Chapter 2

Physico-Mathematical Background

Contents

Symbols

p	fluid pressure (as in (2.1)) (N m^{-2})
ρ	density of fluid (Ns^2 m^{-4})
k	permeability in porous material (m^2)
P	pressure in porous region (as in (2.3)) (N m^{-2})
V	average velocity in porous material (m s^{-1})
$\mathbf{q}$	fluid velocity vector
u, v, w	velocity components along the x, y, z – directions, respectively.
$\mathbf{H}$	magnetic field vector
H	magnitude of the magnetic field (amp m^{-1})
m	magnetic moment
$\mathbf{M}$	magnetization vector
M	magnitude of magnetization (amp m^{-1})
$\mathbf{B}$	magnetic induction/flux vector (weber m^{-2})

Greek symbols

η	fluid viscosity (Ns m^{-2})
τ	shear stress (N m^{-2})
μ	magnetic permeability
$\bar{\mu}$	magnetic susceptibility
μ_0	free space permeability
$\mathbf{\Omega}$	angular velocity vector

This chapter mainly deals with the brief introduction of various physico-mathematical concepts which are necessary to understand subsequent chapters. The materials are taken from various sources [1-22].

2.1 Basic Definitions

Def. 2.1.1 (Fluid)

A fluid is a substance which is capable of flowing and deforms continuously without limit under the action of tangential force or shearing force. This continuous deformation under the application of tangential or shearing forces causes fluid to flow.

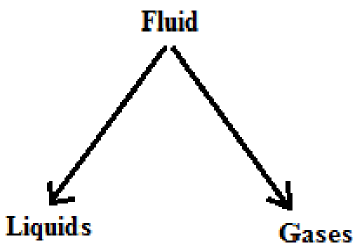

Figure 2.1

Def. 2.1.2 (Pressure and Shear stress)

Pressure is defined as force per unit area.

Mathematically,

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA} \tag{2.1}$$

where δF is a force acting normally on small area δA.

The unit of pressure is N m^{-2}.

If a force δF acting tangent to the small area δA, then (2.1) defines shear stress. It is defined by τ. The shear stress is responsible for fluid motion.

Def. 2.1.3 (Viscosity)

The viscosity of a fluid is a measure of its resistance to flow. The resistance arising from intermolecular forces and internal friction.

The unit of viscosity is Ns m^{-2}.

Thick fluids (such as oil) have relatively high viscosity than thin fluids (such as water).

Def. 2.1.4 (Density)

The mass density ρ of a fluid is the mass of a unit volume of the fluid.

Mathematically,

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}, \tag{2.2}$$

where δm is the small elemental mass, and δv is the volume of the small elemental mass.

The unit of density is $Ns^2\,m^{-4}$.

Def. 2.1.5 (Compressible Fluids)

These are the fluids in which the density of the fluid changes from point to point or in other words the density is not constant.

Def. 2.1.6 (Incompressible Fluids)

These are the fluids in which the density of the fluid is constant.

Def. 2.1.7 (Surface Tension)

It is defined as the tensile force acting on the surface of a liquid when the liquid is in contact with a gas. It is also defined on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension.

The unit of surface tension is N m^{-2}.

Def. 2.1.8 (Ideal Fluids/Inviscid Fluids)

These are the fluids which have no viscosity, surface tension and are incompressible. Since there is no viscosity, there is no shear stress between adjacent fluid layers, and that between the fluid layers and the boundary. Only normal stresses can exist in an ideal fluid flow.

Def. 2.1.9 (Real Fluids/Viscous Fluids/Practical Fluids)

These are the fluids which possesses properties like viscosity, surface tension and compressibility. Due to existence of viscosity shear stress comes into play when fluids are in motion. These fluids are actually available in nature.

Def. 2.1.10 (Shear Strain)

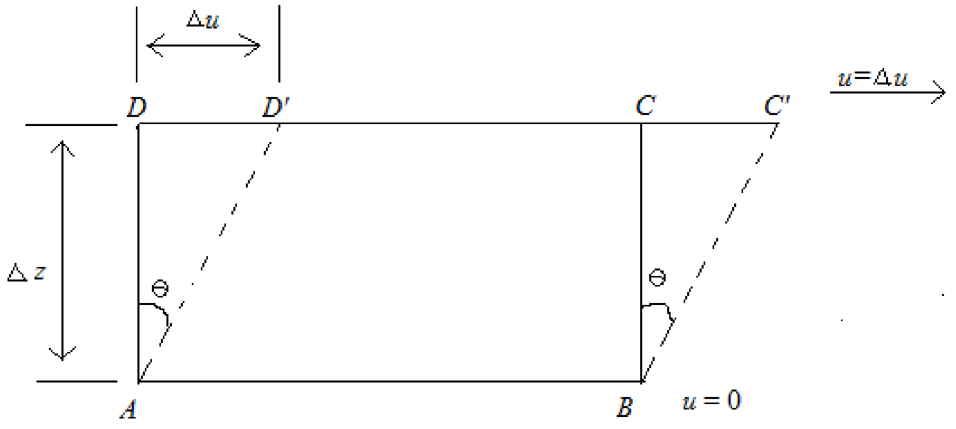

Figure 2.2

The shear strain is the rate of angular deformation of fluid element when the fluid element lies within the range of viscous influence. For a unidirectional flow it can be expressed as $\frac{\Delta u}{\Delta z}$, where Δu is the velocity of upper edge of the rectangular fluid element and Δz is the width of fluid element.

