Chapter 1

Introduction

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1.1 Introduction

Mathematics forms the basis for understanding various physical sciences. Tensor calculus plays an important role in Einstein's theory of general relativity through Einstein's field equations

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -\frac{8\pi G}{c^2}T_{ij}, \tag{1.1.1}$$

where i, j take the values from 0 to 3 and g_{ij}, $\Re_{ij}$, $\Re$, G, c are metric tensor, Ricci tensor, Ricci scalar, Newton's gravitational constant and speed of light in vaccum respectively, T_{ij} is energy-momentum tensor, throughout the thesis we have used geometrized units ($c^2 = G = 1$), unless otherwise stated, and used Einstein's field equations in the form

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -8\pi T_{ij}, \tag{1.1.2}$$

Tensor calculus was originally presented by Ricci in 1892 (Résumé de quelque travaux sur les sytémes variables de fonctions associées á une forme différentielle quadratique, *Bulletin des Sciences Mathématiques* **2** (16):167-189) and later by Tullio Levi-Civita in their classic text *Methods de calcul differentiel absolu et leurs applications* (Methods of absolute differential calculus and their application) in 1900. The Einstein's theory of general relativity is one of the application of tensor calculus.

Einstein's field equations consist of a system of 16 highly nonlinear differential equations. For spherically symmetric spacetime metric the number of equations is reduced to 4. Getting the singularity free exact solution of these nonlinear differential field equations is highly difficult. That is why Tolman [97] said "It is difficult to obtain explicit solutions of Einstein's gravitational field equations, in terms of known analytic functions, on account of their complicated and nonlinear character".

Einstein's field equations connect the geometry of the spacetime with the matter content of the distribution. Therefore geometry plays an important role in general theory of relativity and hence spacetime metric having a definite geometry is of mathematical as well as physical importance in general relativity.

The spacetime metric having geometrical significance was first studied by Karl Schwarzschild [78] and obtained first exact solution of Einstein's field equations for empty spacetime. The study of interior of stellar objects began with Schwarzschild [79] interior solution, in which matter density was assumed to be constant, which is good model for stellar structures in which pressure is relatively low. Schwarzschild used spherical, spherically symmetric spacetime metric to describe interior of relativistic star. In the recent past Vaidya and Tikekar [99], Tikekar and Thomas [93] & Tikekar and Jotania [89] used spheroidal, pseudo spheroidal and paraboloidal spacetimes respectively and found that these spacetimes are useful in describing models of superdense stars. The spacetime metrics used by them are spherically symmetric spacetime metrics of the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{1.1.3}$$

with different ansatz for $e^{\lambda(r)}$. Vaidya and Tikekar [99] considered $e^{\lambda(r)} = \frac{1-K\frac{r^2}{R^2}}{1-\frac{r^2}{R^2}}$, which represents a 3-spheroid

$$\frac{w^2}{b^2} + \frac{x^2+y^2+z^2}{R^2} = 1, \tag{1.1.4}$$

immersed in a 4-dimensional flat space having metric

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{1.1.5}$$

The parametrization

$$\left.\begin{aligned} x &= R\sin\lambda\sin\theta\cos\phi \\ y &= R\sin\lambda\sin\theta\sin\phi \\ z &= R\sin\lambda\cos\theta \\ w &= b\cos\lambda \end{aligned}\right\}, \tag{1.1.6}$$

of 3-spheroid, leads to the spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - \left(\frac{1-K\frac{r^2}{R^2}}{1-\frac{r^2}{R^2}}\right)dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{1.1.7}$$

The exact solution of Einstein's field equations for a perfect fluid on spheroidal spacetime metric (1.1.7) was obtained by Vaidya and Tikekar [99] and Energy conditions were also examined by them. Knutsen [46] examined dynamical stability of model of superdense star on spheroidal spacetime, and found that the model is stable with respect to infinitesimal radial oscillations.

