siI)}{(sn-si-k)!\, k!} \\
&\quad \times\Gamma(A+snB-kI-siI)\Gamma^{-1}(A+siB-siI-kI+I)\, \mathbf{G}(i) \\
&= \mathbf{G}(n) + \sum_{i=0}^{n-1} \sum_{k=0}^{sn-si} (-1)^k\, (A+kB+siB-kI-siI) \\
&\quad \times \frac{\Gamma(A+snB-kI-siI)}{k!\, (sn-si-k)!}\Gamma^{-1}(A+siB-siI-kI+I)\mathbf{G}(i)
\end{aligned}$$

$$= \mathbf{G}(n) + \sum_{i=0}^{n-1} \sum_{k=0}^{sn-si} (-1)^k \binom{sn-si}{k} \frac{(A + kB + siB - kI - siI)}{(sn-si)!}$$

$$\times \Gamma(A + snB - kI - siI)\ \Gamma^{-1}(A + siB - siI - kI + I)\mathbf{G}(i).$$

Here replacing $\Gamma^{-1}(A + siB - siI - kI + I)$ by A_k in the inner most series, and denoting this inner series by B_{sn-si}, we have

$$\begin{aligned} B_{sn-si} \;\; = \;\; & \sum_{k=0}^{sn-si} (-1)^k \binom{sn-sj}{k} (A + kB + siB - kI - siI) \\ & \times \Gamma(A + snB - kI - siI)\ A_k. \end{aligned} \tag{2.3.4}$$

In view of Lemma 2.2.1, the inverse series of this, is given by

$$A_{sn-si} = \sum_{k=0}^{sn-si} (-1)^k \binom{sn-sj}{k} \Gamma^{-1}(A + siB + kB - snI + I)\ B_k, \tag{2.3.5}$$

where A is replaced by $A + siB - siI$ in the lemma.

Now, the choice $B_N = \binom{0}{N} I$ in the series (2.3.5) yields

$$\begin{aligned} A_{sn-si} \;\; &= \;\; \sum_{k=0}^{sn-si} (-1)^k \binom{sn-sj}{k} \Gamma^{-1}(A + siB + kB - snI + I)\ B_k \\ &= \;\; \sum_{k=0}^{sn-si} (-1)^k \binom{sn-sj}{k} \Gamma^{-1}(A + siB + kB - snI + I)\ \binom{0}{k} I \\ &= \;\; \Gamma^{-1}(A + siB - snI + I). \end{aligned} \tag{2.3.6}$$

Thereby $A_k = \Gamma^{-1}(A + siB - siI - kI + I)$ is recovered. Thus, with these A_r and B_r, the series (2.3.4) provides the matrix series orthogonality relation:

$$\begin{aligned} \binom{0}{sn-si} I \;\; = \;\; & \sum_{k=0}^{sn-si} (-1)^k \binom{sn-si}{k} (A + kB + siB - kI - siI) \\ & \times \Gamma(A + snB - kI - siI)\Gamma^{-1}(A + siB - kI - siI + I). \end{aligned} \tag{2.3.7}$$

Using this in (2.3.4), we finally find

$$\overline{\phi}(n) \;=\; \mathbf{G}(n) + \sum_{i=0}^{n-1} \binom{0}{sn - si} \frac{\mathbf{G}(i)}{(sn-si)!}.$$

Thus, $\overline{\phi}(n) = \mathbf{G}(n)$ and hence (2.3.1) $\Rightarrow$ (2.3.2).

(II) The series (2.3.1) implies the series (2.3.3).

We denote the left hand side of the condition (2.3.3) by $\overline{\zeta}(n)$ to get

$$\overline{\zeta}(n) \;=\; \sum_{k=0}^{n} (A + kB - kI) \frac{\Gamma(A + nB - kI)}{(n-k)!} \mathbf{F}(k).$$

Then substituting the series for $\mathbf{F}(k)$ from (2.3.1), we get

$$\begin{aligned}
\overline{\zeta}(n) \;&=\; \sum_{k=0}^{n} (A + kB - kI) \frac{\Gamma(A + nB - kI)}{(n-k)!} \sum_{i=0}^{\lfloor k/s \rfloor} \frac{(-1)^{k-si}}{(k-si)!} \\
&\quad \times \Gamma^{-1}(A + siB - kI + I)\; \mathbf{G}(i) \\
&= \sum_{i=0}^{\lfloor n/s \rfloor} \sum_{k=0}^{n-si} (-1)^k \, \frac{(A + kB + siB - kI - siI)}{(n - si - k)!\; k!} \Gamma(A + nB - kI - siI) \\
&\quad \times\; \Gamma^{-1}(A + siB - siI - kI + I)\; \mathbf{G}(i) \\
&= \sum_{i=0}^{\lfloor n/s \rfloor} \sum_{k=0}^{n-si} (-1)^k \binom{n - si}{k} (A + kB + siB - kI - siI) \\
&\quad \times \Gamma(A + nB - kI - siI) \Gamma^{-1}(A + siB - siI - kI + I)\; \frac{\mathbf{G}(i)}{(n-si)!} \\
&= \sum_{i=0}^{\lfloor n/s \rfloor} \binom{0}{n - si} \frac{\mathbf{G}(i)}{(n-si)!},
\end{aligned}$$

wherein the last expression $\binom{0}{n-si}$ occurs from the series orthogonality relation (2.3.7) when $n \neq si, i = 1, 2, 3 \ldots.$

Thus, the series (2.3.1) implies both series (2.3.2) and (2.3.3).

(III) The series in (2.3.2) and (2.3.3) imply the series (2.3.1).