Mathematically,

$$\lim_{\Delta z \to 0} \frac{\Delta u}{\Delta z} = \frac{du}{dz}.$$

Def. 2.1.11 (Newtonian and Non-Newtonian Fluids)

In Newtonian fluids,

Shear stress $\propto$ Rate of shear strain.

For example, Glycerin, silicone oils, air, gases, etc.

In Non-Newtonian fluids above relation will not follow.

For example, slurries, tooth paste, gel, etc.

Def. 2.1.12 (Porous Medium)

When the solid particles are loosely arranged in a medium, then it is called a porous medium.

For example, natural soil or sand, etc.

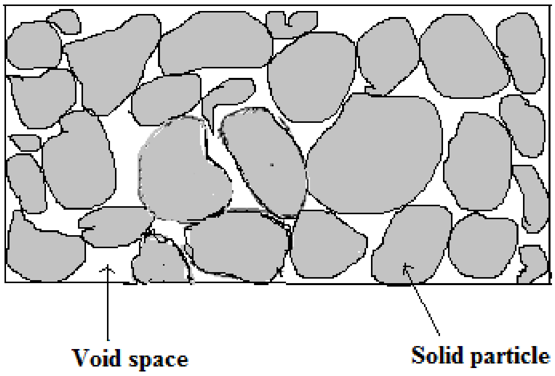

Figure 2.3

Def. 2.1.13 (Porosity)

It is a measure of void spaces in a medium. It is defined as the ratio of pore (void space) volume to the total volume (including solid and void spaces) of a medium. It is a dimensionless quantity.

Def. 2.1.14 (Permeability)

It is a measure of how easily fluid flows through porous media.

The unit of permeability is m^2.

Def. 2.1.15 (Darcy's Law)

Darcy in 1956 governs the law regarding the flow of fluids through porous material. According to this law, the space averaged velocity (Darcian Velocity) V in the porous material is given by

$$V = -\frac{k}{\eta}\nabla P , \tag{2.3}$$

where k, η and P are the permeability, viscosity and pressure in the porous region, respectively.

The negative sign indicates that the flow is in the direction of decreasing pressure.

Def. 2.1.16 (The No-Slip Boundary Condition)

When a viscous fluid flows along a solid surface, the fluid element adjacent to the surface attain the velocity of the surface; in other words the relative velocity of the solid surface and the adjacent fluid particles is zero. This phenomenon is known as no-slip condition.

2.2 Types of Flows

In general, the flow velocity and other hydrodynamic parameters like pressure and density may vary from one point to another at any instant, and also from one instant to another at a fixed point. According to the type of variations, different categories of flows are described as follows.

- **Laminar /Stream Line/Viscous Flows**

This type of flow is characterized by a smooth flow of one lamina of fluid over another. This type of flow occurs when flow is having low velocity or for the liquids having a high viscosity.

- **Turbulent Flows**

These are the flows where fluid moving in a erratic and unpredictable way (i.e. zig-zag way). For example, flow of a river.

- **Rotational and Irrotational Flows**

Rotational flows are the flows where the fluid particles while moving in the direction of flow rotate about their mass centers.

Irrotational flows are the flows in which the fluid particles do not rotate about their own axis.

- **Steady and Unsteady Flows**

Steady flows are the flows where the fluid characteristics like velocity, pressure, density, etc. at a point do not change with respect to time.

Mathematically,

$$\left(\frac{\partial \mathbf{q}}{\partial t}\right)_{(x_0,y_0,z_0)}=0,\quad \left(\frac{\partial p}{\partial t}\right)_{(x_0,y_0,z_0)}=0,\quad \left(\frac{\partial \rho}{\partial t}\right)_{(x_0,y_0,z_0)}=0,$$

where $\mathbf{q}$ is the fluid velocity, p is fluid pressure and ρ is fluid density and (x_0, y_0, z_0) is a fixed point in flow field.

When fluid characteristics like velocity, pressure, density, etc. at a point changes with respect to time, then the flow is called unsteady flow.

Mathematically,

$$\left(\frac{\partial \mathbf{q}}{\partial t}\right)_{(x_0,y_0,z_0)}\neq 0,\quad \left(\frac{\partial p}{\partial t}\right)_{(x_0,y_0,z_0)}\neq 0,\quad \left(\frac{\partial \rho}{\partial t}\right)_{(x_0,y_0,z_0)}\neq 0$$

- **Uniform and Nonuniform Flows**

Uniform flows are those type of flows where the flow parameters like pressure, velocity, density, etc. at given time do not change with respect to space (length of direction of flow). Mathematically,

$$\left(\frac{\partial p}{\partial s}\right)_{t=\text{constant}}=0,\quad \left(\frac{\partial \mathbf{q}}{\partial s}\right)_{t=\text{constant}}=0,\quad \left(\frac{\partial \rho}{\partial s}\right)_{t=\text{constant}}=0\ .$$

Nonuniform flows are the flows where parameters like pressure, velocity, density etc. at a given time change with respect to space (length of direction of flow).

Mathematically,

$$\left(\frac{\partial p}{\partial s}\right)_{t=\text{constant}}\neq 0,\quad \left(\frac{\partial \mathbf{q}}{\partial s}\right)_{t=\text{constant}}\neq 0,\quad \left(\frac{\partial \rho}{\partial s}\right)_{t=\text{constant}}\neq 0\,.$$

- **Compressible and Incompressible Flows**

When the density changes are appreciable, then the flow is called compressible otherwise it is called incompressible flow.

