Chapter 6

Wavelets and Preconditioners

In this chapter, we have made an attempt to solve Dirichlet-Poisson's equation in two dimension using preconditioning concepts. We consider the representation of elliptic differential operators in wavelet bases and preconditioner based on wavelet methods. The method for solving Poisson's equation for two dimensions and three dimensions with Dirichlet boundary conditions, has been implemented in Matlab based on work of G. Beylkin (see [Bey94]). In Galerkin approach, we get an ill-conditioned system. Here, we have made an attempt to show that the condition number of the reduced matrix is of size $O(1)$ using wavelet preconditioning.

As we know that the wavelets are smooth and well localized functions derived from dilations and translations of a single function ψ, called the mother wavelet, in the following way:

$$\psi_{j,k} = 2^{j/2}\psi\left(2^j x - k\right); \quad j,k \in \mathbf{Z}.$$

The interesting thing about wavelets is that they provide unconditional basis for different spaces such as L^2, Sobolev spaces, and Hölder spaces. The first orthogonal basis of wavelets was constructed by J. O. Strömberg. Later, the concept of Multi Resolution Analysis introduced by Y. Meyer (see [Mey93]) and S. Mallat (see [Mal91]), leads to Fast Wavelet Transform (FFT). In 1988, I. Daubechies [Dau88] constructed orthonormal wavelets with compact support.

It is well known that wide class of operators (Calderon-Zygmund and Pseudo-Differential Operators) have almost sparse representations in wavelet bases which permits a number of fast algorithms for applying these operators to functions, solving integral equations etc. In 1989, G. Beylkin, R. Coifman, and V. Rokhlin [BCR91] used this fact to developed techniques to compress integral equations and thereby append the field to differential equations.

For the last couple of years, the interest for using wavelets applications to Partial Differential equations grow and lot of research papers were published. For the Elliptic problems with Dirichlet or Neumann boundary conditions, S. Jaffard examined the use of wavelets as a test function in Galerkin's method and showed the existence of a diagonal preconditioner which makes the condition number of the corresponding matrix bounded by a constant (see [Jaf92]). One drawback of the Jaffard's result was the absence of an explicit description of the wavelets to be used and the fact that they did not have compact support which makes them unstable for practical use. Until now, there has been several constructions of wavelets but non of them have succeeds finally, to solve the Boundary Value Problem.

G. Beylkin developed a method to represent a differential operators in the wavelet bases constructed in [Bey94]. This method leads to fast algorithms for evaluating these operators acting on the functions, and therefore, suggest an alternative to common methods for the discretization of differential equations. The discretization of differential equations in the Galerkin's method leads to a sparse matrix with large condition number. For a second order elliptic problem, the condition number is of order $O(1/h^2)$, where h is the size of discretization. To avoid such ill conditioning, G. Beylkin used the preconditioning in [Bey94] to obtain a condition number of $O(1)$. In this chapter, we have made an attempt to obtain condition number of $O(1)$ for second order elliptic problem in two dimension.

6.1 Numerical Approach to Poisson Equation

We shall use wavelets in the Galerkin's method to solve Poisson's equations with Dirichlet boundary conditions on a bounded domain $\Omega \subset R^n$. The Sobolev space is

$$H^s(\Omega) = \left\{ f \in L^2(\mathbf{R}) : D^\alpha f \in L^2(\mathbf{R}), \forall |\alpha| \leq s, s \in \mathbf{N} \right\},$$

with norm

$$\|f\|_{H^s}^2 = \int_\Omega \sum_{|\alpha| \leq s} |D^\alpha f(\bar{x})|^2 \, d\bar{x}.$$

The operator D^α is defined as

$$D^\alpha f(\bar{x}) = \frac{\partial^{|\alpha|} f(\bar{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}; \quad with \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Furthermore, a continuous function $f \in$ Hölder space $C^r(\Omega); 0 \leq r \leq 1$, if there exists a constant c such that

$$|f(x) - f(y)| \leq c \, |x - y|^r; \quad \forall x, y \in \Omega.$$

For $m < r \leq m+1$, let $r = m + s$, then the functions $f \in C^r(\Omega)$ iff

$$D^\alpha f \in C^r(\Omega); \quad |\alpha| = m.$$

Finally, we say that $f \in C_0^r(\Omega)$ if $f \in C^r(\mathbf{R}^n)$ and vanishing outside Ω.

