# Chapter VI

# Absolute Indexed Generalized Nörlund Summability of Double Orthogonal Series

## 6.1 Introduction

Let

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} \tag{6-1}$$

be a given double infinite series. Suppose $\{s_{mn}\}$ be a sequence of partial sums of the series (6-1).

Suppose the sequence $\{p_n\}$ and $\{q_n\}$ are denoted by $p$ and $q$ respectively. Then the convolution of $p$ and $q$ denoted by $(p * q)_n$ and is defined as follows:

$$R_{mn} \coloneqq (p * q)_n = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m-i,n-j}q_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}q_{m-i,n-j}$$

The generalized Nörlund transform i.e. $(N, p_n, q_n)$ transform of the sequence $\{s_{mn}\}$ is $t_{mn}^{pq}$ and is defined by

$$t_{mn}^{pq} = \frac{1}{R_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m-i,n-j}q_{ij}s_{ij} \tag{6-2}$$

The following notations were used by Krasniqi, Xh. Z. 2011(2) while estimating the Nörlund summability of double orthogonal series:

$$R_{mn}^{\upsilon\mu} = \sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n} p_{m-i,n-j}q_{ij}\,;$$

$$R_{mn}^{00} = R_{mn};$$

$$R_{m,n-1}^{\upsilon n} = R_{m-1,n-1}^{\upsilon n} = 0\,;\, 0 \le \upsilon \le m;$$

$$R_{m,n-1}^{m\mu} = R_{m-1,n-1}^{m\mu} = 0\,;\, 0 \le \mu \le n;$$

$$\bar{\Delta}_{11}\left(\frac{R_{mn}^{\upsilon\mu}}{R_{mn}}\right) = \frac{R_{m,n}^{\upsilon\mu}}{R_{m,n}} - \frac{R_{m-1,n}^{\upsilon\mu}}{R_{m-1,n}} - \frac{R_{m,n-1}^{\upsilon\mu}}{R_{m,n-1}} + \frac{R_{m-1,n-1}^{\upsilon\mu}}{R_{m-1,n-1}}$$

We define the following

$$\bar{R}_{mn} := \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}q_{ij}$$

Simillarly, the generalized $(\bar{N}, p_n, q_n)$ transform of the sequence $\{s_{mn}\}$ is $\bar{t}_{mn}^{pq}$ and is defined by

$$\bar{t}_{mn}^{pq} = \frac{1}{\bar{R}_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}q_{ij}s_{ij} \tag{6-3}$$

We use the following notations:

$$\bar{R}_{mn}^{\upsilon\mu} = \sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n} p_{ij}q_{ij}\,;$$

$$\bar{R}_{mn}^{00} = \bar{R}_{mn};$$

$$\bar{R}_{m,n-1}^{\upsilon n} = \bar{R}_{m-1,n-1}^{\upsilon n} = 0\,;0 \le \upsilon \le m;$$

$$\bar{R}_{m,n-1}^{m\mu} = \bar{R}_{m-1,n-1}^{m\mu} = 0\,;0 \le \mu \le n;$$

$$\bar{\Delta}_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right) = \frac{\bar{R}_{m,n}^{\upsilon\mu}}{\bar{R}_{m,n}} - \frac{\bar{R}_{m-1,n}^{\upsilon\mu}}{\bar{R}_{m-1,n}} - \frac{\bar{R}_{m,n-1}^{\upsilon\mu}}{\bar{R}_{m,n-1}} + \frac{\bar{R}_{m-1,n-1}^{\upsilon\mu}}{\bar{R}_{m-1,n-1}}$$

The series (6-1) is $\left|N^{(2)},p,q\right|_k$ for $k \ge 1$, if the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(mn)^{k-1}\left|t_{m,n}^{p,q} - t_{m-1,n}^{p,q} - t_{m,n-1}^{p,q} + t_{m-1,n-1}^{p,q}\right|^k < \infty$$

with the condition

$$t_{m,-1}^{p,q} = t_{-1,n}^{p,q} = t_{-1,-1}^{p,q} = 0, \quad m,n = 0,1,\ldots$$

