Chapter IV

Absolute General Matrix Summability of an Orthogonal Series

4.1 Introduction

In this section, we shall discuss the matrix summability of general infinite series. We have also discussed the concepts of orthogonal series and orthogonal expansion.

4.1.1 General Matrix Summability

Suppose,

$$\sum_{n=1}^{\infty} a_n \tag{4-1}$$

be a given infinite series. Let $\{s_n\}$ be the partial sum of (4-1). Let $A \coloneqq (A_{nv})$ be a given normal matrix. i.e. $A \coloneqq (A_{nv})$ be a lower triangular matrix having non-zero elements in diagonal. Then, A defines sequence-to- sequence transformation, which maps the sequence $s \coloneqq \{s_n\}$ to $As \coloneqq \{A_n(s)\}$, where

$$A_n(s) \coloneqq \sum_{\nu=0}^{n} a_{n\nu}\, s_\nu, n = 0,1,2,\ldots \tag{4-2}$$

(See Krasniqi, Xh. Z. 2012(1), Krasniqi, Xh. Z. et al. 2012, Tanovic-Miller, N. 1979)

The following definition is due to Flett, T. M. 1957.
Let,

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

The series (4-1) is said to be summable $|A|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty \tag{4-3}$$

It is important to note that if, we consider

$$a_{n\nu} = \frac{p_{n-\nu}}{P_n}$$

then $|A|_k$ summability reduces to $|N, p_n|_k$ summability.
Similarly, if we consider

$$a_{n\nu} = \frac{p_\nu}{P_n}$$

then $|A|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

Flett, T. M. 1957 have extended the above definition by introducing the parameter δ (See Krasniqi, Xh. Z. 2012(1))

The series (4-1) is said to be summable $|A;\ \delta|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|\bar{\Delta} A_n(s)|^k < \infty \qquad (4\text{-}4)$$

In more extended form, the following definition is due to Özarslan, et al. 2011
Let $\{\Phi_n\}$ be a sequence of positive real numbers. We say that (4-1) is summable $\Phi - |A\,;\delta|_k$; $k \geq 1$ and $\delta \geq 0$ if

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1}|\bar{\Delta} A_n(s)|^k < \infty \qquad (4\text{-}5)$$

If we take $\delta = 0$ and $\Phi_n = n$ for all values of n, then $\Phi_n - |A, \delta|_k$ summability reduces to $|A|_k$ summability.
If we take $\Phi_n = n$ for all values of n, then $\Phi_n - |A, \delta|_k$ summability reduces to $|A; \delta\,|_k$ summability.
We associate two lower matrices $\bar{A} \coloneqq (\bar{a}_{nv})$ $\hat{A} \coloneqq (\hat{a}_{nv})$ for given normal matrix A: (a_{nv}). The matrices $\bar{A}$ and $\hat{A}$ are as follows:

$$\bar{a}_{nv} \coloneqq \sum_{i=v}^{n} a_{ni,} \quad {}_{n,i=0,1,2\ldots}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00},\ \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v},\ n = 1,2, \ldots$$

Here, $\bar{A}$ and $\hat{A}$ are well known matrices of series-to-sequence transformation and series-to-series transformation respectively.

(See Krasniqi, Xh. Z. 2012(1), Krasniqi, Xh. Z. et al. 2012)

4.1.2 Orthogonal Series and Generalized Matrix Summability

Let $\{\varphi_n(x)\}$ be an orthonormal system of functions defined in the interval $[a, b]$.The orthogonal series is given by

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \qquad (4\text{-}6)$$

where $\{c_n\}$ is a sequence of real numbers.
The series (4-6) is called an orthonormal expansion for any $f(x)$, if c_n is represented by

$$c_n = \int_a^b f(x)\varphi_n(x)dx, \quad n = 0,1,2 \ldots$$

and it is denoted by,

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x)$$

Suppose the sequence $\{p_n\}$ and $\{q_n\}$ are denoted by p and q respectively. Then the convolution $(p * q)_n$ of p and q is defined by,

$$(p * q)_n = \sum_{k=0}^{n} p_{n-k}\, q_k = \sum_{k=0}^{n} p_k\, q_{n-k}$$

where $(p * q)_n \neq 0$, for all n.