2.3 Fundamental Equations of Fluid Dynamics

2.3.1 Material Derivative and Acceleration

Let the position of a particle at any instant t in a flow field be given by the space coordinates (x, y, z) with respect to a rectangular Cartesian frame of reference. The velocity components u, v, w of the particle along the x, y and z –directions, respectively, can then be written in the Eulerian form as

$$\begin{aligned} u &= u\,(x, y, z, t) \\ v &= v\,(x, y, z, t) \\ w &= w(x, y, z, t) \end{aligned}$$

At any infinitesimal time interval Δt, let the particle move to a new position given by the coordinates $(x+\Delta x, y+\Delta y, z+\Delta z)$, and its velocity components are given by $u+\Delta u, v+\Delta v$ and $w+\Delta w$. Therefore,

$$\begin{aligned} u+\Delta u &= u\,(x+\Delta x,\ y+\Delta y,\ z+\Delta z,\ t+\Delta t) \\ v+\Delta v &= v\,(x+\Delta x,\ y+\Delta y,\ z+\Delta z,\ t+\Delta t) \\ w+\Delta w &= w\,(x+\Delta x,\ y+\Delta y,\ z+\Delta z,\ t+\Delta t) \end{aligned}$$

Using Taylor series for right hand side of the above equations,

$$u+\Delta u = u(x, y, z, t) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z, \Delta t$$

$$v+\Delta v = v(x, y, z, t) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z + \frac{\partial v}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z, \Delta t$$

$$w+\Delta w = w(x, y, z, t) + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \frac{\partial w}{\partial t}\Delta t + \text{higher order terms in } \Delta x, \Delta y, \Delta z, \Delta t$$

Simplifying the above equations, dividing both side by Δt and neglecting higher order terms in $\Delta x, \Delta y, \Delta z$ and Δt, we get

$$\frac{\Delta u}{\Delta t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$
$$\frac{\Delta v}{\Delta t} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$
$$\frac{\Delta w}{\Delta t} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Considering the limiting forms of the equations, we have

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

(2.4)

From the above equations,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \ . \qquad (2.5)$$

Here, $\frac{D}{Dt}$ is called the total differential with respect to time. The first term $\frac{\partial}{\partial t}$ on the right hand side is called temporal or local derivative and the last three terms are together known as convective derivative.

From equation (2.4), the components of material or substantial acceleration are given as

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Thus,

Material or substantial acceleration

= Temporal (or local acceleration) + Convective acceleration

Similarly, the components of acceleration in cylindrical polar coordinate system can be written as

$$a_r = \frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$
$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$
$$a_z = \frac{Dv_z}{Dt} = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}$$

where the term $-\frac{v_\theta^2}{r}$ in the radial component appears due to an inward radial acceleration arising from a change in the direction of v_θ (velocity component in the azimuthal direction θ). This is known as centripetal acceleration. The term $\frac{v_r v_\theta}{r}$ represents a component of acceleration in azimuthal direction caused by a change in direction of v_r with θ.

2.3.2 Angular Velocity Vector

In Cartesian coordinate system, considering *u*, *v*, *w* as the components of flow velocities in *x*, *y*, *z* directions respectively. Then the components of angular velocity are given as

$$\Omega_{xy} = \Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$\Omega_{yz} = \Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$
$$\Omega_{zx} = \Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

where Ω_{xy} represents rotation in *xy*-plane which is equivalent to say Ω_z, which represents rotation with respect to the *z*-axis. Similarly Ω_{yz} represents rotation in *yz*-plane which is equivalent to say Ω_x, which represents rotation with respect to the *x*-axis and Ω_{zx} represents rotation in *zx*-plane which is equivalent to say Ω_y, which represents rotation with respect to the *y*-axis. Therefore, the angular velocity of fluid flow is represented by a vector as

$$\boldsymbol{\Omega} = \Omega_x \hat{\mathbf{i}} + \Omega_y \hat{\mathbf{j}} + \Omega_z \hat{\mathbf{k}}$$

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}$$

$$\boldsymbol{\Omega} = \frac{1}{2}\left\{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}\right\}$$

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\boldsymbol{\Omega} = \frac{1}{2}(\nabla \times \mathbf{q})$$

where $\mathbf{q} = (u, v, w)$ is a flow velocity vector, the quantity $\nabla \times \mathbf{q}$ is known as the vorticity of flow, which is a mathematical measure of rotationalities in the flow field.

When $\nabla \times \mathbf{q} = \mathbf{0}$, then the flow is said to be irrotational.