Put

$$\psi(n) = \sum_{k=0}^{n} \frac{(A + kB - kI)\Gamma(A + nB - kI)}{(n-k)!} \mathbf{F}(k), \tag{2.3.8}$$

then

$$\psi(sn) = \sum_{k=0}^{sn} \frac{(A + kB - kI)\Gamma(A + snB - kI)}{(sn-k)!} \mathbf{F}(k),$$

and from (2.3.3), $\psi(n) = O$ for $n \neq sm, (m = 1, 2, 3, \ldots)$. Now, from Lemma 2.2.1 with $N = n$, the series (2.3.8) possesses the inverse series:

$$\mathbf{F}(n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma^{-1}(A + kB - nI + I)}{(n-k)!} \psi(k). \tag{2.3.9}$$

But $\psi(sn) = \mathbf{G}(n)$, hence from the inverse pair (2.3.8) and (2.3.9) we obtain relation:

$$\mathbf{G}(n) = \sum_{k=0}^{sn} \frac{(A + kB - kI)\Gamma(A + snB - kI)}{(sn-k)!} \mathbf{F}(k)$$

implies

$$\mathbf{F}(n) = \sum_{k=0}^{\lfloor n/s \rfloor} (-1)^{n-sk} \frac{\Gamma^{-1}(A + skB - nI + I)}{(n-sk)!} \mathbf{G}(k).$$

This completes the converse part and hence the proof of the theorem. □

Remark 2.3.1. *It may be noted that $s = 1$, the theorem coincides with Lemma 2.2.1.*

2.4 General Class of Matrix Polynomials

Let $\{M_j; j = 0, 1, \ldots, N\}$ be a sequence of matrices in $\mathbb{C}^{p\times p}$ which may include the scalars if any. Put $\mathbf{G}(k) = \Gamma(A - rkI + I)\ M_k\ x^k$ and $sB = -(r - s)I$ in the theorem to get

$$\begin{aligned}\mathbf{F}(n) &= \sum_{k=0}^{\lfloor n/s\rfloor} \frac{(-1)^{n-sk}}{(n-sk)!}\Gamma^{-1}(A-rkI-nI+skI+I)\Gamma(A-rkI+I)\ M_k\ x^k \\ &= \sum_{k=0}^{\lfloor n/s\rfloor}(-1)^{n-sk}\frac{[(A-rkI+I)_{-n+sk}]^{-1}}{(n-sk)!}\ M_k\ x^k,\end{aligned}$$

since $\Gamma^{-1}(P-jI)$ and $\Gamma(P)$ commute for $j=0,1,2,\ldots$.

In view of the formula (1.3.18), we replace A by $-A$, n by rk and k by $-n+sk$ to get

$$(-A)_{rk}(-A+rkI)_{n-sk} \;=\; (-1)^{n-sk}(-A)_{rk}[(I+A-rkI)_{-n+sk}]^{-1},$$

that is, $(-1)^{n-sk}(I-A-rkI)^{-1}_{-n+sk}=(-A+rkI)_{n-sk}$.

Thus, we get

$$F(n)=\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-A+rkI)_{n-sk}}{(n-sk)!}\ M_k\ x^k.$$

This suggests a general class of matrix polynomials which may be denoted and defined as follows.

Definition 2.4.1. *For a matrix $C\in\mathbb{C}^{p\times p}$, $r\in\mathbb{Z}$ and $s\in\mathbb{N}$,*

$$H_n(C,r,s;x)=\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(C+rkI)_{n-sk}}{(n-sk)!}\ M_k\ x^k. \tag{2.4.1}$$

The inverse matrix series of this is obtained with the help of the same substitutions: $A=-C, \mathbf{G}(k)=\Gamma(-C-rkI+I)\ M_k\ x^k$, $sB=-(r-s)I$ in (2.3.2). Hence using the above formulas, we get

$$\begin{aligned}&\Gamma(-C-rnI+I)\ M_n\ x^n \\ &= \sum_{k=0}^{sn}\frac{(-C+k(-(r-s)I/s)-kI)}{(sn-k)!}\ \Gamma(-C-(r-s)nI-kI)H_k(C,r,s;x)\end{aligned}$$

$$= \sum_{k=0}^{sn} \frac{(-C-(rk/s)I)}{(sn-k)!} \Gamma(-C-(r-s)nI-kI) H_k(C,r,s;x). \tag{2.4.2}$$

Using the formula (1.3.18), that is

$$(A)_{n-k} = (-1)^k \ n! \ (A)_n \ [(I-A-nI)_k]^{-1}$$

with $A = -C$, k is replaced by $sn-k$ and n is replaced rn, we obtain

$$\begin{aligned} \Gamma(-C-(r-s)nI-kI) &= (-1)^{sn-k} \Gamma(-C-rnI)(I+C+rnI)^{-1}_{k-sn} \\ &= \Gamma(-C-rnI)(-C-rnI)_{sn-k}, \end{aligned}$$

the equation (2.4.2) reduces to the form:

$$(-C-rnI)\Gamma(-C-rnI) \ M_n \ x^n$$

$$= \sum_{k=0}^{sn} \frac{(-C-(rk/s)I)}{(sn-k)!} \Gamma(-C-rnI)(-C-rnI)_{sn-k} H_k(C,r,s;x).$$

That is,

$$M_n \ x^n = \sum_{k=0}^{sn} \frac{(-C-(rk/s)I)}{(sn-k)!} (I-C-rnI)_{sn-k-1} H_k(C,r,s;x). \tag{2.4.3}$$

The equation (2.3.3) is assumed to hold.

Hence forth the condition (2.3.3) will be omitted while deducing the particular inverse pairs.