The wavelets were constructed by S. Jaffard and Y. Meyer to provide bases for the Hölder space $C_0^r(\Omega)$; $r \in \mathbf{Z}$ and Sobolev space

$$H_0^s(\Omega) = \{f \in H^s(\Omega) : f = 0 \ \ outside \ \ \Omega\}.$$

In their construction, they consider the space V_p of functions that are C^{2m-2}, vanishing outside Ω and are polynomials of degree $2m-1$ in each variables in the cubes

$$k2^{-p} + 2^{-p}[0,1]^n; \ \ with \ \ k \in \mathbf{Z}^n.$$

They show that there exists an L^2 orthonormal basis of V_p composed of functions $\psi_{j,k}$ such that

$$|D^\alpha \psi_{j,k}| \leq 2^{j\alpha} 2^{nj/2} \exp\left(-\gamma 2^j |x - k2^{-j}|\right)$$

for $|\alpha| \leq 2m-2$ and a position γ. The wavelets are indexed by $j = 0, 1, \cdots, p$ and by $k \in \mathbf{Z}^n$ such that

$$k2^{-j} + (m+1)2^{-j}[0,1] \subset \bar{\Omega}.$$

This shows that $\psi_{j,k}$ and its partial derivatives are essentially centered around $k2^{-j}$ with a width of 2^{-j}. S. Jaffard and Y. Meyer proved the following proposition:

Proposition 6.1.1 *If a function $f \in H_0^1(\Omega)$, then the following condition for the wavelet coefficients $c_{j,k}(= \langle f, \psi_{j,k}\rangle)$ holds*

$$C_1 \sum |2^j c_{j,k}|^2 \leq \|f\|_{H_0^1}^2 \leq C_2 \sum |2^j c_{j,k}|^2 .$$

Proof: See [And98].

Consider now the following BVP for Poisson equation, where $u \in H^2(\Omega)$

$$\left.\begin{aligned} -\Delta u = f \ \ in \ \ \Omega \\ u = g \ \ on \ \ \partial\Omega \end{aligned}\right\} \tag{6.1}$$

where $\partial\Omega$ is the boundary of Ω and $f \in L^2(\mathbf{R})$.

Before we use a Galerkin method to solve the problem (6.1), it has to be reduced to a homogeneous problem. This can be done if there exist a smooth function $\tilde{g}$ that extends g inside Ω. Then, $\tilde{u} = u - \tilde{g}$ will be the solution of the following problem:

$$\left.\begin{array}{c} -\Delta\tilde{u} = \tilde{f} \;\; in \;\; \Omega \\ \tilde{u} = 0 \;\; on \;\; \partial\Omega \end{array}\right\} \tag{6.2}$$

where we replace f by $\tilde{f} = f - \Delta\tilde{g}$. We shall give how variation form to equation (6.2) when we use Green's formula

$$-\int_{\Omega} v\Delta w dx = -\int_{\Omega} \nabla v \cdot \nabla w dx + \int_{\partial\Omega} v\frac{\partial w}{\partial n} ds. \tag{6.3}$$

Multiply $\tilde{u}$ in (6.2) by a function $v \in H_0^1(\Omega)$ and integrating, we arrive at the following variational form of (6.2)

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx; \;\; \forall v \in H_0^1(\Omega).$$

Now, if we use the fact that the wavelets provide bases for H_0^1, the Galerkin approximation will consists in finally $u \in V_j$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx; \;\; \forall v \in V_j.$$

When the functions are expressed by their constitution in the wavelet basis, we are left with the problem

$$M\bar{x} = \bar{y} \tag{6.4}$$

where the stiffness matrix M is given by

$$M_{(j,k)(j,k')} = \langle \nabla\psi_{j,k}, \nabla\psi_{j,k'} \rangle$$

and the values $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = (\langle u, \psi_{j,k} \rangle),$$

$$\bar{y} = (\langle f, \psi_{j,k} \rangle).$$

Next, we shall show that there exist a diagonal matrix D such that the condition number of the matrix $D^{-1}MD^{-1}$ is bounded. Recall that condition number of a matrix is defined as the ratio of the largest and smallest singular values and controls the rate of the convergence of a number of iterative algorithms for solving linear systems.

Let A be the vector $c_{j,k}$ of the wavelet coefficients, then

$$A^T M A = \langle \nabla f, \nabla f \rangle.$$

Now, we have

$$C_1 \sum \left|2^j c_{j,k}\right|^2 \leq A^T M A \leq C_2 \sum \left|2^j c_{j,k}\right|^2 .$$

Hence, if D is the diagonal matrix defined by

$$D_{(j,k)(j,k')} = 2^j \delta_{(j,j)} \delta_{(k,k')}$$

then

$$C_1 \left\|A\right\|^2 \leq A^t D^{-1} M D^{-1} A \leq C_2 \sum \left\|A\right|^2 .$$

Further, we have the following theorem:

Theorem 6.1.2 *For a Galerkin method using the wavelets constructed by S. Jaffard and Y. Meyer to approximate the solution of (6.2), the condition number of $D^{-1}MD^{-1}$ is bounded by C_1C_2 (see [And98]).*