Tikekar and Thomas [93] have taken $e^{\lambda(r)}$ in the form $e^{\lambda(r)} = \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}$. With this choice of $e^{\lambda(r)}$, the $t = constant$ sections of spacetime metric (1.1.3) have geometry of 3-pseudo spheroid with cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2+y^2+z^2}{R^2} = 1, \tag{1.1.8}$$

immersed in four-dimensional Euclidean space with metric (1.1.5). The space part of the metric is obtained by introducing parametric equations

$$\left.\begin{aligned} x &= R\sinh\lambda\sin\theta\cos\phi \\ y &= R\sinh\lambda\sin\theta\sin\phi \\ z &= R\sinh\lambda\cos\theta \\ w &= b\cosh\lambda \end{aligned}\right\}. \tag{1.1.9}$$

Tikekar and Thomas [93] also have found that the model of stars on pseudo spheroidal spacetime are stable under radial modes of pulsation.

Tikekar and Jotania [89] used $e^{\lambda(r)} = 1+\frac{r^2}{R^2}$. The 3-space of spacetime metric (1.1.3), obtained as $t = constant$, has geometry of 3-paraboloid immersed in 4-dimensional flat space having metric (1.1.5) with Cartesian equation $x^2 + y^2 + z^2 = 2wR$.

Ever since Schwarzschild [78] obtained solution for Einstein's field equations, a number of exact solutions of Einstein's field equations were obtained describing models of isotropic stars, anisotropic stars, collapsing stars accompanied by radiation, charged stars, charged anisotropic stars and core-envelope models of superdense stars.

A method was developed by Tolman [97] to find exact solution of Einstein's field equations in terms of known functions for static fluid spheres. Delgaty and Lake [13] analysed physical plausibility conditions for 127 solutions of Einstein's field equations and found that only 16 of them satisfies all the conditions and only for 9 solutions sound speed is decreasing with radius. Pant and Sah [69] generalized Tolman's I, IV and V solutions and the de Sitter solution, also obtained class of new static solutions assuming equation of state. Durgapal [21] obtained class of new exact solutions for spherically symmetric static fluid spheres with the ansatz $e^{\nu} \propto (1 + x)^n$, and found that for each integer value of n, one can have new exact solution. Tikekar [88] obtained new exact solution for a static fluid sphere on spheroidal spacetime. Chattopadhyay and Paul [10] obtained the solutions of static compact stars on higher dimensional spacetime. The space part of spacetime metric considered by them is $(D - 1)$ pseudo spheroid immersed in D-dimensional Euclidean space.

The locally anisotropic equation of state for relativistic spheres was considered by Bowers and Liang [7]. Pant and Sah [68] obtained analytic solution for charged fluid on spherically symmetric spacetime, in their analysis, if charge is absent, the solution is Tolman's solution VI with $B = 0$. Consenza *et. al.* [12] developed the procedure to obtain solution of Einstein's field equations for anisotropic matter from known solutions of isotropic matter. The charged analog of Vaidya-Tikekar [99] solution on spheroidal spacetime was obtained by Patel and Koppar [70]. Bayin [5] found the solution for anisotropic fluid sphere by generalizing equation of state $p = \alpha\rho$ and also studied radiating anisotropic fluid sphere. Tikekar and Thomas [94] found exact solution of Einstein's field equations for anisotropic fluid sphere on pseudo spheroidal spacetime. The key feature of their model is the high variation of density from centre to boundary of stellar configuration also radial and tangential pressure are equal at the centre and boundary of the star. Mak and Harko [61] obtained classe of exact anisotropic solutions of Einstein's field equations on spherically symmetric spacetime metric. Komathiraj and Maharaj [47] studied analytical models of quark stars where they found a class of solutions of Einstein-Maxwell system by considering linear equation of state. Karmakar *et. al.* [43] analysed the role of pressure anisotropy for Vaidya-Tikekar [99] model. The exact solutions for Einstein-Maxwell system were extensively studied by Komathiraj and Maharaj [48][57] & Thirukkanesh and Maharaj [85].