2.5 Particular Cases

We begin with the special cases of the polynomial (2.4.1). In first place, we obtain a matrix analogue of the *extended Jacobi polynomial* [104] which occurs with the help of the substitutions $r - s = l \in \mathbb{N} \cup \{0\}$ and

$$M_k = (-1)^{sk}\Gamma(C + rkI)\,(A_1)_k \cdots (A_p)_k\,[(B_1)_k]^{-1} \cdots [(B_q)_k]^{-1}/k!,$$

where the matrices $B_i + jI$ are invertible for all $i = 1, 2, \ldots, q$, and $j \in \mathbb{N} \cup \{0\}$. We adopt the convention that (P) represents the array of finitely many parameters $P_1, P_2 \ldots P_i$, say. With these substitutions and notations, we find from $H_n(C, r, s; x)$, the extended Jacobi matrix polynomial denoted here by $\Gamma(C + nI)\mathcal{F}^{C}_{n,l,s}[(A); (B) : x]/n!$ in the form:

$$\begin{aligned}\mathcal{F}^{C}_{n,l,s}[(A); (B) : x] &= \sum_{k=0}^{\lfloor n/s \rfloor} (-nI)_{sk}(C + nI)_{lk}\,(A_1)_k \cdots (A_p)_k \\ &\quad \times [(B_1)_k]^{-1} \cdots [(B_q)_k]^{-1} \frac{x^k}{k!}.\end{aligned} \tag{2.5.1}$$

The inverse series of this is deduced as follows.
We re-write the formula (1.3.18) by replacing n by sn, k by $k + 1$ and A by $I - C - rnI$, then we get

$$(I - C - rnI)_{sn-k-1} = (-1)^{k+1}(I - C - rnI)_{sn}[(C + rnI - snI)_{k+1}]^{-1}.$$

Next, using the formula:

$$(I - A)_k = (-1)^k[(A)_{-k}]^{-1}$$

with $A = C + rnI$ and $k = sn$, the product $(I - C - rnI)_{sn}$ assumes the form:

$$(I - C - rnI)_{sn-k-1} = (-1)^{sn+k+1}\Gamma(C + rnI)\Gamma^{-1}(C + (r - s)nI + kI + I).$$

Hence in view of the inverse series (2.4.3), we find the inverse series of the polynomial (2.5.1) in the form:

$$(A_1)_n \cdots (A_p)_n\,[(B_1)_n]^{-1} \cdots [(B_q)_n]^{-1} \frac{x^n}{n!}$$

$$= \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!\ k!}(C + LkI + kI)[(C + kI)_{ln+1}]^{-1}\mathcal{F}^{C}_{k,l,s}[(A); (B) : x], \quad (2.5.2)$$

where $L = l/s, l = r - s$.

The particular case $r = s$ (that is, $l = 0$) of (2.5.1) leads us to the matrix form of the Brafman polynomial [8, Eq.(52), p. 186]:

$$\mathcal{B}_{n,s}[(A); (B) : x] = \sum_{k=0}^{\lfloor n/s \rfloor} (-nI)_{sk}\ (A_1)_k \cdots (A_p)_k\ [(B_1)_k]^{-1} \cdots [(B_q)_k]^{-1}\ \frac{x^k}{k!}.$$

The inverse series of this follows at once from (2.5.2) in the form:

$$(A_1)_n \cdots (A_p)_n\ [(B_1)_n]^{-1} \cdots [(B_q)_n]^{-1} \frac{x^n}{n!} = \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!\ k!} \mathcal{B}_{k,s}[(A); (B) : x].$$

A worth mentioning instance of (2.5.1) is the matrix analogue of the Jacobi polynomial. This may be obtained by taking $C = A + B + I, p = 0,\ q = 1, B_1 = B + I$ and $x \to ((1+x)/2)$. With this, the matrix polynomial $\mathcal{F}^{A+B+I}_{n,l,s}\left[-; B + I; \frac{1+x}{2}\right]$ reduces to the extended form of the Jacobi matrix polynomial: $(-1)^n\ [(B + I)_n]^{-1}\ n!\ P^{(A,B)}_{n,s,l}(x)$ which is given by (cf. [21, Eq.(16), p.793] with $s = 1$ and $l = 1$)

$$\begin{aligned} P^{(A,B)}_{n,s,l}(x) &= (-1)^n \frac{[(B + I)_n]}{n!} \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-nI)_{sk}}{k!}(A + B + nI + I)_{lk} \\ &\quad \times [(B + I)_k]^{-1} \left(\frac{1 + x}{2}\right)^k. \end{aligned} \quad (2.5.3)$$

This evidently extends the classical Jacobi polynomial stated in [80, Eq.(3), p. 254].

The inverse matrix series:

$$\begin{aligned} \frac{[(B + I)_n]^{-1}}{n!} \left(\frac{1 + x}{2}\right)^n &= \sum_{k=0}^{sn} (-1)^k \frac{(-snI)_k}{(sn)!}(A + B + LkI + kI + I) \\ &\quad \times [(A + B + kI + I)_{ln+1}]^{-1}[(B + I)_k]^{-1} P^{(A,B)}_{k,s,l}(x) \end{aligned}$$

which is believed to be new, follows from (2.5.2) with the same substitutions.