The series (6-1) is $\left|\bar{N}^{(2)},p,q\right|_k$ for $k \ge 1$, if the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(mn)^{k-1}\left|\bar{t}_{m,n}^{p,q} - \bar{t}_{m-1,n}^{p,q} - \bar{t}_{m,n-1}^{p,q} + \bar{t}_{m-1,n-1}^{p,q}\right|^k < \infty$$

with the condition

$$\bar{t}_{m,-1}^{p,q} = \bar{t}_{-1,n}^{p,q} = \bar{t}_{-1,-1}^{p,q} = 0, \quad m,n = 0,1,\ldots$$

## 6.2 Double Orthogonal Series and Double Orthogonal Expansion

Let $\{\varphi_{mn}(x)\}$; $m,n = 0,1,2,\ldots$ be an orthonormal system defined on an interval $(a,b)$. We consider double orthogonal series

$$\sum_{n=0}^{\infty} c_{mn}\varphi_{mn}(x) \qquad \text{(6-4)}$$

where, $c_{mn}$ be sequence of the real numbers. If the coefficient $c_{mn}$ in (6-4) are represented by

$$c_{mn} = \int_a^b f(x)\,\varphi_{mn}(x);\;\; m,n = 0,1,\ldots$$

for certain function $f(x)$, then we say that the (6-4) is an orthogonal expansion of $f(x)$ and we shall express this relation by

$$f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \varphi_{mn}(x) \qquad (6\text{-}5)$$

## 6.3 Absolute Indexed Generalized Nörlund Summability of Double Orthogonal Series

Okuyuma, Y. 2002 have proved the following theorem:

**Theorem 6.1**

If the series

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left( \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 \left[ c_j \right]^2 \right\}^{\frac{1}{2}}$$

converges then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|$ almost everywhere.

Krasniqi, Xh. Z. 2011(2) have proved the following theorem for absolute generalized Nörlund summability with index $k$ of the orthogonal series (6-4).

**Theorem 6.2**

If

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{v=1}^{m} \left[ \bar{\Delta}_{11} \left( \frac{R_{mn}^{v0}}{R_{mn}} \right) \right]^2 |a_{v0}|^2 \right\}^{\frac{k}{2}};$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{\mu=1}^{n} \left[ \bar{\Delta}_{11} \left( \frac{R_{mn}^{0\mu}}{R_{mn}} \right) \right]^2 |a_{0\mu}|^2 \right\}^{\frac{k}{2}};$$

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{v=1}^{m} \sum_{\mu=1}^{n} \left[ \bar{\Delta}_{11} \left( \frac{R_{mn}^{v\mu}}{R_{mn}} \right) \right]^2 |a_{v\mu}|^2 \right\}^{\frac{k}{2}}$$

converges for $1 \le k \le 2$, then the orthogonal series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \varphi_{mn}(x)$$

is $\left|N^{(2)},p,q\right|_k$ summable almost everywhere.

In this chapter, we have extended the theorem of Krasniqi, Xh. Z. 2011(2) which is as follows:

**Theorem 6A**

If

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right)\right\}^2|c_{\upsilon 0}|^2\right]^{\frac{k}{2}}; \qquad (6\text{-}6)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\mu=1}^{n}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right)\right\}^2\left|c_{0\mu}\right|^2\right]^{\frac{k}{2}}; \qquad (6\text{-}7)$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right\}^2\left|c_{\upsilon\mu}\right|^2\right]^{\frac{k}{2}} \qquad (6\text{-}8)$$

converges for $1 \le k \le 2$, then the orthogonal series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}\,\varphi_{mn}(x)$$

is $\left|\bar{N}^{(2)},p,q\right|_k$ summable almost everywhere.

## 6.4 Proof of Theorems

### Proof of Theorem 6A

Let $1 < k < 2$.