Non-adiabatic gravitational collapse of a radiating star on the background of spheroidal spacetime was studied by Tikekar and Patel [91]. Tikekar and Patel [92] also studied non-adiabatic gravitational collapse of a charged radiating stellar structure, where they formulated equations governing shear free non-adiabatic collapse of spherical charged anisotropic matter in the presence of heat flow in the radial direction. Maharaj and Govender [56] considered effect of shear in charged radiating gravitational collapse. Non-adiabatic charged gravitational collapse by considering effect of viscosity is studied by Prisco *et. al.* [73]. The gravitation collapse with heat flux and shear on spherically symmetric spacetime metric is studied by Rajah and Maharaj [75], they found that gravitational behavior is described by Riccati equation, also found two new closed form solution. Misthry *et. al.* [63] found several new classes of exact solutions for radiative collapse.

Koppar and Patel [49] obtained the models of stars with two density distributions. Paul and Tikekar [72] obtained core-envelope models of stars on spheroidal spacetime and core-envelope models of stars on pseudo spheroidal spacetime are obtained by Tikekar and Thomas [95].

These studies show that core-envelope models having core and envelope with different physical features, collapse of radiating stars, dynamical stability of models of superdense stars and anisotropic stars are important in general theory of relativity. In this thesis we have studied core-envelope models of superdense stars on pseudo spheroidal spacetime, core-envelope model of a collapsing radiating star, dynamical stability of the model of superdense stars on paraboloidal spacetime, core-envelope models of superdense stars on paraboloidal spacetime, anisotropic stars on paraboloidal spacetime and also generated quadratic equation of state for anisotropic models of superdense stars on paraboloidal spacetime.

Following preliminaries are provided to facilitate comprehension of the results presented in later chapters.

1.2 Preliminaries

Definition 1.2.1. *The expression $a_1x^1 + a_2x^2 + ... + a_kx^k$ is denoted by symbol $\sum_{i=1}^{k} a_ix^i$. By* ***summation convention*** *we mean that if a suffix occurs twice in a term, once in the lower position and once in the upper position then that suffix implies summation over the range under consideration. Hence we drop the summation sign and write a_ix^i to denote above expression.*

Definition 1.2.2. *If a suffix occurs twice in a term, once in the lower position and once in the upper position, then that suffix is called* ***dummy suffix,*** *Hence $a_ix^i = a_jx^j$.*

Definition 1.2.3. *If A^i be the set of quantities defined in the coordinate system $(x^1, x^2, ..., x^n)$ and A'^α be the set of quantities defined in the coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities A^i is said to be* ***contravariant vector*** *or* ***contravariant tensor of rank 1*** *if it satisfies the transformation law*

$$A'^\alpha = A^i \frac{\partial x'^\alpha}{\partial x^i}.$$

Example 1.2.1. *dx^i is a contravariant vector. If we transform dx^i from coordinate system $(x^1, x^2, ..., x^n)$ to $(x'^1, x'^2, ..., x'^n)$, the transformation gives*

$$dx'^\alpha = dx^i \frac{\partial x'^\alpha}{\partial x^i},$$

which obays the law of transformation of coordinates.

Definition 1.2.4. *If A_j be the set of quantities defined in the coordinate system $(x^1, x^2, ..., x^n)$ and A'_β be the set of quantities defined in the coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities A_j is said to be* ***covariant vector*** *or* ***covariant tensor of rank 1*** *if it satisfies the transformation law*

$$A'_\beta = A_j \frac{\partial x^j}{\partial x'^\beta}.$$

Example 1.2.2. *$\frac{\partial\phi}{\partial x^j}$ is a covariant vector. If we transform $\frac{\partial\phi}{\partial x^j}$ from coordinate system $(x^1, x^2, ..., x^n)$ to $(x'^1, x'^2, ..., x'^n)$, the transformation gives*

$$\frac{\partial\phi}{\partial x'^\beta} = \frac{\partial\phi}{\partial x^j}\frac{\partial x^j}{\partial x'^\beta},$$

which obays the law of transformation of coordinates.