We shall use the following notations.

$R_n := (p * q)_n$, $R_n^j := \sum_{m=j}^{n} p_{n-m} q_m$, $R_n^{n+1} = 0$; $R_n^0 = R_n$

The generalized Nörlund mean of (4-6) is the sequence $\{t_n^{p,q}\}$, which is as follows:

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_{n-k}\, q_k s_k \qquad (4\text{-}7)$$

The series (4-6) is $|N, p, q|$ summable, if

$$\sum_{n=0}^{\infty} \left| t_n^{p,q} - t_{n-1}^{p,q} \right| < \infty.$$

4. 2 Generalized Matrix Summability of Orthogonal Series

In this section, we shall discuss the matrix summablity as well as generalized matrix summablity of an orthogonal series.

Okuyama Y. has proved the following theorems (Okuyama, Y. 2002):

Theorem 4.1

If the series

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n\, \varphi_n(x)$$

is summable $|N,\ p,\ q|$ almost everywhere.

Theorem 4.2

Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) \omega^{(1)}(n)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|N, p, q|$ summable almost everywhere, where $\omega^{(1)}(n)$ is defined by

$$\omega^{(1)}(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$$

The following two theorems are due to Krasniqi, Xh. Z. et al. 2012.

Theorem 4.3

If the series

$$\sum_{n=1}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{j=0}^{n} \left|\hat{a}_{n,j}\right|^2 \left|c_j\right|^2 \right\}^{k/2}$$

converges for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|A|_k$ summable almost everywhere.

Theorem 4.4

Let $1 \le \mathrm{k} \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converge. If the following series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \omega^{(k)}(A; n)$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \in |A|_k$$

everywhere
where

$$\omega^{(k)}(A;j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left|\hat{a}_{n,j}\right|^2$$

The following two theorems are due to Krasniqi, Xh. Z. et al. 2012(1).

Theorem 4.5

If the series

$$\sum_{n=0}^{\infty} \left\{ n^{2\left(\delta+1-\frac{1}{k}\right)} \sum_{j=0}^{n} \left|\hat{a}_{n,j}\right|^2 \left|c_j\right|^2 \right\}^{\frac{k}{2}}$$

converges for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|A;\delta|_k$ summable almost everywhere.

Theorem 4.6

Let $1 \le \mathrm{k} \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converge. If the following series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \omega^{(k)}(A,\delta;n)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |A;\delta|_k$$

everywhere

where

$$\omega^{(k)}(A,\delta;j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{2\left(\delta+\frac{1}{k}\right)} \left|\hat{a}_{n,j}\right|^2$$

In this chapter, we have extended the two theorems of Krasniqi, Xh. Z. et al. 2012 and Krasniqi, Xh. Z. 2012(1) , which are as follows:

Theorem 4A

If the series

$$\sum_{n=1}^{\infty} \left\{ {\Phi_n}^{2\delta+2-\frac{2}{k}} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 \left|c_j\right|^2 \right\}^{\frac{k}{2}} \tag{4-8}$$

converges for $1 \le k \le 2$,then orthogonal series

$$\sum_{n=0}^{\infty} c_n \, \varphi_n(x)$$

is $\Phi - |A\colon \delta|_k$ summable almost everywhere.

Theorem 4B

Let $1 \le \mathrm{k} \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{\Phi_n}\right\}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{\Phi_n \Omega(n)}$$

converges.