The Gegenbauer matrix polynomial turns out to be a particular case of (2.5.3) which occurs when $A = B = F - \frac{I}{2}$. We straight away obtain the polynomial (cf. [80, Eq. (15), p.279]):

$$\mathcal{C}_n^F(x; l, s) = (-1)^n \frac{[(F + I/2)_n]}{n!} \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-nI)_{sk}}{k!} (2F + nI)_{lk} [(F + I/2)_k]^{-1} \left(\frac{1+x}{2}\right)^k.$$

With the same substitutions, we get the inverse matrix series:

$$\begin{aligned} \frac{[(F + I/2)_n]^{-1}}{n!} \left(\frac{1+x}{2}\right)^n &= \sum_{k=0}^{sn} (-1)^k \frac{(-snI)_k}{(sn)!} (2F + LkI + kI) \\ &\times [(2F + kI)_{ln+1}]^{-1} [(F + I/2)_k]^{-1} \mathcal{C}_k^F(x; l, s). \end{aligned}$$

The immediate instance $F = I/2$ of this polynomial is the *extended* Legendre matrix polynomials which is denoted here by $P_{n,l,s}(x)$, is given by (cf. [80, Eq. (3), p.166])

$$P_{n,l,s}(x) = (-1)^n \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-nI)_{sk}}{k!} (I + nI)_{lk} [(I)_k]^{-1} \left(\frac{1+x}{2}\right)^k$$

which possesses the inverse matrix series:

$$\begin{aligned} \frac{[(I)_n]^{-1}}{n!} \left(\frac{1+x}{2}\right)^n &= \sum_{k=0}^{sn} (-1)^k \frac{(-snI)_k}{(sn)!} (LkI + kI + I) \\ &\times [(kI + I)_{ln+1}]^{-1} [(I)_k]^{-1} P_{n,l,s}(x). \end{aligned}$$

This is believed to be new.

Another extended matrix analogue of the Legendre polynomials: $P_n(x, C)$ given in (cf. [93, Eq.(2.3), p. 439] with $r = 2, s = 1$) which can be deduced from (2.5.3) by putting $A = -C + I$ and $B = C - I$. The explicit representation together with its inverse series is given below.

$$P^{C}_{n,s,l}(x) = \frac{[(C)_n]}{n!} \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-nI)_{sk}}{k!} (nI+I)_{lk}[(C)_k]^{-1} \left(\frac{1-x}{2}\right)^k$$

and

$$\frac{[(C)_n]^{-1}}{n!} \left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!} (LkI+kI+I)[(kI+I)_{ln+1}]^{-1}[(C)_k]^{-1} P^{C}_{k,s,l}(x).$$

The further reducibility with $A = C - I, B = -C$ leads to the *extended* Chebyshev matrix polynomials: $(C)_n T_{n,s}(x, C)/n!$ [21, Section 6, p.801].

The Laguerre matrix polynomial however occurs from the theorem directly. In fact, the choice $B = O$ (thereby $A \neq \ell I, \ell = 0, 1, \ldots$) and $F(n) \to \Gamma^{-1}(A - nI + I)(A+I)^{-1}_n F(n)$ and $G(n) \to (A+I)^{-1}_{sn} G(n)$ transform the theorem into the form:

$$\left.\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-1)^{n-sk}}{(n-sk)!} (A+I)_n [(A+I)_{sk}]^{-1}\, G(k), \\ G(n) &= \sum_{k=0}^{sn} \frac{(A+I)_{sn}}{(sn-k)!} [(A+I)_k]^{-1}\, F(k), \end{aligned}\right\}$$

where $A+jI$ are assumed to be invertible for $j \in \mathbf{N}$. From this, we get the *extended* Laguerre matrix polynomial (cf. [52, Eq.(3.7), p.58] with $s = 1$) along with its inverse (cf. [61] with $s = 1$):

$$L^{(A,\lambda)}_{n,s}(x) = \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-1)^{n-sk}}{(n-sk)!} (A+I)_n [(A+I)_{sk}]^{-1} \frac{\lambda^k x^k}{k!};$$

$$\frac{\lambda^n x^n}{n!} = \sum_{k=0}^{sn} \frac{(A+I)_{sn}}{(sn-k)!} [(A+I)_k]^{-1}\, L^{(A,\lambda)}_{k,s}(x).$$

Re-writing (2.5.3) and its inverse as

$$P^{(A,B)}_{n,s,l}(x) \;=\; \frac{[(B+I)_n]}{n!} \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(-nI)_{sk}}{k!} (A+B+nI+I)_{lk} [(B+I)_k]^{-1} \left(\frac{1-x}{2}\right)^k;$$

$$\frac{[(B+I)_n]^{-1}}{n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{sn}(-1)^k\frac{(-snI)_k}{(sn)!}(A+B+LkI+kI+I)$$
$$\times[(A+B+kI+I)_{ln+1}]^{-1}[(B+I)_k]^{-1}P_{k,s,l}^{(A,B)}(x),$$

and putting $A=-B-I$, we obtain an extended matrix version of the non constant Chebyshev polynomial of first kind denoted by $T_{n,s,l}(x,B)$ which is stated below along with inverse series.

$$T_{n,s,l}(x,B) = \frac{[(B+I)_n]}{n!}\sum_{k=1}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}}{k!}(nI)_{lk}[(B+I)_k]^{-1}\left(\frac{1-x}{2}\right)^k;$$

$$\frac{[(B+I)_n]^{-1}}{n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=s}^{sn}(-1)^k\frac{(-snI)_k}{(sn)!}(LkI+kI)$$
$$\times[(kI)_{ln+1}]^{-1}[(B+I)_k]^{-1}T_{k,s,l}(x,B).$$

Also putting $A=B=-I/2$, we obtain an extended matrix version of the non constsnt Chebyshev polynomial of first kind denoted by $T_{n,s,l}(x)$; whereas putting $A=B=I/2$, this reduces to the extended matrix version of the Chebyshev polynomial of second kind denoted by $U_{n,s,l}(x)$. They are respectively stated below along with their inverse series (cf. [80, Eq.(1) and (2), p.301] with $s=1, l=1$).

1. $T_{n,s,l}(x) = \frac{[(I/2)_n]}{n!}\sum_{k=1}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}}{k!}(nI)_{lk}[(I/2)_k]^{-1}\left(\frac{1-x}{2}\right)^k;$

$$\frac{[(I/2)_n]^{-1}}{n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=s}^{sn}(-1)^k\frac{(-snI)_k}{(sn)!}(LkI+kI)[(kI)_{ln+1}]^{-1}$$
$$\times[(I/2)_k]^{-1}T_{k,s,l}^{(A,B)}(x),$$

Here we have used the double series identities (1.3.31) and (1.3.32).

