Chapter V

Approximation by $(E, 1)$ means of Walsh- Fourier series

5.1 Introduction

We shall consider the well-known Rademacher functions.

We may define the Rademacher functions on $I = [0,1)$ as mentioned below:

$$r_0(x) := \begin{cases} 1 & \text{if} \quad x \in [\,0,\ \frac{1}{2}) \\ -1 & \text{if} \quad x \in [\frac{1}{2}\,,\ 1) \end{cases}$$

$$r_0(x+1) := r_0(x),$$

$$r_n(x) := r_0(2^n x),\ n \geq 1 \text{ and } x \in [0,1)$$

The Walsh orthonormal system $\{w_k(x) : k \geq 0\}$ is defined on the unit interval $I = [0,1)$.

Rademacher system of functions constitutes a part of Walsh's system.

When $k = 0$, we shall take $w_0(x) = 1$
Let k be an integer and $k \geq 1$, then

$$k := \sum_{i=0}^{\infty} k_i 2^i, \qquad k_i = 0 \text{ or } 1$$

is the dyadic representation of k.

The Walsh's system $w_k(x)$ is represented by

$$w_k(x) := \prod_{i=0}^{\infty} [r_i(x)]^{k_i} = [r_0(x)]^{k_0}[r_1(x)]^{k_1} \ldots$$

Let $\mathcal{P}_n$ be the collection of Walsh's polynomials of order less than n. i.e. function of the form

$$P(x) := \sum_{k=0}^{n-1} c_k w_k(x) \tag{5-1}$$

where, $\{c_k\}$ is sequence of real numbers and $n \geq 1$.

The best approximation of $f \in L^P(I), 1 \leq p \leq \infty$ by polynomials in $\mathcal{P}_n$ is defined by

$$E_n(f;\ L^p\ ) := \inf_{P \in \mathcal{P}_n} ||f - P||_p$$

where

$$||f||_p := \left\{\int_0^1 |f(x)|^p dx\right\}^{\frac{1}{p}}, 1 \leq p < \infty$$

$$||f||_\infty := sup\{|f(x)|: x \in I\}$$

The modulus of continuity of a function $f \in L^P$ in $L^P, 1 \leq p < \infty$ is given by

$$\omega_p(f, \delta) := \sup_{|t| < \delta} \|\tau_t f - f\|_p, \qquad \delta > 0$$

where $\tau_t f(x) := f(x + t), x, t \in I$ which is the direct translation by t.

(See Móricz, F. et al. 1992, Móricz, F. et al. 1996)

5.2 Walsh Fourier Series

The Walsh-Fourier series for a given function $f \in L^1$, is defined by

$$\sum_{k=0}^{\infty} a_k w_k(x) \tag{5-2}$$

where

$$a_k = \int_0^1 f(t)\, w_k(t) dt$$

The n^{th} partial sums of series (5-2) for $n \geq 1$ is

$$s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x)$$

We may also represent $s_n(f, x)$ by

$$s_n(f, x) = \int_0^1 f(x + t) D_n(t) dt$$

where

$$D_n(t) := \sum_{k=0}^{n-1} w_k(t), \;\; n \geq 1$$

is the Walsh-Dirichlet kernel of order n.

The $(E, 1)$ mean of series (5-2) is

$$T_n(f,x) = \frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k} s_k(f,x)$$

We shall consider the representation of $T_n(f,x)$ in a slightly different way:

$$T_n(f,x) = \int_0^1 f(x+t)\bar{L}_n(t)dt \qquad (5\text{-}3)$$

where

$$\bar{L}_n(t) = \frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k} D_k(t)\ , n \geq 1 \qquad (5\text{-}4)$$

is called the $(E,1)$ kernel.