Definition 1.2.5. *If $A^{i_1,i_2,......,i_p}$ be the set of quantities defined in coordinate system $(x^1, x^2, ..., x^n)$ and $A'^{\alpha_1,\alpha_2,.......,\alpha_p}$ be the set of quantities defined in coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities $A^{i_1,i_2,......,i_p}$ is said to be* ***contravariant tensor of rank p*** *if it satisfies the transformation law*

$$A'^{\alpha_1,\alpha_2,.......,\alpha_p} = A^{i_1,i_2,......,i_p}\frac{\partial x'^{\alpha_1}}{\partial x^{i_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{i_2}}.....\frac{\partial x'^{\alpha_p}}{\partial x^{i_p}}.$$

Definition 1.2.6. *If $A_{j_1,j_2,......,j_q}$ be the set of quantities defined in coordinate system $(x^1, x^2, ..., x^n)$ and $A'_{\beta_1,\beta_2,.......,\beta_q}$ be the set of quantities defined in coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities $A_{j_1,j_2,......,j_q}$ is said to be* ***covariant tensor of rank q*** *if it satisfies the transformation law*

$$A'_{\beta_1,\beta_2,.......,\beta_q} = A_{j_1,j_2,......,j_p}\frac{\partial x^{j_1}}{\partial x'^{\beta_1}}\frac{\partial x^{j_2}}{\partial x'^{\beta_2}}.....\frac{\partial x^{j_q}}{\partial x'^{\beta_q}}.$$

Definition 1.2.7. *If $A^{i_1,i_2,......,i_p}_{j_1,j_2,......,j_q}$ be the set of quantities defined in coordinate system $(x^1, x^2, ..., x^n)$ and $A'^{\alpha_1,\alpha_2,.......,\alpha_p}_{\beta_1,\beta_2,.......,\beta_q}$ be the set of quantities defined in coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities $A^{i_1,i_2,......,i_p}_{j_1,j_2,......,j_q}$ is said to be* ***mixed tensor of rank p+q*** *if it satisfies the transformation law*

$$A'^{\alpha_1,\alpha_2,.......,\alpha_p}_{\beta_1,\beta_2,.......,\beta_q} = A^{i_1,i_2,......,i_p}_{j_1,j_2,......,j_q}\frac{\partial x'^{\alpha_1}}{\partial x^{i_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{i_2}}.....\frac{\partial x'^{\alpha_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x'^{\beta_1}}\frac{\partial x^{j_2}}{\partial x'^{\beta_2}}.....\frac{\partial x^{j_q}}{\partial x'^{\beta_q}}.$$

Definition 1.2.8. *A covariant tensor A_{ij} is said to be* ***symmetric*** *if*

$$A_{ij} = A_{ji},$$

similarly a contravariant tensor $A^{\alpha\beta}$ is said to be ***symmetric*** *if*

$$A^{\alpha\beta} = A^{\beta\alpha}.$$

Definition 1.2.9. *A covariant tensor A_{ij} is said to be* ***anti-symmetric*** *if*

$$A_{ij} = -A_{ji},$$

similarly a contravariant tensor $A^{\alpha\beta}$ is said to be ***anti-symmetric*** *if*

$$A^{\alpha\beta} = -A^{\beta\alpha}.$$

Definition 1.2.10. *Consider two tensors $A^{\alpha}_{\beta\gamma}$ and $B^{\alpha}_{\beta\gamma}$ of same rank and character then their* ***sum*** *is a tensor $C^{\alpha}_{\beta\gamma}$ of same rank and character defined as*

$$C^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma} + B^{\alpha}_{\beta\gamma}.$$