For $n=0$, $T_{0,s,l}(x,B):=I$ and $T_{0,s,l}(x):=I$.

2. $U_{n,s,l}(x) = \frac{[(3I/2)_n]}{n!}\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}}{k!}(nI+2I)_{lk}[(3I/2)_k]^{-1}\left(\frac{1-x}{2}\right)^k;$

$$\frac{[(3I/2)_n]^{-1}}{n!}\left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{sn}(-1)^k\frac{(-snI)_k}{(sn)!}(LkI+kI+2I)[(kI+2I)_{ln+1}]^{-1}$$
$$\times[(3I/2)_k]^{-1}\,U_{k,s,l}(x).$$

2.6 Wilson Matrix Polynomial and Racah Matrix Polynomial

The inverse pair of the theorem is capable of providing the matrix analogues of the well known orthogonal polynomials in ${}_4F_3$-function forms; namely, the Wilson polynomial and the Racah polynomial (or 6-j coefficients) [110, Eq.(1.1.1) and (1.2.1), p. 24 and 26] (also [5, Eq. (a), (b), p. 47]). For deducing these matrix polynomials, the following inverse pair is first obtained from the theorem with the aid of the substitutions $r - s = l, sB = -lI$, and $F(n) \to (-1)^n \Gamma^{-1}(A - nI + I)F(n)/n!$.

$$\left.\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/s \rfloor} (-nI)_{sk}[(A - nI + I)_{(s-r)k}]^{-1}\, G(k), \\ G(n) &= \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!}\, (A - (lk/s)I - kI)(A - kI + I)_{(s-r)n-1} F(k). \end{aligned}\right\}.$$

Now in view of the formula (1.3.18),

$$[(A - nI + I)_{(s-r)k}]^{-1} = (-1)^{(s-r)k}(-A + nI)_{(r-s)k},$$

and

$$(A - kI + I)_{(s-r)n-1} = (-1)^{(r-s)n+1}[(-A + kI)_{(r-s)n+1}]^{-1}.$$

Thus, the above pair changes to

$$\left.\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/s \rfloor} (-nI)_{sk}(-A + nI)_{lk}\, G(k), \\ G(n) &= \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!}\, (A + (rk/s)I)[(-A + kI)_{ln+1}]^{-1} F(k). \end{aligned}\right\}. \tag{2.6.1}$$

Now if A, B, C and D are the positive stable matrices in $\mathbb{C}^{p\times p}$ and the inverse of the each of the matrices $A+B+jI, A+C+jI, A+D+jI$ exists for all $j = 0, 1, 2, \ldots$

then with A and $G(n)$ are respectively replaced by $A+B+C+D+I$, and

$$\frac{(A+ixI)_n(A-ixI)_n}{n!}\ [(A+B)_n]^{-1}[(A+C)_n]^{-1}[(A+D)_n]^{-1},$$

the inverse pair (2.6.1) yields an *extended* Wilson matrix polynomial:

$$\begin{aligned}&[(A+B)_n]^{-1}[(A+C)_n]^{-1}[(A+D)_n]^{-1}\ P_{n,l,s}(x^2)\\&=\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}}{k!}(A+B+C+D+nI+I)_{lk}(A+ixI)_k(A-ixI)_k\\&\times[(A+B)_k]^{-1}[(A+C)_k]^{-1}[(A+D)_k]^{-1};\end{aligned}$$

and its inverse series

$$\begin{aligned}&\frac{(A+ixI)_n(A-ixI)_n}{n!}(A+B)_n^{-1}[(A+C)_n]^{-1}[(A+D)_n]^{-1}\\=&\sum_{k=0}^{sn}\frac{(-snI)_k}{(sn)!}(A+B+C+D+(r/s)kI+I)\\&\times[(A+B+C+D+kI+I)_{ln+1}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\&\times[(A+D)_k]^{-1}P_{k,l,s}(x^2).\end{aligned}\tag{2.6.2}$$

Next, in (2.6.1) if A is replaced by $A+B+I$ and

$$G(k)=\frac{(-xI)_k(xI+D+E+I)_k}{k!}[(A+I)_k]^{-1}[(B+E+I)_k]^{-1}[(D+I)_k]^{-1},$$

then we find the *extended* Racah matrix polynomial:

$$\begin{aligned}&R_{n,l,s}(x(xI+D+E+I);A,B,D,E)\\=&\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}}{k!}(A+B+nI+I)_{lk}(-xI)_k(xI+D+E+I)_k\\&\times[(A+I)_k]^{-1}[(B+E+I)_k]^{-1}[(D+I)_k]^{-1};\end{aligned}$$

whose inverse series is

$$\frac{(-xI)_n(xI+D+E+I)_n}{n!}\ [(A+I)_n]^{-1}(B+E+I)_n^{-1}[(D+I)_n]^{-1}$$

$$= \sum_{k=0}^{sn} \frac{(-snI)_k}{(sn)!}(A+B+(rk/s)I+I)\left[(A+B+kI+I)_{ln+1}\right]^{-1} \times R_{k,l,s}(x(xI+D+E+I);A,B,D,E), \tag{2.6.3}$$

wherein the matrices $A+jI, A+B+jI, B+E+jI$ and $D+jI$ are all assumed to be invertible for $j=0,1,2,\ldots$.

Since the Wilson polynomial and Racah polynomial encompass several polynomials belonging to Askey scheme; namely the polynomilas of Hahn, dual Hahn, Meixner, Krawtchouk, Charlier, Jacobi etc. (see [5, p. 46] for complete reducibility chart), their extended matrix polynomials' versions would follow directly from these two matrix polynomials together with their inverse series relations.