If

$$\sum_{n=1}^{\infty} |c_n|^2 (\Omega(\mathrm{n}))^{\frac{2}{k}-1} \, \omega^{(k)}(A,\delta\,;\,\Phi_n)) \tag{4-9}$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

is $\Phi - |A\,;\delta|_k$ summable almost everywhere, where $\omega^{(k)}(A,\delta\,;\,\Phi_n)$ is

$$\omega^{(k)}(A,\delta\,;\,\Phi_n) := \frac{1}{\left[\Phi_j\right]^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} \left[\Phi_n\right]^{2\left(\delta+\frac{1}{k}\right)} \left|\hat{a}_{n,j}\right|^2$$

Proof of Theorem 4A

Let $s_\upsilon(x)$ be the υ^{th} sequence of partial sums

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

then the matrix transformation $A_n(s)$ of the partial sums $s_v(x)$ is given by

$$\begin{aligned}
A_n(s)(x) &= \sum_{v=0}^{n} a_{nv}(x) s_v(x) \\
&= \sum_{v=0}^{n} a_{nv} s_v(x) \\
&= \sum_{v=0}^{n} a_{nv} \sum_{k=0}^{v} c_k \varphi_k\,(x) \\
&= \sum_{k=0}^{n} c_k \varphi_k\,(x) \sum_{v=k}^{n} a_{nv} \\
&= \sum_{k=0}^{n} c_k \varphi_k\,(x) \bar{a}_{nk}
\end{aligned}$$

Hence,

$$\begin{aligned}
\bar{\Delta} A_n(s)(x) &= \sum_{k=0}^{n} \bar{a}_{nk}\, c_k \varphi_k(x) - \sum_{k=0}^{n-1} \bar{a}_{n-1,k}\, c_k \varphi_k(x) \\
&= \bar{a}_{nn} c_n \varphi_n(x) + \sum_{k=0}^{n-1} (\bar{a}_{n,k} - \bar{a}_{n-1,k}) c_k \varphi_k(x) \\
&= \hat{a}_{nn} c_n \varphi_n(x) + \sum_{k=0}^{n-1} \hat{a}_{n,k}\, c_k \varphi_k(x) \\
&= \sum_{k=0}^{n} \hat{a}_{n,k}\, c_k \varphi_k(x)
\end{aligned}$$

There are two facts to consider the values of $k \in [1,2]$.

(1) We shall take $1 < k < 2$ because by definition of $|A|_k$ summability, we have $k \geq 1$ and Hölder's inequality is applied for $p = \frac{2}{k} > 1$, $q = \frac{2}{2-k} > 1$, so $k < 2$.

(2) For $k = 1\,,2$ we may apply Schwarz's inequality.

Therefore, we may take $1 \leq k \leq 2$.

By the Hölder's inequality and by definition of orthogonality, we have

$$\begin{aligned}
\int_a^b \left|\bar{\Delta} A_n(s)(x)\right|^k dx &= \int_a^b |A_n(s)(x) - A_{n-1}(s)(x)|^k dx \\
&= \int_a^b 1 \cdot |A_n(s)(x) - A_{n-1}(s)(x)|^k dx
\end{aligned}$$

$$= \left\{ \int_a^b (1)^{\frac{2}{2-k}} dx \right\}^{1-\frac{k}{2}} \cdot \left\{ \int_a^b \{ |A_n(s)(x) - A_{n-1}(s)(x)|^k \}^{\frac{2}{k}} dx \right\}^{\frac{k}{2}}$$

$$= \left\{ \int_a^b (1)^{\frac{2}{2-k}} dx \right\}^{1-\frac{k}{2}} \left(\sum_{k=0}^{n} |\hat{a}_{nk}\, c_k \varphi_k(x)|^2 \right)^{\frac{k}{2}}$$

By orthonormality, we have

$$= (b-a)^{1-\frac{k}{2}} \left(\sum_{k=0}^{n} |\hat{a}_{nk}\ |^2 |c_k|^2 \right)^{\frac{k}{2}}$$