2.3.3 Derivation of Continuity Equation

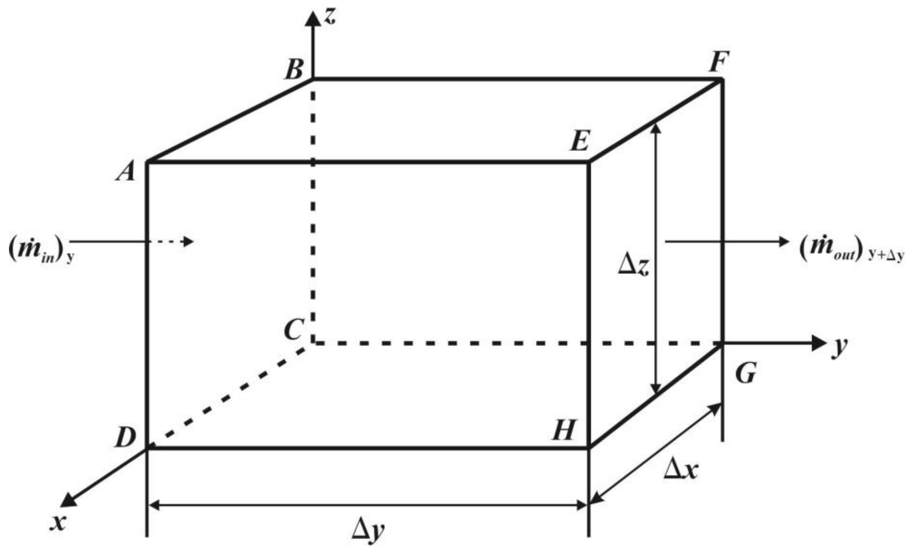

Figure 2.4

Consider a small fluid element of parallelopiped shape in the flow having density ρ as shown in the figure 2.4. Consider the flow along the *y*-direction. Let the fluid enter across the face *ABCD* with a velocity *v*. Therefore, the mass of fluid entering the control volume (CV) along through the face *ABCD* is given by

$$(\dot{m}_{in})_y = \rho v \Delta x \Delta z \tag{2.6}$$

The mass of fluid leaving the control volume through face *EFGH* is given by

$$(\dot{m}_{out})_{y+\Delta y} = (\dot{m}_{in})_y + \frac{\partial}{\partial y}(\dot{m}_{in})_y \Delta y + \text{higher order term(h.o.t.) of } \Delta y$$

$$= \rho v \Delta x \Delta \mathrm{z} + \frac{\partial}{\partial y}(\rho v)\Delta x \Delta \mathrm{y} \Delta \mathrm{z} + \text{h.o.t.}$$

(Using (2.6))

Hence, the rate at which mass accumulates due to flow in the y –direction, neglecting higher order terms, is

$$(\dot{m}_{in})_y - (\dot{m}_{out})_{y+\Delta y} = -\frac{\partial}{\partial y}(\rho v)\Delta x \Delta y \Delta z \tag{2.7}$$

Similarly, the rate at which mass accumulates due to flow in the *x*–direction is

$$(\dot{m}_{in})_x - (\dot{m}_{out})_{x+\Delta x} = -\frac{\partial}{\partial x}(\rho u)\Delta x \Delta y \Delta z\ , \tag{2.8}$$

and the rate at which mass accumulates due to flow in the z –direction is

$$(\dot{m}_{in})_z - (\dot{m}_{out})_{z+\Delta z} = -\frac{\partial}{\partial z}(\rho w)\Delta x \Delta y \Delta z\ . \tag{2.9}$$

But

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t}(m_{CV}) = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z. \tag{2.10}$$

Using equations (2.7), (2.8), (2.9) into (2.10) we obtained,

$$-\frac{\partial}{\partial x}(\rho u)\Delta x \Delta y \Delta z - \frac{\partial}{\partial y}(\rho v)\Delta x \Delta y \Delta z - \frac{\partial}{\partial z}(\rho w)\Delta x \Delta y \Delta z = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\Delta x \Delta y \Delta z = 0.$$

Taking limit as $\Delta x, \Delta y, \Delta z \to 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \qquad (2.11)$$

which is valid for any size of control volume.

- The vector form of this continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{q}) = 0, \quad \text{where } \mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$

For incompressible fluid,

$$\nabla \bullet \mathbf{q} = 0.$$

2.3.4 Derivation of Navier-Stokes Equation

Consider an infinitely small parallelepiped of fluid element as shown in figure 2.5 in a fluid motion. Let the parallelepiped is having sides $\Delta x, \Delta y$ and Δz along Cartesian coordinate axes in *x*, *y* and *z* respectively.

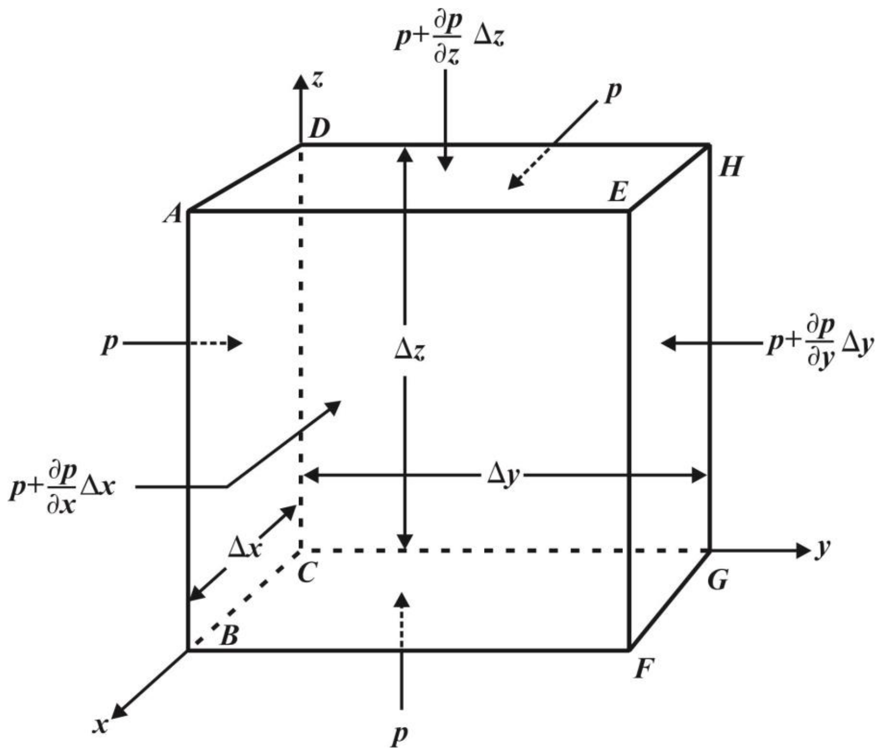

Figure 2.5

Then the forces acting on the fluid element are as follows.