2.7 Application

In this section, the matrix generating functions will be derived from the first series of the inverse pair (2.6.1); whereas from the second series, certain matrix summation formulas will be obtained.

2.7.1 Generating Function Relations

Theorem 2.7.1. *For a positive stable matrix C in $\mathbb{C}^{p\times p}$ and $|t|<1$, the following generating function relation holds.*

$$\sum_{n=0}^{\infty}\frac{(C)_n}{n!}\,F(n)t^n = (1-t)^{-C}\sum_{k=0}^{\infty}(C)_{rk}\,G(k)\left(\frac{(-t)^s}{(1-t)^r}\right)^k.$$

Proof. Taking $A=C$ in the first series of (2.6.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(C)_n}{n!}\,F(n)\,t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-nI)_{sk}(C)_n(C+nI)_{lk}}{n!}G(k)t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-1)^{sk}\,(C)_{n+lk}}{(n-sk)!}G(k)t^n\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{sk}(C)_{n+rk}}{n!}G(k)t^{n+sk}$$

$$= \sum_{k=0}^{\infty}(C)_{rk}\left(\sum_{n=0}^{\infty}\frac{(C+rkI)_n}{n!}t^n\right)G(k)(-t)^{sk}$$

$$= \sum_{k=0}^{\infty}(C)_{rk}\ (1-t)^{-C-rkI}\ G(k)(-t)^{sk}$$

$$= (1-t)^{-C}\sum_{k=0}^{\infty}(C)_{rk}\ G(k)\left(\frac{(-t)^s}{(1-t)^r}\right)^k.$$

□

Corollary 2.7.1. *For* $|t| < 1$,

$$\sum_{n=0}^{\infty}F(n)\ t^n = (1-t)^{-I}\sum_{k=0}^{\infty}(rk)!\,G(k)\left(\frac{(-t)^s}{(1-t)^r}\right)^k.$$

This corollary is the immediate instance $C = I$ of Theorem 3.7.1.

Theorem 2.7.2. *For the invertible matrices* $C+nI,\ n = 0,1,\ldots$ *and* $|t| < 1$,

$$\sum_{n=0}^{\infty}F(n)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\ (C)_{rk}\ {}_1F_1\left(C+rkI;C+skI;t\right)\left[(C)_{sk}\right]^{-1}G(k)(-t)^{sk}.$$

Proof. From the first series of (2.6.1), we have the left hand side

$$\sum_{n=0}^{\infty}F(n)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}(-nI)_{sk}(C+nI)_{lk}G(k)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/s\rfloor}\frac{(-1)^{sk}\ (C)_{n+lk}}{(n-sk)!}[(C)_n]^{-1}G(k)t^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{sk}(C)_{n+rk}}{n!}[(C)_{n+sk}]^{-1}\ G(k)t^{n+sk}$$

$$= \sum_{k=0}^{\infty}(C)_{rk}\left(\sum_{n=0}^{\infty}\frac{(C+rkI)_n}{n!}[(C+skI)_n]^{-1}\ t^n\right)$$
$$\times[(C)_{sk}]^{-1}G(k)(-t)^{sk}$$

$$= \sum_{k=0}^{\infty}\ (C)_{rk}\ {}_1F_1\left(C+rkI;C+skI;t\right)\left[(C)_{sk}\right]^{-1}G(k)(-t)^{sk}.$$

□

Now $G(n) = M_n x^n$ implies $F(n) = H_n(C, r, s, x)$, and thereby Theorem 3.7.1 yields the generating function relation:

$$\sum_{n=0}^{\infty} \frac{(C)_n}{n!} H_n(C, r, s, x)\ t^n \quad = \quad \sum_{k=0}^{\infty} (C)_{rk}\ (1-t)^{-C}\ M_k \left(\frac{x(-t)^s}{(1-t)^r}\right)^k .$$

From Corollary 2.7.1, we get

$$\sum_{n=0}^{\infty} H_n(C, r, s, x)\ t^n = (1-t)^{-I} \sum_{k=0}^{\infty} (rk)!\ M_k \left(\frac{(-t)^s x}{(1-t)^r}\right)^k ,$$

where $|t| < 1$, and from Theorem 3.7.2, we have

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(C, r, s, x) \frac{t^n}{n!} \quad &= \quad \sum_{k=0}^{\infty}\ (C)_{rk}\ {}_1F_1\left(C + rkI; C + skI; t\right) \\ &\quad \times [(C)_{sk}]^{-1}\ M_k\ \left((-t)^s x\right)^k . \end{aligned}$$

The generating function relations of the extended Wilson matrix polynomial and extended Racah matrix polynomial may be deduced as follows. Taking $A + B + C + D - I = R, A + ixI = z_1 I, A - ixI = z_2 I, A + B = A_1, A + C = A_2, A + D = A_3,$ then the generating function relation occurs.

$$\begin{aligned} &\sum_{n=0}^{\infty} [(A_1)_n]^{-1} [(A_2)_n]^{-1} [(A_3)_n]^{-1} P_{n,l,s}(x^2) \frac{t^n}{n!} \\ = \ & \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\lfloor n/s \rfloor} (-1)^{sk} \frac{(R + nI)_{lk}}{(n - sk)!k!} (z_1 I)_k (z_2 I)_k [(A_1)_n]^{-1} [(A_2)_n]^{-1} [(A_3)_n]^{-1} \\ = \ & \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \Gamma(R + nI + rkI) \Gamma^{-1}(R + nI + skI) (z_1 I)_k (z_2 I)_k [(A_1)_n]^{-1} [(A_2)_n]^{-1} \\ & \times [(A_3)_n]^{-1} \frac{(-t)^{sk}}{k!} \\ = \ & \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} (R + nI)_{rk} (R + nI)_{sk}^{-1} (z_1 I)_k (z_2 I)_k [(A_1)_n]^{-1} [(A_2)_n]^{-1} [(A_3)_n]^{-1} \frac{(-t)^{sk}}{k!} \\ = \ & \sum_{n=0}^{\infty} \frac{t^n}{n!}\ {}_{r+2}F_{s+3}(\prec r; R + nI \succ, z_1 I, z_2 I; \prec s; R + nI \succ, A_1, A_2, A_3; (-t)^s), \end{aligned}$$

where $r \le s + 1$ or $r = s + 2$ and $|t| < 1$ for convergence. On the other hand, if $r - s = l, A + B + I = Q, Q + nI = Q_n, D + E + I = B_1, A + I = B_2, B + E + I = B_3,$

and $D+I=B_4$, then the generating function relation holds.