Definition 1.2.11. *The* ***outer product*** *of a tensor $A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n}$ having covariant rank n and contravariant rank m with a tensor $B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}$ having covaraint rank p and contravariant rank q, is a tensor having covariant rank $n+q$ and contravariant rank $m+p$ and is defined as*

$$C^{i_1,i_2,\ldots,i_m,\alpha_1,\alpha_2,\ldots,\alpha_p}_{j_1,j_2,\ldots,j_n\beta_1,\beta_2,\ldots,\beta_q} = A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}.$$

Definition 1.2.12. *The process of setting one covariant and one contravariant suffixes equal is called* ***contraction****. Contraction reduce the tensor rank by 2.*

Definition 1.2.13. *The* ***inner product*** *of a tensor $A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n}$ having covariant rank n and contravariant rank m with a tensor $B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}$ having covaraint rank p and contravariant rank q, is a tensor having covariant rank $n+q-1$ and contravariant rank $m+p-1$ and is defined as*

$$C^{i_2,\ldots,i_m,\alpha_1,\alpha_2,\ldots,\alpha_p}_{j_1,j_2,\ldots,j_n,\beta_2,\ldots,\beta_q} = A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{i_1\beta_2,\ldots,\beta_q} = A^{i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_2,\ldots,\beta_q}$$

Hence ***inner product*** *of two tensors is their outer product followed by contraction.*

Definition 1.2.14. ***Quotient law of tensors*** *says that, a set of quantities, whose inner product with an arbitrary tensor is a tensor, is tensor itself.*

Definition 1.2.15. ***Kronecker Delta*** *is denoted by δ^i_j and is defined as*

$$\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

which is mixed tensor of rank 2.

Definition 1.2.16. *The metric of the form*

$$ds^2 = g_{ij}dx^i dx^j,$$

where i and j varies from 1 to n is called ***Riemannian metric*** *in n dimensional* ***Riemannian space****, g_{ij} is called* ***fundamental tensor****. The* ***reciprocal*** *of g_{ij} is denoted by g^{ij} and is defined as*

$$g^{ij} = \frac{cofactor\ of\ g_{ij}\ in\ |g_{ij}|}{g},$$

where $g = |g_{ij}|$.

Definition 1.2.17. *The process of multiplying covariant tensor A_i with contravariant metric tensor g^{ij} is called* ***raising an index*** *and resultant is contravariant tensor*

$$A^j = g^{ij}A_i.$$

Definition 1.2.18. *The process of multiplying contravariant tensor A^i with covariant metric tensor g_{ij} is called* ***lowering an index*** *and resultant is covariant tensor*

$$A_j = g_{ij}A^i.$$

Definition 1.2.19. ***Christoffel symbol of first kind*** *is denoted by $\Gamma_{\mu\nu\sigma}$ and is defined as*

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) = \frac{1}{2}\left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}\right).$$

The lower suffix preceding by comma denote a derivative in this way. ***Christoffel symbol of second kind*** *is denoted by $\Gamma^\sigma_{\mu\nu}$ and is defined as*

$$\Gamma^\sigma_{\mu\nu} = g^{\sigma\beta}\Gamma_{\mu\nu\beta}.$$

Both the Christoffel symbol are not tensor quantities.

Definition 1.2.20. ***Geodesic*** *is a curve for which variation in the arc length is* *zero if end points are kept fixed that is*

$$\delta \int_P^Q ds = 0 \Rightarrow \int_P^Q ds \text{ is stationary.}$$

The equation of **geodesic** *for Riemannian metric* $ds^2 = g_{ij}dx^i dx^j$ *is*

$$\frac{d^2x^k}{ds^2} + \frac{dx^i}{ds}\frac{dx^j}{ds}\Gamma^k_{ij} = 0.$$

Definition 1.2.21. ***Covariant derivative*** *of covariant tensor* A_μ *of rank* 1 *is*

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\alpha_{\mu\nu}A_\alpha = A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu}A_\alpha.$$