Moricz, F. et al. 1992 proved the following theorem:

Theorem 5.1

Let $f \in L^p, 1 \leq p \leq \infty$, let $n = 2^m + k,\ 1 \leq k \leq 2^m, m \geq 1$ and let $\{q_k ; k \geq 0\}$ be a sequence of non-negative numbers such that

$$\frac{n^{\gamma-1}}{Q_n^\gamma}\sum_{k=0}^{n-1} q_k^\gamma = O(1) \qquad (5\text{-}5)$$

for some $1 < \gamma \leq 2$

If $\{q_k\}$ is non-decreasing, then

$$||t_n(f) - f||_p \leq \frac{5}{2Q_n}\sum_{j=o}^{m-1} 2^j q_{n-2^j}\,\omega_p\left(f, 2^{-j}\right) + O\{\omega_p(f, 2^{-m})\} \qquad (5\text{-}6)$$

if $\{q_k\}$ is non-increasing then

$$||t_n(f) - f||_p \leq \frac{5}{2Q_n}\sum_{j=o}^{m-1}\left(Q_{n-2^j+1} - Q_{n-2^{j+1}+1}\right)\omega_p\left(f, 2^{-j}\right) + O\{\omega_p(f, 2^{-m})\} \qquad (5\text{-}7)$$

Moricz, F. et al. 1996 proved the following theorem:

Theorem 5.2

Let $f \in L^p, 1 \leq p \leq \infty$, let $n \coloneqq 2^m + k,\ 1 \leq k \leq 2^m, m \geq 1$ and let $\{q_k ; k \geq 0\}$ be a sequence of non-negative numbers.

If $\{p_k\}$ is non-decreasing and satisfies the conditions

$$\frac{np_n}{P_n} = O(1) \tag{5-8}$$

then

$$||\bar{t}_n(f) - f||_p \leq \frac{3}{P_n} \sum_{j=o}^{m-1} 2^j p_{2^{j+1}-1}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\} \tag{5-9}$$

if $\{p_k\}$ is non-increasing then

$$||\bar{t}_n(f) - f||_p \leq \frac{3}{P_n} \sum_{j=o}^{m-1} 2^j p_{2^j}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\} \tag{5-10}$$

We have generalized the result of Moricz, F. et al. 1992 and Moricz, F. et al. 1996 for $(E, 1)$ summability. Our result is as follows:

Theorem 5A

Let $f \in L^p$, $1 \leq p \leq \infty$, let $n \coloneqq 2^m + k$, $1 \leq k \leq 2^m$, $m \geq 1$,then

$$||T_n(f) - f||_p \leq \frac{3}{2^n} \sum_{j=o}^{m-1} 2^j \binom{n}{2^{j+1}-1} \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\} \tag{5-11}$$

Yano, Sh. 1951(1) proved that Walsh-Fejér kernel

$$K_n(t) \coloneqq \frac{1}{n} \sum_{k=1}^{n} D_k(t) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \omega_k(t), \quad n \geq 1$$

is quasi-positive and $K_{2^m}(t)$ is even positive.

(See Moricz, F. et al. 1992)

We use following lemmas to prove our theorem.

Lemma 5A (Moricz, F. et al. 1992)

Let $m \geq 0$ and $n \geq 1$, then $K_{2^m}(t) \geq 0$ for all $t \in I$

$$\int_0^1 |K_n(t)|\, dt \leq 2 \quad \text{and} \quad \int_0^1 K_{2^m}(t) dt = 1$$

Lemma 5B

Let $n = 2^m + k,\ 1 \le k \le 2^m\ and\ m \ge 1$ then

$$Q_n\bar{L}_n(t) = -\sum_{j=0}^{m-1} r_j(t)w_{2^j-1}(t)\sum_{i=1}^{2^i-1} i\left(\binom{n}{2^{j+1}-i} - \binom{n}{2^{j+1}-i-1}\right)K_i(t)$$

$$-\sum_{j=0}^{m-1} r_j(t)\omega_{2^j-1}(t)\,2^j\binom{n}{2^j}K_{2^j}(t)$$

$$+\sum_{j=0}^{m-1}\left(2^{2^{j+1}-1} - 2^{2^j-1}\right)D_{2^{j+1}}(t)$$

$$+(2^n - 2^{n-k-1})D_{2^m}(t) + r_m(t)\sum_{i=1}^{k}\binom{n}{2^m+i}D_i(t) \qquad (5\text{-}12)$$

Lemma 5C (Moricz, F. et al. 1992)