$$\begin{aligned}
&\sum_{n=0}^{\infty} R_{n,l,s}(x(xI+D+E),A,B,C,D,E)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty} t^n \sum_{k=0}^{\lfloor n/s\rfloor} \frac{(-1)^{sk}}{(n-sk)!k!}(Q+nI)_{lk}(-xI)_k(xI+B_1)_k[(B_2)_k]^{-1}\\
&\times[(B_3)_k]^{-1}[(B_4)_k]^{-1}\\
=&\sum_{n=0}^{\infty} \frac{t^n}{n!}\sum_{k=0}^{\infty}\Gamma(Q+nI+rkI)\Gamma^{-1}(Q+nI+skI)(-xI)_k(xI+B_1)_k\\
&\times[(B_2)_k]^{-1}[(B_3)_k]^{-1}[(B_4)_k]^{-1}\frac{(-t)^{sk}}{k!}\\
=&\sum_{n=0}^{\infty} \frac{t^n}{n!}\sum_{k=0}^{\infty}(Q+nI)_{rk}(Q+nI)_{sk}^{-1}(-xI)_k(xI+B_1)_k[(B_2)_k]^{-1}\\
&\times[(B_3)_k]^{-1}[(B_4)_k]^{-1}\frac{(-t)^{sk}}{k!}\\
=&\sum_{n=0}^{\infty} \frac{t^n}{n!}\ {}_{r+2}F_{s+3}(\prec r;Q_n\succ,-xI,xI+B_1;\prec s;Q_n\succ,B_2,B_3,B_4;(-t)^s)
\end{aligned} \tag{2.7.1}$$

in which $r\le s+1$ or $r=s+2$ and $|t|<1$ for convergence.

2.7.2 Summation Formulas

The inverse series (2.4.3) with the assumption that the finite sequence $\{M_n\}$ of matrices in $\mathbb{C}^{p\times p}$ contains all invertible matrices, is rewritten as

$$\begin{aligned}
x^nI \ =\ &\sum_{k=0}^{sn}[M_n]^{-1}\,\frac{(-C-(rk/s)I)}{(sn-k)!}\,(I-C-rnI)_{sn-k-1}\\
&\times H_k(C,r,s;x).
\end{aligned} \tag{2.7.2}$$

Now, multiplying this by $1/n!$ and then taking the infinite sum, it gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{x^n}{n!}I=e^xI=&\sum_{n=0}^{\infty}\sum_{k=0}^{sn}[M_n]^{-1}\,\frac{(-C-(rk/s)I)}{(sn-k)!n!}\,(I-C-rnI)_{sn-k-1}\\
&\times H_k(C,r,s;x).
\end{aligned} \tag{2.7.3}$$

From this, a number of particular sums can be deduced by assigning particular values to x. For instance, $x = 0$ in (2.7.3) furnishes the matrix sum:

$$-I = \sum_{n=0}^{\infty}[M_n]^{-1}\sum_{k=0}^{sn}\frac{(C+(rk/s)I)}{(sn-k)!}(I-C-rnI)_{sn-k-1}\ M_0\frac{(C)_n}{(n!)^2}.$$

Next, assuming $|x| < 1$, and taking summation n from 0 to ∞ in (2.7.2), then there occurs the sum:

$$\left(\frac{-1}{1-x}\right)I = \sum_{n=0}^{\infty}[M_n]^{-1}\sum_{k=0}^{sn}\frac{(C+(rk/s)I)}{(sn-k)!}(I-C-rnI)_{sn-k-1} \times H_k(C,r,s;x). \tag{2.7.4}$$

Here also, by assigning the different values to x from $(-1,1)$, a number of particular summation formulas can be deduced. For example, for $x = \frac{1}{2}$, it reduces to

$$-2I = \sum_{n=0}^{\infty}[M_n]^{-1}\sum_{k=0}^{sn}\frac{(C+(rk/s)I)}{(sn-k)!}(I-C-rnI)_{sn-k-1}H_k(C,r,s;1/2).$$

The summation formulas involving the Wilson matrix polynomial and the Racah matrix polynomial are obtained from their respective inverse series (2.6.2) and (2.6.3). They are stated below.