Here semi-colon denote covariant derivative and comma denote ordinary derivative. ***Covariant derivative*** *of contravariant tensor* A^μ *of rank* 1 *is*

$$A^\mu_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu}A^\alpha = A^\mu_{,\nu} + \Gamma^\mu_{\alpha\nu}A^\alpha.$$

Covariant derivative *of covariant tensor* $A_{\mu\nu}$ *of rank* 2 *is*

$$A_{\mu\nu;\gamma} = \frac{\partial A_{\mu\nu}}{\partial x^\gamma} - A_{\alpha\nu}\Gamma^\alpha_{\mu\gamma} - A_{\mu\alpha}\Gamma^\alpha_{\nu\gamma} = A_{\mu\nu,\gamma} - A_{\alpha\nu}\Gamma^\alpha_{\mu\gamma} - A_{\mu\alpha}\Gamma^\alpha_{\nu\gamma}.$$

Covariant derivative *of contravariant tensor* $A^{\mu\nu}$ *of rank* 2 *is*

$$A^{\mu\nu}_{;\gamma} = \frac{\partial A^{\mu\nu}}{\partial x^\gamma} + A^{\alpha\nu}\Gamma^\mu_{\alpha\gamma} + A^{\mu\alpha}\Gamma^\nu_{\alpha\gamma} = A^{\mu\nu}_{,\gamma} + A^{\alpha\nu}\Gamma^\mu_{\alpha\gamma} + A^{\mu\alpha}\Gamma^\nu_{\alpha\gamma}.$$

In general the ***covariant derivative*** *of tensor having contravariant rank* l *and covariant rank* m *is*

$$\begin{aligned}
A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m;\beta} &= \frac{\partial A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}}{\partial x^\beta} + A^{\alpha\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_1}_{\alpha\beta} + ... + A^{\mu_1.....\mu_{l-1}\alpha}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_l}_{\alpha\beta} - \\
&\quad A^{\mu_1\mu_2.....\mu_l}_{\alpha\nu_2.....\nu_m}\Gamma^\alpha_{\nu_1\beta} - ..... - A^{\mu_1\mu_2.....\mu_l}_{\nu_1.....\nu_{m-1}\alpha}\Gamma^\alpha_{\nu_m\beta} \\
&= A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m,\beta} + A^{\alpha\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_1}_{\alpha\beta} + ... + A^{\mu_1.....\mu_{l-1}\alpha}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_l}_{\alpha\beta} - \\
&\quad A^{\mu_1\mu_2.....\mu_l}_{\alpha\nu_2.....\nu_m}\Gamma^\alpha_{\nu_1\beta} - ..... - A^{\mu_1\mu_2.....\mu_l}_{\nu_1.....\nu_{m-1}\alpha}\Gamma^\alpha_{\nu_m\beta}.
\end{aligned}$$

Definition 1.2.22. *The* ***Riemann-Christoffel tensor*** *or* ***curvature tensor*** *is* *defined as*

$$\Re^{\beta}_{\mu\nu\sigma} = \frac{\partial\Gamma^{\beta}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\beta}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\sigma} = \Gamma^{\beta}_{\mu\sigma,\nu} - \Gamma^{\beta}_{\mu\nu,\sigma} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\sigma}.$$

Definition 1.2.23. *The **Bianci relation** is given by*

$$\Re^{\beta}_{\mu\nu\sigma;\tau} + \Re^{\beta}_{\mu\sigma\tau;\nu} + \Re^{\beta}_{\mu\tau\nu;\sigma} = 0,$$

which states that Riemann-Christoffel tensor satisfies these differential equations and symmetry conditions.