Thus,

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \int_a^b \left|\overline{\Delta} A_n(s)(x)\right|^k dx \le (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \left(\sum_{k=0}^{n} |\hat{a}_{nk}\ |^2 |c_k|^2 \right)^{\frac{k}{2}} \qquad \text{(4-10)}$$

$$= (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left\{ \Phi_n^{2\delta+2-\frac{2}{k}} \sum_{k=0}^{n} |\hat{a}_{nk}\ |^2 |c_k|^2 \right\}^{\frac{k}{2}}$$

Hence

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \int_a^b \left|\overline{\Delta} A_n(s)(x)\right|^k dx \le (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left\{ \Phi_n^{2\delta+2-\frac{2}{k}} \sum_{k=0}^{n} |\hat{a}_{nk}\ |^2 |c_k|^2 \right\}^{\frac{k}{2}}$$

Thus, using (4-8)

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \int_a^b \left|\overline{\Delta} A_n(s)(x)\right|^k dx < \infty$$

Now, $\left|\overline{\Delta} A_n(s)(x)\right|^k$ is non-negative and by Beppo Levi's theorem,

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \left|\overline{\Delta} A_n(s)(x)\right|^k < \infty$$

almost everywhere.
Hence the proof.

Proof of Theorem 4B
From (4-7), we have

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \int_a^b \left|\overline{\Delta} A_n(s)(x)\right|^k dx$$

$$\le (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \left[\sum_{j=0}^{n} \left|\hat{a}_{nj}\ \right|^2 |c_j|^2 \right]^{\frac{k}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \frac{1}{(\Phi_n \Omega(n))^{1-\frac{k}{2}}} \left(\Phi_n^{2\delta+1} (\Omega(n))^{\frac{2}{k}-1} \sum_{j=0}^{n} \left|\hat{a}_{nj}\right|^2 \left|c_j\right|^2 \right)^{\frac{k}{2}}$$

Hence by Hölder's inequality

$$\leq (b-a)^{1-\frac{k}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{(\Phi_n \Omega(n))} \right)^{1-\frac{k}{2}} \left(\sum_{n=1}^{\infty} \Phi_n^{2\delta+1} (\Omega(n)^{\frac{2}{k}-1} \sum_{j=0}^{n} \left|\hat{a}_{nj}\right|^2 \left|c_j\right|^2 \right)^{\frac{k}{2}}$$

Since,

$$\sum_{n=1}^{\infty} \frac{1}{(\Phi_n \Omega(n))} < \infty$$

$$\leq M_1 \left(\sum_{j=1}^{\infty} \left|c_j\right|^2 \sum_{n=j}^{\infty} \Phi_n^{2\delta+\frac{2}{k}} \left(\frac{\Omega(\mathrm{n})}{\Phi_\mathrm{n}} \right)^{\frac{2}{k}-1} \left|\hat{a}_{nj}\right|^2 \right)^{\frac{k}{2}}$$

Since $\left\{\frac{\Omega(n)}{\Phi_n}\right\}$ is non-increasing,

$$\leq M_1 \left(\sum_{j=1}^{\infty} \left|c_j\right|^2 \left(\frac{\Omega(\mathrm{j})}{\Phi_j} \right)^{\frac{2}{k}-1} \sum_{n=j}^{\infty} \Phi_n^{2\delta+\frac{2}{k}} \left|\hat{a}_{nj}\right|^2 \right)^{\frac{k}{2}}$$

$$= M_1 \left\{ \sum_{j=1}^{\infty} \left|c_j\right|^2 [\Omega(\mathrm{j})]^{\frac{2}{k}-1} \omega^{(k)}(A, \delta; \Phi_j) \right\}^{\frac{k}{2}}$$

Hence, by condition (4-9)

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \int_a^b \left|\overline{\Delta} A_n(s)(x)\right|^k dx < \infty$$

Hence, by Beppo Levi's theorem,

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1} \left|\overline{\Delta} A_n(s)(x)\right|^k$$

Hence, the proof follows.