In the following, the detailed calculation of forces acting in the *y*–direction is shown. The forces acting in the *x* and *z*–directions written similarly.

(i) Normal forces due to pressure

$$p\Delta x\Delta z-\left(p+\frac{\partial p}{\partial y}\Delta y\right)\Delta x\Delta z=-\frac{\partial p}{\partial y}\Delta x\Delta y\Delta z. \qquad (2.11)$$

(ii) Body forces due to gravity

The body force per unit mass of the fluid in the y –direction is given by

$$m\,g_y=g_y\;\rho\;\Delta x\Delta y\Delta z, \qquad (2.12)$$

where g_y is component of gravitational force or body force $\mathbf{g}$ in the y – directions.

(iii) Inertia forces

The inertia force on the fluid mass along the *y*–direction is given by

$$\text{mass}\times\text{acceleration}=\rho\Delta x\Delta y\Delta z\frac{dv}{dt} \qquad (2.13)$$

(iv) Shear forces

Let S_x, S_y and S_z be the viscous forces in the *x*, *y* and *z*–directions, respectively. Then, the shear force acting on the parallelepiped along the y –direction is given by

$$S_y\,\rho\,\Delta x\Delta y\Delta z. \qquad (2.14)$$

According to Newton's second law of the motion, and using (2.11)-(2.14),

$$-\frac{\partial p}{\partial y}\Delta x\Delta y\Delta z+g_y\,\rho\,\Delta x\Delta y\Delta z\;+S_y\,\rho\,\Delta x\Delta y\Delta z=\rho\,\Delta x\Delta y\Delta z\,\frac{dv}{dt}$$

$$g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{dv}{dt}-S_y\,. \qquad (2.15)$$

Similarly, g_x and g_z can be obtained as follows.

$$g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} - S_x\,, \qquad g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - S_z. \tag{2.16}$$

Now, the values of S_x , S_y and S_z can be obtained as under.

The shear forces acting on the face *ABCD* and *EFGH* are given by respectively

$$-\eta\,\frac{\partial v}{\partial y}\,\Delta x\Delta z$$

and

$$\eta\,\frac{\partial}{\partial y}\left(v + \frac{\partial v}{\partial y}\Delta y\right)\Delta x\Delta z = \eta\left(\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}\Delta y\right)\Delta x\Delta z.$$

Therefore, the resultant force acting along the y –direction is given by

$$-\eta\frac{\partial v}{\partial y}\,\Delta x\Delta z + \eta\left(\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}\Delta y\right)\Delta x\Delta z = \eta\frac{\partial^2 v}{\partial y^2}\Delta x\Delta y\Delta z \tag{2.17}$$

Similarly, the y–components of resultant shear force acting on the faces *CDHG*, *ABFE* and *BCGF*, *ADHE* are given by respectively

$$\eta\,\frac{\partial^2 v}{\partial x^2}\Delta x\Delta y\Delta z\,, \tag{2.18}$$

and

$$\eta\frac{\partial^2 v}{\partial z^2}\Delta x\Delta y\Delta z. \tag{2.19}$$

Using equations (2.17) - (2.19),

$$\eta\frac{\partial^2 v}{\partial x^2}\Delta x\Delta y\Delta z + \eta\frac{\partial^2 v}{\partial y^2}\Delta x\Delta y\Delta z + \eta\frac{\partial^2 v}{\partial z^2}\Delta x\Delta y\Delta z = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\Delta x\Delta y\Delta z$$

Therefore, the shear force per unit mass is obtained as

$$S_y = \frac{\eta}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right).$$

Similarly,

$$S_x = \frac{\eta}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \text{and} \quad S_z = \frac{\eta}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

Putting these values of S_x, S_y and S_z in equations (2.15) and (2.16), we get

$$g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} - \frac{\eta}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2.20}$$

$$g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{dv}{dt} - \frac{\eta}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{2.21}$$

and

$$g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - \frac{\eta}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{2.22}$$

Equations (2.20) - (2.22) are called **Navier-Stokes Equations** for viscous flow.

- Vector form of the Navier-Stokes Equations is

$$\rho \frac{D\mathbf{q}}{Dt} = -\nabla p + \rho\,\mathbf{g} + \eta \nabla^2 \mathbf{q}, \tag{2.23}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \bullet \mathbf{q} \quad \text{where} \quad \mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}.$$

In general,

$$\rho \frac{D\mathbf{q}}{Dt} = \rho\mathbf{F} - \nabla p + \eta \nabla^2 \mathbf{q}\,, \tag{2.24}$$

where **F** is the body force in general.