Definition 1.2.24. *The Riemann-Christoffel tensor with the contraction is called **Ricci tensor**, that is*

$$\Re_{\mu\nu} = \Re^{\beta}_{\mu\nu\beta}.$$

*The explicit form of **Ricci tensor** is given by*

$$\Re_{\mu\nu} = \frac{\partial\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha},$$

*again contractin we get **Ricci scalar***

$$\Re = \Re^{\nu}_{\nu} = g^{\mu\nu}\Re_{\mu\nu}.$$

Definition 1.2.25. *The **principle of covariance** says that the laws must be expressible in a form which is independent of the particular spacetime coordinate chosen that is laws of nature remains invariant with respect to any spacetime coordinate system.*

Definition 1.2.26. *The **principle of equivalance** says that at every spacetime point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated cartesian coordinate systems in the absence of gravitation.*

Definition 1.2.27. *The **energy-momentum tensor** for a perfect fluid is of the form*

$$T_{ij} = (\rho + p)\, u_i u_j - g_{ij} p,$$

where ρ and p denotes density and pressure of fluid respectively and $u_i = \frac{dx^i}{dt}$.

Definition 1.2.28. *The **Einstein's Tensor** is defined as*

$$G_{ij} = \Re_{ij} - \frac{1}{2}\Re g_{ij}.$$

Theorem 1.2.1. ***Schwarzschild exterior solution:*** *For empty spacetime the static spherically symmetric spacetime metric*

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(dr^2 + \sin^2\theta d\phi^2\right),$$

possesses a solution of the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

□

Proof. Consider the spherically symmetric spacetime metric of the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(dr^2 + \sin^2\theta d\phi^2\right), \tag{1.2.1}$$

Here the coordinates are

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^4 = \phi, \tag{1.2.2}$$

the components of fundamental tensor for spacetime metric (1.2.1) are

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad g_{\mu\nu} = 0 \; for \; \mu \neq \nu, \tag{1.2.3}$$

and

$$g = g_{00}g_{11}g_{22}g_{33} = -e^{\lambda+\nu}r^2\sin^2\theta. \tag{1.2.4}$$

The Christoffel's symbol of second kind is given by

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\beta}\left(g_{\nu\beta,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}\right). \tag{1.2.5}$$

The non-vanishing components of Christoffel's symbol of second kind are

$$\Gamma^1_{00} = \nu' e^{\nu-\lambda} \quad \Gamma^0_{10} = \tfrac{\nu'}{2} \quad \Gamma^1_{11} = \tfrac{\lambda'}{2}$$

$$\Gamma^2_{12} = \tfrac{1}{r} \quad \Gamma^1_{22} = -re^{-\lambda} \quad \Gamma^3_{13} = \tfrac{1}{r}$$

$$\Gamma^3_{23} = \cot\theta \quad \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda} \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

The Ricci tensor is given by

$$\Re_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{1.2.6}$$

The non-vanishing components of Ricci tensor are

$$\Re_{00} = e^{\nu-\lambda}\left(-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r}\right), \tag{1.2.7}$$

$$\Re_{11} = \frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r}, \tag{1.2.8}$$

$$\Re_{22} = e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1, \tag{1.2.9}$$

$$\Re_{33} = \Re_{22}\sin^2\theta. \tag{1.2.10}$$

For empty spacetime Einstein's field equations are given by $\Re_{ij} = 0$, therefore

$$e^{\nu-\lambda}\left(-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r}\right) = 0, \tag{1.2.11}$$

$$\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} = 0, \tag{1.2.12}$$

$$e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1 = 0, \tag{1.2.13}$$

$$\left\{e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1\right\}\sin^2\theta = 0. \tag{1.2.14}$$