This leads us to the following problem instead of (6.4). Solve

$$M_1 \bar{x_1} = \bar{y_1}$$

where the matrix M_1 is given by

$$M_1 = D^{-1} M D^{-1}$$

and the vectors are given by

$$\bar{x_1} = D\bar{x},$$

$$\bar{y_1} = D^{-1}\bar{y}.$$

6.1.1 Numerical Results for One Dimension

In this section, we apply the methods discussed in the previous section to the following problem:

$$-u'' = f; \quad in \ [0, 1]$$

$$u(0) = u(1).$$

We know how to represent linear operators in a wavelets basis for $L^2(\mathbf{R})$. Since, we do not have basis for the specific interval $[0, 1]$, we have to find a function $\tilde{u}(x)$ defined on the real line satisfying the condition

$$-\tilde{u}'' = f(x); \quad in \ 0 \leq x \leq 1.$$

If we denote the function $\tilde{u}$ to be 1-periodic, then the condition

$$\tilde{u}(0) = \tilde{u}(1)$$

is automatically fulfilled and the restriction to the interval $[0,1]$ of the function $\tilde{u}(x)$ solves the original problem.

Suppose we are given the scaling coefficients for the function f at level N. Now, we look for an approximation to $\tilde{u}$ in the subspace V_N and therefore, express the operator d^2/dx^2 in the non standard form. Since, we only consider 1-periodic functions, we may restrict ourselves to the functions

$$\phi_{j,k}(x); \quad for \quad k = 0, 1, \cdots, 2^j - 1 \quad j = 0, 1, \cdots, N-1$$

in equation

$$\gamma_{i,l}^j = 2^j \int_{\mathbf{R}} \phi(2^j x - i) \frac{d^2}{dx^2} \phi(2^j x - l) dx = 4^j r_{i-l}.$$

Furthermore, we want the basic functions to be differentiable; this means that the fillength L in equation

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{L-1} n_k e^{-ik\xi},$$

must be greater than or equal to 10. This gives rise to a system of linear equations for the wavelet coefficients of the function $\tilde{u}$, where the complex matrix has sparse status. To solve this linear system of equations, we use the different methods like: CGS, PCG, BICG, BICGStab, GMRES, QMR, etc. with the preconditioning described earlier. Table 6.1 compares the condition number κ_p with preconditioning and κ without preconditioning:
For the numerical experiments, we let the function f be given by

N	κ	κ_p
5	0.0026 e5	4.3952
6	0.0100 e5	4.6355
7	0.0410 e5	4.7583
8	0.1640 e5	4.9133
9	0.6559 e5	5.0405
10	2.6200 e5	5.1694

Table 6.1: **Comparison of Condition Number without and with preconditioning.**

$$f(x) = \sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x),$$

and hence the exact solution is given by

$$u(x) = -\frac{1}{64\pi^2} \left[16\sin(2\pi x) + 4\sin(4\pi x) + \frac{16}{9}\sin(6\pi x) + \sin(8\pi x) \right].$$

All calculations were made with Matlab, where we use predefined functions to handel sparse matrices. The program uses the level N and vector of the scaling coefficients of f as input. The scaling coefficients of f are evaluated by a quadrature formula.

6.1.2 Numerical Results for Two Dimension

We consider the Dirichlet boundary value problem

$$\Delta u = -2; \quad in\ (0,1)\times(0,1) \tag{6.5}$$

$$u = 0 \ \ on\ \ x = 0,\ \ y = 0,\ \ x = 1,\ \ and\ \ y = 1.$$

The analytic solution of this problem is given by

$$u(x,y) = (1-x)x - \frac{8}{\pi^3}\sum \frac{\sinh[(2n-1)\pi(1-y)] + \sinh[(2n-1)\pi y]}{\sinh[(2n-1)\pi]} \cdot \frac{\sin[(2n-1)\pi x]}{(2n-1)^3} \tag{6.6}$$

	No Preconditioning		With Preconditioning	
N	Iterations	Error	Iterations	Error
8	30	19.1234 e-6	5	5.2005 e-5
9	79	3.80 e-5	7	3.6000 e-5
10	170	1.8321 e-5	5	3.0535 e-5
12	400	6.5163 e-5	6	4.123 e-5

Table 6.2: **Comparison of Condition Number without and with preconditioning for the mentioned problem**

Table 6.2 shows the result of preconditioning and without preconditioning on the solution of the above problem. We have used wavelet preconditioning and GMRES method to solve the linear system of equations. The number of iterations in all methods are observed with a permissible tolerance of 10^{-5}.

6.1.3 Conclusion

- We have solved Dirichlet-Poisson problem using wavelet approach with preconditioning aspects.
- The iterative methods which we have used are: CGS, BICG, GMRES, BICG.

- The preconditioners which we have used are: Jacobi, ILU, and wavelet preconditioners.
- The best preconditioner is Wavelet preconditioner and best iterative methods is GMRES.
- The condition number of the resulting matrix after using preconditioner is of size $O(1)$.