2.4 Basic Idea and Types of Bearings

A bearing is a system of machine elements. It is used to support an applied load by reducing friction between the upper and lower surfaces. The moving surfaces (upper or lower) in machinery involve relative sliding or rolling motion. Examples of relative motion are linear sliding motion (such as in machine tools) and rotation motion (such as in motor vehicle wheels). Most bearings are used to support rotating shafts in machines. All the forces on the shaft must be supported by the bearing and the forces on the bearing is referred to as bearing load.

The load on the shaft can be divided into radial and axial components. The axial component, also known as thrust load, is in the direction of the shaft axis. Whereas radial load component is in the direction normal to the shaft axis.

If the bearing supports a radial load, it is called radial or journal bearing. If the bearing supports a thrust or axial load, it is called thrust bearing and if bearing supports both radial and axial load it is called conical bearing.

2.4.1 Rolling Element Bearings

The main feature of this type of bearing is rolling motion. This bearing involves less friction and wear.

For examples ball bearings or cylindrical rolling-element bearings.

2.4.2 Hydrodynamic Bearings

In these type of bearings, sliding surfaces are completely separated by fluid film. To support the bearing load, the fluid film is maintained at high pressure. This mechanism is known as thin film of lubrication. The pressure in the lubrication film is generated by hydrodynamic action due to the rapid relative motion of the sliding surfaces. The film behaves like a viscous wedge and because of this high pressure is generated and hence load-carrying capacity. Also wear is prevented because of complete separation of sliding surfaces.

2.4.3 Hydrostatic Bearings

In these type of bearings two sliding surfaces are separated by a film which is maintained by very high pressure. The pressure is generated in this film by an external pump. The fluid under this high pressure carries the high load and preventing the very high friction and wear. Because of the pressure generated by external pump this type of bearing is costly compare to the other bearing.

2.4.4 Electromagnetic Bearings

In these type of bearings, magnetic force is used to support the load. Several electromagnets are attached on the two sliding surfaces. The load carrying capacity is generated with the help of magnetic field. With the help of active feedback control, two sliding surfaces can be separated and wear is completely prevented. These type of bearings are under developmental stage.

2.5 Types of Lubrication

There are different types of lubrication which are as under.

2.5.1 Hydrodynamic (Fluid Film) Lubrication

Here, two surfaces in motion are completely separated by thin fluid film of lubricant, that is surface to surface contact is completely avoided by lubricant film. In this type of lubrication, certain minimum speed of the mating surfaces are required to generate the pressure in the lubricant film.

2.5.2 Boundary Lubrication

In this type of lubrication, thin fluid film between the two surfaces is not sufficient and it cannot prevent the surface asperities from striking with each other. Of course, this contacts are occasional, for instance during the start or stop of the bearing operation, during short supply of lubricant for some unexpected reason.

2.5.3 Elasto-Hydrodynamic Lubrication

In this type of lubrication, due to high pressure distribution in the fluid film there is an elastic deformation in the contact area between surfaces. The pressure in this case is much higher than regular hydrodynamic bearings. While machine is in operation, the viscosity of the lubricant film increases and elasto-hydrodynamic film is generated between two surfaces which are in relative motion, as a result of their deforming elastically against the build up of oil pressure.

2.5.4 Hydrostatic Lubrication

Hydrostatic lubrication consists in pushing lubricant between two surfaces which are in relative motion by an external pressurization system. This can be used when the hydrodynamic lubrication is not very effective. Since any contact is prevented between surfaces which are in relative motion, hydrostatic lubrication produces some favourable effects, especially very low friction when the relative velocity of the surfaces is low.

2.6 Derivation of Generalized Reynolds Equation

The Reynolds equation is the mathematical statement of the classical theory of lubrication and it was formulated by Osborne Reynolds.

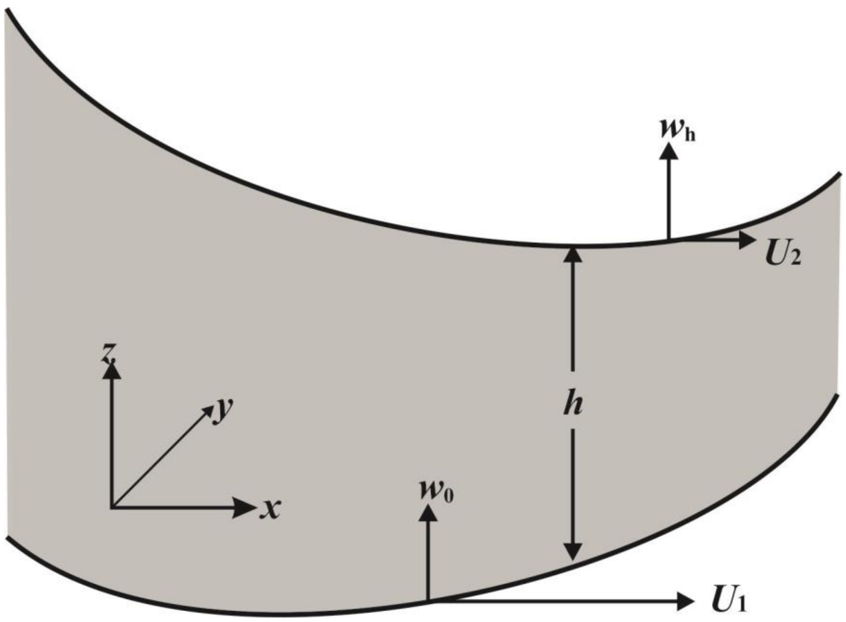

Figure 2.6

Using equations (2.20), (2.21) under the standard assumptions, yields

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial z^2} \quad \text{and} \quad \frac{\partial p}{\partial y} = \eta \frac{\partial^2 v}{\partial z^2}$$

Solving

$$\eta \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$$

implies

$$u = \frac{1}{2\eta} \frac{\partial p}{\partial x} z^2 + c_1 z + c_2 \quad . \tag{2.25}$$

Also solving

$$\eta \frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y}$$

implies

$$v = \frac{1}{2\eta} \frac{\partial p}{\partial y} z^2 + c_3 z + c_4 \tag{2.26}$$

Here c_1, c_2, c_3, c_4 are either constants or at most, functions of *x* and *y*.