If $g \in \mathcal{P}_{2^m}, f \in L^p$, where $m \ge 0$ and $1 \le p \le \infty$, then for $1 \le p < \infty$

$$\left\{\int_0^1\left|\int_0^1 r_m(t)g(t)[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le 2^{-1}\omega_p(f,2^{-m})\int_0^1|g(t)|dt \qquad (5\text{-}13)$$

While for $p = \infty$

$$sup\left\{\left|\int_0^1 r_m(t)g(t)[f(x+t)-f(x)]dt\right|dt\ ;\ \ x \in I\right\}$$

$$\le 2^{-1}\omega_\infty(f,2^{-m})\int_0^1|g(t)|dt \qquad (5\text{-}14)$$

5.3 Proof of Lemmas

Proof of Lemma 5B

The proof of this lemma is on the same line of Skvorcov, V. 1981, Moricz, F. et al. 1992 and Moricz, F. et al. 1996

Using (5-4), we have

$$2^n \bar{L}_n(t) = 2^n \frac{1}{2^n} \sum_{i=1}^{n} \binom{n}{i} D_i(t)$$

$$= \sum_{i=1}^{n} \binom{n}{i} D_i(t)$$

$$= \sum_{i=1}^{2^m-1} \binom{n}{i} D_i(t) + \sum_{i=2^m}^{2^m+k} \binom{n}{i} D_i(t)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} \binom{n}{2^j+i} \left(D_{2^j+i}(t)\right) + \sum_{i=0}^{k} \binom{n}{2^j+i} D_{2^m+i}(t)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} \binom{n}{2^j+i} \left(D_{2^j+i}(t) - D_{2^j+i}(t)\right)$$

$$+ \sum_{j=0}^{m-1} D_{2^{j+1}}(t) \sum_{i=0}^{2^j-1} \binom{n}{2^j+i} + \sum_{i=0}^{k} \binom{n}{2^j+i} D_{2^m+i}(t) \qquad \text{(5-15)}$$

As it is well known that

$$D_{2^m+i}(t) = D_{2^m}(t) + r_m(t) D_i(t), \qquad 1 \le i \le 2^m \qquad \text{(5-16)}$$

(See Shipp, et al. 1990)

Now

$$\omega_{2^i-1-l}(t) = \omega_{2^i-1}(t)\omega_l(t), \quad 0 \le l < 2^i$$

Hence,

$$D_{2^{j+1}}(t) - D_{2^j+i}(t) = r_j(t) \sum_{l=i}^{2^j-1} \omega_l(t) = r_j(t) \sum_{l=0}^{2^j-i-1} \omega_{2^j-i-l}(t)$$

$$= r_j(t)\omega_{2^j-1}(t)D_{2^j-i}(t), \qquad 0 \le i < 2^j \tag{5-17}$$

Substitute (5-16) and (5-17) into (5-15)

$$\begin{aligned} Q_n\bar{L}_n(t) = &-\sum_{j=0}^{m-1} r_j(t)\omega_{2^j-1}(t)\sum_{i=0}^{2^j-1}\binom{n}{2^j+i}D_{2^j-i}(t) \\ &+\sum_{j=0}^{m-1}\left(2^{2^{j+1}-1}-2^{2^j-1}\right)D_{2^{j+1}}(t) + (2^n - 2^{n-k-1})D_{2^m}(t) \\ &+ r_m(t)\sum_{i=1}^{k}\binom{n}{2^m+i}D_i(t) \end{aligned} \tag{5-18}$$

Using summation by part, we have

$$\sum_{i=0}^{2^j-1}\binom{n}{2^j+i}D_{2^j-i}(t) = \sum_{i=1}^{2^j-1} iK_i(t)\left(\binom{n}{2^{j+1}-i}-\binom{n}{2^{j+1}-i-1}\right)+2^jK_{2^j}(t)\binom{n}{2^j}$$

Substituting this in to (5-18) gives (5-12).