With $A+B+C+D+I = R$, and applying $\sum t^n$ both sides in (2.6.2), gives

$$\begin{aligned}&{}_2F_3\left(A+ixI, A-ixI; A+B, A+C, A+D; t\right)\\ &= \sum_{n=0}^{\infty}\frac{t^n}{(sn)!}\sum_{k=0}^{sn}(-snI)_k(R+(r/s)kI)[(R+kI)_{ln+1}]^{-1}[(A+B)_k]^{-1}[(A+C)_k]^{-1}\\ &\times[(A+D)_k]^{-1}P_{k,l,s}(x^2).\end{aligned} \tag{2.7.5}$$

Similarly, considering $\sum t^n$ both sides in (3.6.2), yields the sum:

$$\begin{aligned}&{}_2F_3\left(-xI, xI+D+E+I; A+I, B+E+I, D+I; t\right)\\ &= \sum_{n=0}^{\infty}\frac{t^n}{(sn)!}\sum_{k=0}^{sn}(-snI)_k(A+B+(rk/s)I+I)\ [(A+B+kI+I)_{ln+1}]^{-1}\\ &\times R_{k,l,s}(x(xI+D+E+I); A, B, D, E).\end{aligned} \tag{2.7.6}$$

Here it is noteworthy that for the particular values of x, a number of sums may be obtained. For example, for $x = 0$, the sum (3.7.3) simplifies to

$$
\begin{aligned}
& {}_2F_3(A, A; A+B, A+C, A+D; t) \\
= & \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{k=0}^{sn} (-snI)_k (R + (rk/s)I)[(R+kI)_{ln+1}]^{-1} \\
& \times \sum_{j=0}^{\lfloor k/s \rfloor} \frac{(-kI)_{sj}}{j!} (R+kI)_{lj} (A)_j (A)_j [(A+B)_j]^{-1} [(A+C)_j]^{-1} [(A+D)_j]^{-1} \\
= & \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{sn-sj} (-1)^k \binom{sn-sj}{k} \frac{(k+sj)!}{(sn-sj)!k!j!} (R + (r/s)kI + rjI) \\
& \times [(R+kI+sjI)_{ln+1}]^{-1} (R+kI+sjI)_{lj} [(A)_j]^2 [(A+B)_j]^{-1} [(A+C)_j]^{-1} \\
& \times [(A+D)_j]^{-1} t^n \\
= & \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{j=0}^{\infty} \sum_{k=0}^{sn} (-1)^k \binom{sn}{k} \frac{(k+sj)!}{k!j!} (R + (r/s)kI + rjI) \\
& \times (R+kI+sjI)^{-1}_{ln-sj+1} (R+kI+sjI)_{lj} [(A)_j]^2 [(A+B)_j]^{-1} [(A+C)_j]^{-1} \\
& \times [(A+D)_j]^{-1}
\end{aligned}
$$

On the other hand, when $x = 0$, then since $R_{n,l,s}(0; A, B, C, D) = I$, the sum (3.7.3) gets reduced to the elegant form:

$$
I = \sum_{n=0}^{\infty} \frac{t^n}{(sn)!} \sum_{k=0}^{sn} (-snI)_k (A + B + (rk/s)I + I) \; [(A+B+kI+I)_{ln+1}]^{-1}.
$$

2.8 Matrix Analogues of Riordan's Inverse Pairs

The Gould classes (1) and (2) [82, Table-2, p. 52] and the Legendre-Chebyshev classes (3) and (7) [82, Table-6, p. 69] due to John Riordan can be extended to the matrix forms by means of Theorem 3.3.1. The matrix form of the Gould class (1) occurs from the theorem when $G(k) \to \Gamma(A + skB - skI + I)G(k)$; whereas the Legendre-Chebyshev class (3) occurs if $B = C + I$ is taken in the theorem.
The Gould class (2) and the Legendre-Chebyshev class (7) are yielded by the following alternative form of the theorem. It is obtained by replacing first A by $A + I$ and then $F(n)$ by $(A + nB - nI + I)^{-1}F(n)$ and $G(n)$ by $\Gamma(A + snB -$

$snI+I)G(n)$. With these, the inverse pair of the theorem changes to

$$\left.\begin{aligned} F(n) &= \sum_{k=0}^{\lfloor n/s\rfloor}(-1)^{n-sk}(A+nB-nI+I)\Gamma^{-1}(A+skB-nI+2I)\\ &\quad\times\Gamma(A+skB-skI+I)\frac{G(k)}{(n-sk)!},\\ G(n) &= \sum_{k=0}^{sn}\Gamma^{-1}(A+snB-snI+I)\Gamma(A+snB-kI+I)\frac{F(k)}{(sn-k)!}. \end{aligned}\right\}(2.8.1)$$

This pair itself is the Gould class (2). The Legendre-Chebyshev class (7) is the case $B=C+I$ of (2.8.1). They are tabulated in Table-2.

Table-2. $F(n)=\sum\dfrac{a_{n,k}}{(n-sk)!}\,G(k);\; g(n)=\sum(-1)^{sn-k}\dfrac{b_{n,k}}{(sn-k)!}\,F(k)$

Theorem / Inv.Pair No.	B	$a_{n,k}$	$b_{n,k}$	Matrix analogue of Class (No.)
Theorem -1	B	$\Gamma^{-1}(A+skB-nI+I)$ $\times\Gamma(A+skB-skI+I)$	$\Gamma^{-1}(A+snB-snI+I)$ $\times(A+kB-kI)$ $\times\Gamma(A+snB-kI)$	Gould Class (1)
(2.8.1)	B	$(A+nB-nI+I)$ $\times\Gamma^{-1}(A+skB-nI+2I)$ $\times\Gamma(A+skB-skI+I)$	$\Gamma^{-1}(A+snB-snI+I)$ $\times\Gamma(A+snB-kI+I)$	Gould Class (2)
Theorem -1	$C+I$	$\Gamma^{-1}(A+skC+skI-nI+I)$ $\times\Gamma(A+skC+I)$	$\Gamma^{-1}(A+snC+I)$ $\times(A+kC)$ $\times\Gamma(A+snC+snI-kI)$	Legendre -Chebyshev Class (3)
(2.8.1)	$C+I$	$(A+nC+I)$ $\times\Gamma^{-1}(A+skC+skI-nI+2I)$ $\times\Gamma(A+skC+I)$	$\Gamma^{-1}(A+snC+I)$ $\times\Gamma(A+snC+snI-kI+I)$	Legendre -Chebyshev Class (7)