Using boundary conditions for *u* and *v* as

$$u = U_1, v = 0 \text{ at } z = 0$$
$$u = U_2, v = 0 \text{ at } z = h$$

equations (2.25) and (2.26) become,

$$u = \frac{1}{2\eta} \frac{\partial p}{\partial x} (z^2 - hz) + \left(1 - \frac{z}{h}\right) U_1 + \frac{z}{h} U_2 \tag{2.27}$$

$$v = (z^2 - hz) \frac{1}{2\eta} \frac{\partial p}{\partial y} \tag{2.28}$$

where U_1 and U_2 represent the velocity of the bearing surfaces.

Substituting the value of equation (2.27) and (2.28) in the equation of continuity

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

and integrating it across the film thickness $(0, h)$,

$$(w)_0^h = -\int_0^h \frac{\partial u}{\partial x} dz - \int_0^h \frac{\partial v}{\partial y} dz$$

Using the Leibnitz's rule for integration,

$$w_h - w_0 = \frac{1}{12}\frac{\partial}{\partial x}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial x}\right) + \frac{1}{12}\frac{\partial}{\partial y}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial y}\right) - \frac{1}{2}(U_1 - U_2)\frac{\partial h}{\partial x} - \frac{h}{2}\frac{\partial (U_1 + U_2)}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{\eta}\frac{\partial p}{\partial y}\right) = \underbrace{6(U_1 - U_2)\frac{\partial h}{\partial x}}_{\text{Represents the wedge effect}} + \overbrace{6h\frac{\partial (U_1 + U_2)}{\partial x}}^{\text{Represents the stretch effect}} + \underbrace{12(w_h - w_0)}_{\text{Represents the squeeze effect}} \tag{2.29}$$

This equation is known as Reynolds equation in which left-hand side represents approximately the average curvature of the pressure distribution surface. If it is negative then it implies that the pressure distribution is upward convex or in other words pressure generated is positive.

2.7 Ferromagnetic Concepts

Def. 2.7.1 (Magnetic Dipole)

The Magnetic dipoles commonly exist in magnetic materials. In magnetic material isolated pole does not exist. A magnetic pole creates a magnetic field around it, which produces a force on a second pole. Experiment shows that this force F is directly proportional to the product of the pole strength p and its field strength H; that is, $F=kpH$ where k is the proportionality constant.

Def. 2.7.2 (Magnetic Field)

Charles Coulomb, in 1785, observed that unlike poles experience a force that is proportional to the product of their pole strengths p_1 and p_2 and inversely proportional to the square of the distance d between them.

$$F = k\frac{p_1 p_2}{d^2}$$

This force is known as magnetic field. It is denoted by **H**. The magnitude of the magnetic field (magnetic fields strength) is denoted by H and its unit of measurement is amp m^{-1}.

Def. 2.7.3 (Magnetic Moment)

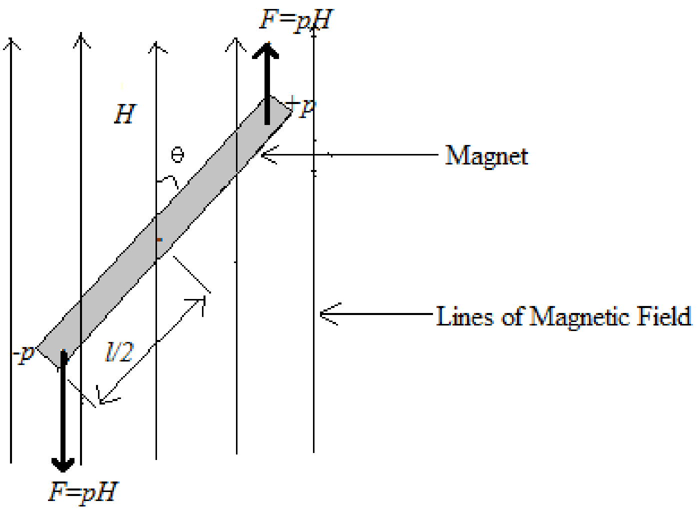

Figure 2.7

Consider a magnet with poles of strength p located near each end and separated by a distance l. Suppose a magnet is placed at an angle θ to a uniform field H. Then the torque acts on the magnet, tending to turn it parallel to the field. The moment of this torque which is known as magnetic moment is given by $\mathbf{p}\times\mathbf{H} = pH\sin\theta$ by assuming magnet of unit length.

Def. 2.7.4 (Magnetization / Intensity of Magnetization)

A quantity that describes the degree to which magnets are magnetized is called the magnetization (or intensity of magnetization) M. It can be considered as $M = \frac{m}{v}$ where m is the magnetic moment per unit volume v. Its unit is amp m^{-1}.