Equations (1.2.13) and (1.2.14) are dependent and hence there three independent equations (1.2.11) - (1.2.14). Dividing (1.2.11) by $e^{\nu-\lambda}$ and adding in (1.2.12) gives

$$\lambda' + \nu' = 0, \tag{1.2.15}$$

whose solution is

$$\lambda + \nu = K, \tag{1.2.16}$$

where K is constant of integration. For a large value of r, space must be approximately flate that is as $r \to \infty$, the unknowns $\lambda, \nu \to 0$. Hence $K = 0$ and therefore

$$\lambda + \nu = 0 \Rightarrow \lambda = -\nu. \tag{1.2.17}$$

From equation(1.2.13)

$$e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) = 1, \tag{1.2.18}$$

Using equation (1.2.17) in equation (1.2.18),

$$e^{-\lambda}\left(1 + r\nu'\right) = 1 \Rightarrow \frac{d}{dr}\left(re^{\nu}\right) = 1 \Rightarrow re^{\nu} = r - 2m, \tag{1.2.19}$$

where m is constant of integration, therefore

$$e^{\nu} = 1 - \frac{2m}{r} \; and \; e^{\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}, \tag{1.2.20}$$

therefore complete solution is of the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.2.21}$$

□

Remark 1.2.1. *Schwarzschild exterior solution gave rise to corrections in Newtonian theory for planetary motion. These corrections are notable in the case of mercury, the planet nearest to sun.*

Remark 1.2.2. *Schwarzschild exterior solution has two singularities at* $r = 0$ *and at* $r = 2m$.

1.3 Layout of the thesis

The thesis is organized as follows:

Chapter 1 deals with general introduction of the thesis.

Chapter 2 deals with core-envelope model of superdense stars with the feature - core consisting of isotropic fluid distribution and envelope consisting of anisotropic fluid distribution on pseudo spheroidal spacetime. In the case of superdense stars core-envelope models with isotropic pressure in the core and anisotropic pressure in the envelope, may not be unphysical. We have used existing solution of Tikekar and Thomas [95] for developing the model. The core radius is found to be $b = \sqrt{2}R$, R being geometric parameter and for positivity of tangential pressure $p_{\perp}$ it is required that $\frac{a^2}{R^2} > 2$, where a is the radius of the star. This requirement restrict the value of density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)} \leq 0.093$. Further it is observed that thickness of the envelope decreases with increasing value of λ, and radius of the star increases as λ increases.

Chapter 3 describes the non-adiabatic gravitational collapse of a spherical distribution of matter having radial heat flux on pseudo spheroidal spacetime. The star is divided into two regions: core consisting of anisotropic fluid distribution and an envelope consisting of isotropic fluid distribution, various aspects of the collapse have been studied. The variation of polytropic index γ with respect to time, at the centre and on the boundary is calculated for the model with density variation parameter $\lambda = 0.05$. The polytropic index at the centre is less than $\frac{4}{3}$ and at the boundary is much higher than $\frac{4}{3}$ during initial stage of collapse. This indicates that the central region is dynamically unstable. Assuming the evolution of heat flow is governed by Maxwell-Cattaneo heat transport equation and by making suitable assumptions equation governing temperature profile have been derived.

In Chapter 4, we investigate stability of superdense star on paraboloidal spacetime. The stability of models of stars on paraboloidal spacetime is investigated by integrating Chandrasekhar's pulsation equation and it is found that the models with $0.26 < \frac{m}{a} < 0.36$ will be stable under radial modes of pulsation.

In Chapter 5, we have reported two core-envelope models with the feature core consisting of isotropic fluid and envelope consisting of anisotropic fluid distribution on the background of paraboloidal spactime. For both the models thickness of envelope increases as $\frac{m}{a}$ increases. A noteworthy feature these models is, they admits thin envelope, hence is significant in the study of glitches and star quakes.

Chapter 6 describes an anisotropic model of superdense star on paraboloidal spacetime. A particular choice is made on radial pressure p_r to integrate Einstein's field equations. The central pressure in this model is $\frac{p_0}{R^2}$. The bounds on p_0 is obtained such that physical plausibility conditions are satisfied and Herrera's [35] overtuning method is applied to check the stability of the star. it is found that prescribed model is stable for $\frac{1}{3} < p_0 < 0.3944$. Further equation of state is generated and we found that model satisfies quadratic equation of state.

Chapter 6 is followed by summery of thesis, appendices, list of publications and references used during the course of research.