5.4 Proof of Theorems

Proof of Theorem 5A

Let $1 \le p \le \infty$. Using (5-3), (5-12) and by Minkowski's inequality, we have

$$Q_n\|T_n(f)-f\|_p$$

$$:= \left\{\int_0^1\left|\int_0^1 Q_n\bar{L}_n(t)\,[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le \sum_{j=0}^{m-1}\left\{\int_0^1\left|\int_0^1 r_j(t)g_j(t)\,[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$+\sum_{j=0}^{m-1}\left\{\int_0^1\left|\int_0^1 r_j(t)h_j(t)\,[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$+\sum_{j=0}^{m-1}\left(2^{2^{j+1}-1}-2^{2^j-1}\right)\left\{\int_0^1\left|\int_0^1 D_{2^{j+1}}(t)[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$+(2^n-2^{n-k-1})\left\{\int_0^1\left|\int_0^1 D_{2^m}(t)\,[f(x+t)-f(x)]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$+2^n\left\{\int_0^1\left|\int_0^1 r_m(t)\frac{1}{2^n}\sum_{i=1}^{k}\binom{n}{2^m+i}\,D_i\,(t)[\,f(x+t)-f(x)\,]dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\coloneqq A_{1n}+\ A_{2n}\ +A_{3n}\ +\ \ A_{4n}\ +\ \ A_{5n} \tag{5-19}$$

where,

$$g_j(t)\coloneqq\ w_{2^j-1}(t)\sum_{i=1}^{2^j-1} i\left(\binom{n}{2^{j+1}-i}-\binom{n}{2^{j+1}-i-1}\right)K_i(t)$$

$$h_j(t)\ \coloneqq w_{2^j-1}(t)2^j\binom{n}{2^j}K_{2^j}(t), 0\le j\le m$$

Using Lemma 5A,

$$\int_0^1\left|g_j(t)\right|dt=\int_0^1\left|w_{2^j-1}(t)\sum_{i=1}^{2^j-1} i\left(\binom{n}{2^{j+1}-i}-\binom{n}{2^{j+1}-i-1}\right)K_i(t)\right|dt$$

$$\le 2\sum_{r=2^j+1}^{2^{j+1}-1}(2^{j+1}-r)\ \left|\left(\binom{n}{r}-\binom{n}{r-1}\right)K_i(t)\right|$$

$$\le 2^{j+1}\binom{n}{2^{j+1}-1}$$

Using Lemma 5C,

$$A_{1n}\ \le\sum_{j=0}^{m-1}2^j\binom{n}{2^{j+1}-1}\omega_p\left(\,f\,,2^j\,\right) \tag{5-20}$$

Hence, using Lemma 5A and Lemma 5C, we get

$$A_{2n} \leq 2^{(-1)} \sum_{j=0}^{m-1} 2^j \binom{n}{2^j} \omega_p \left(f , 2^{-j} \right) \tag{5-21}$$

Since

$$D_{2^m}(t) = \begin{cases} 2^m & \text{if } t \in \ [0,\ 2^{-m}) \\ 0 & \text{if } \ t \in \ [2^{-m}, 1) \end{cases}$$

By using generalized Minkowski's inequality,

$$A_{3n} \leq \sum_{j=0}^{m-1} \left(\binom{n}{2^{j+1} - 1} - \binom{n}{2^j - 1} \right) \times \int_0^1 D_{2^{j+1}}(t) \left\{ \int_0^1 |\{ f(x+t) - f(x) \}|^p \, dx \right\}^{\frac{1}{p}} dt$$

$$\leq \sum_{j=0}^{m-1} \left(\binom{n}{2^{j+1} - 1} - \binom{n}{2^j - 1} \right) \omega_p\left(f, 2^{-j}\right)$$

$$\leq \sum_{j=0}^{m-1} 2^j \left(\binom{n}{2^{j+1} - 1} \right) \omega_p\left(f, 2^{-j}\right) \tag{5-22}$$

Hence,

$$A_{4n} \leq \left(\binom{n}{k} - \binom{n}{n-k-1} \right) \omega_p(f, 2^{-m}) \tag{5-23}$$

Similarly, by using Lemma 5B and 5D, we obtain

$$A_{5n} \leq 2^{(-1)} 2^n \omega_p(f, 2^{-m}) \int_0^1 |L_k(t)| dt$$

$$\leq C 2^{n-1} \, \omega_p(f, 2^{-m}) \tag{5-24}$$

Combining (5-19) to (5-24) yields (5-11).