Def. 2.7.5 (Magnetic Induction / Flux Density)

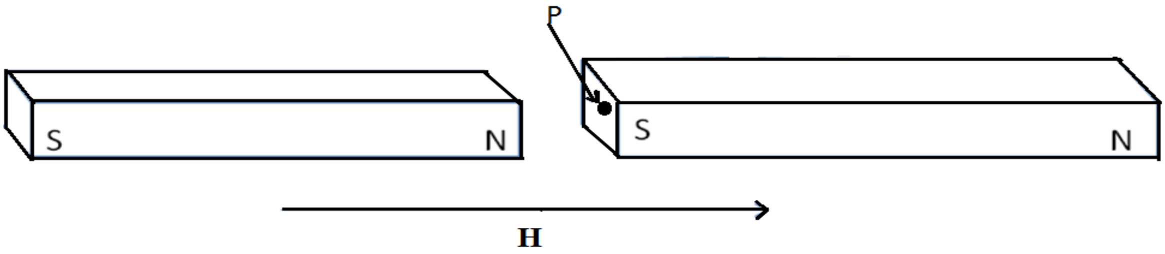

Figure 2.8

A permanent magnet with magnetization M is placed in an applied field **H** oriented at an angle to **M**. Then each square meter of the surface there are $\mu_0 M$ lines passing through it. These are known as lines of magnetization. These lines add to the force $\mu_0 H$ due to the applied field **H** and the combined group of lines crossing the gap are called lines of magnetic flux or magnetic induction **B**. Therefore **B**= μ_0 (**H** + **M**). The unit of magnetic induction is weber m^{-2} which is known as tesla (T)

Def. 2.7.6 (Magnetic Susceptibility)

It is used to explain the magnetization of material, and is defined as the ratio of magnetization M to the magnetic field strength H .

Mathematically,

$$\bar{\mu} = \frac{M}{H},$$

which is a dimensionless quantity.

The susceptibility describes the magnitude of magnetic response shown by a magnetic material at low field strengths. A large value of $\bar{\mu}$ corresponds to strongly magnetic material, while a small value corresponds to a weak magnetic material. Free space has a value of $\bar{\mu} = 0$.

Def. 2.7.7 (Magnetic Permeability)

It is defined as the ratio of amount of magnetic flux density B to the applied magnetic field intensity H.

Mathematically,

$$\mu = \frac{B}{H}.$$

2.8 Historical Development of Ferrofluids

In 1779, Gown knight attempted to produce Magnetic Fluid (MF) or Ferrofluid (FF) by suspending fine iron fillings in water which was not successful. Next in 1940, Bitter prepared a magnetic colloid which was stable under gravity but unstable in the presence of magnetic field. With the help of this behavior, he observed the domain boundaries on the surface of ferromagnetic material. Later, Bitter colloid was refined by Elmore who studied its physical properties. In this improved colloids the magnetic particles concentration was very low and hence the magnetization was also very low.

In 1963, Papell who was working with NASA established a method of stabilizing MFs against aggregation. He was interested to mix this MF with rocket fuel so that it can be used in very low gravitational field with the help of externally applied magnetic field. Papell's MF achieved very high saturation magnetization and low viscosity in the presence of magnetic field. His fluid consisted of finely divided particles of magnetite in kerosene. He used oleic acid as dispersing agent to avoid the particles from clumping together. During this time R. E. Rosensweig and his associates were synthesized the magnetic fluids that was stronger than Papell's magnetic fluid. Even in the strong magnetic field these fluids did not solidify and it remained as a fluid. It also achieved very high magnetization. A mathematical model based on this fluid was first proposed by Rosensweig and Neuringer. Because of this, a new branch of fluid mechanics had been established which is known as ferrohydrodynamics.

Rosensweig and his colleagues established very first Ferrofluid (FF) industry.

In India too, an increasing trend of research interest in magnetic fluid has been evident through the pioneering contribution by Mehta et.al. [20]. Recent contributions from diversified viewpoints are due to Pursi et. al. [21] and Vaidyanathan et. al. [22].

2.8.1 Brief Idea about Ferrofluids

Ferrofluids (FFs) or Magnetic fluids (MFs) are stable colloidal suspensions containing fine ferromagnetic particles dispersing in a non-conducting liquid. In the case of application of external magnetic field **H**, FFs experiences a force $(\mathbf{M} \bullet \nabla)\ \mathbf{H}$. Moreover, due to no coincidence of two rotation namely liquid and magnetic particles, the state of stress is not always symmetric, means magnetization vector **M** is not always parallel to magnetic field vector **H**, which arise magnetic body-torque density $\mu\mathbf{M}\times\mathbf{H}$.

Ferrofluids are not found in nature, but it can be synthesized as per requirement. The most important advantage of FFs over the conventional is that the FFs can be retained at the desired location with the help of a magnetic field. Due to this reason FFs gained widespread popularity among the researchers working on lubrication theory of bearings. Moreover, the use of FF lubrication also adds an additional importance from nano science point of view.

2.8.2 Forces Used in the Equation of Motion for Ferrofluids

In classical fluid dynamics mainly pressure, gravity and viscous forces are considered. Whereas in ferrohydrodynamics, additional magnetic body force and magnetic body torque density are also considered.

Our research problems are mainly concentrated on the use of Shliomis [14, 16] FF flow model with uniform and variable magnetic fields. The detailed discussion of the model is presented in the corresponding chapter.

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