The indexed generalized $(\bar{N},p_n,q_n)$ mean $\bar{t}_{mn}^{pq}$ of double orthogonal series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}\,\varphi_{mn}(x)$$

is

$$\bar{t}_{mn}^{pq}=\frac{1}{\bar{R}_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{ij}q_{ij}\,s_{ij}$$

$$=\frac{1}{\bar{R}_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{ij}q_{ij}\sum_{\upsilon=0}^{i}\sum_{\mu=0}^{j}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)$$

$$=\frac{1}{\bar{R}_{mn}}\sum_{\upsilon=0}^{m}\sum_{\mu=0}^{n}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)\sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n}p_{i,j}q_{ij}$$

$$=\frac{1}{\bar{R}_{mn}}\sum_{\upsilon=0}^{m}\sum_{\mu=0}^{n}\bar{R}_{mn}^{\upsilon\mu}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)$$

where,

$$\bar{R}_{mn}^{\upsilon\mu}=\sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n}p_{ij}q_{ij}$$

Since,

$$\bar{R}_{m,n-1}^{\upsilon n}=\bar{R}_{m-1,n-1}^{\upsilon n}=0\,;0\le\upsilon\le m;$$

$$\bar{R}_{m,n-1}^{m\mu}=\bar{R}_{m-1,n-1}^{m\mu}=0\,;0\le\mu\le n;$$

Now,

$$\Delta_{11}\bar{t}_{m,n}^{p,q}(x)=\bar{t}_{m,n}^{p,q}(x)-\bar{t}_{m-1,n}^{p,q}(x)-\bar{t}_{m,n-1}^{p,q}(x)+\bar{t}_{m-1,n-1}^{p,q}(x)$$

$$=\frac{1}{\bar{R}_{mn}}\sum_{\upsilon=0}^{m}\sum_{\mu=0}^{n}\bar{R}_{mn}^{\upsilon\mu}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)-\frac{1}{\bar{R}_{m-1,n}}\sum_{\upsilon=0}^{m-1}\sum_{\mu=0}^{n}\bar{R}_{m-1,n}^{\upsilon\mu}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)$$

$$-\frac{1}{\bar{R}_{m,n-1}}\sum_{\upsilon=0}^{m}\sum_{\mu=0}^{n-1}\bar{R}_{m,n-1}^{\upsilon\mu}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)+\frac{1}{\bar{R}_{m-1,n-1}}\sum_{\upsilon=0}^{m-1}\sum_{\mu=0}^{n-1}\bar{R}_{m-1,n-1}^{\upsilon\mu}c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)$$

$$=\sum_{\upsilon=1}^{m}\left(\frac{\bar{R}_{m,n}^{\upsilon 0}}{\bar{R}_{m,n}}-\frac{\bar{R}_{m-1,n}^{\upsilon 0}}{\bar{R}_{m-1,n}}-\frac{\bar{R}_{m,n-1}^{\upsilon 0}}{\bar{R}_{m,n-1}}+\frac{\bar{R}_{m-1,n-1}^{\upsilon 0}}{\bar{R}_{m-1,n-1}}\right)c_{\upsilon 0}\varphi_{\upsilon 0}(x)$$

$$+\sum_{\mu=1}^{n}\left(\frac{\bar{R}_{m,n}^{0\mu}}{\bar{R}_{m,n}}-\frac{\bar{R}_{m-1,n}^{0\mu}}{\bar{R}_{m-1,n}}-\frac{\bar{R}_{m,n-1}^{0\mu}}{\bar{R}_{m,n-1}}+\frac{\bar{R}_{m-1,n-1}^{0\mu}}{\bar{R}_{m-1,n-1}}\right)c_{0\mu}\varphi_{0\mu}(x)$$

$$+\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left(\frac{\bar{R}_{m,n}^{\upsilon\mu}}{\bar{R}_{m,n}}-\frac{\bar{R}_{m-1,n}^{\upsilon\mu}}{\bar{R}_{m-1,n}}-\frac{\bar{R}_{m,n-1}^{\upsilon\mu}}{\bar{R}_{m,n-1}}+\frac{\bar{R}_{m-1,n-1}^{\upsilon\mu}}{\bar{R}_{m-1,n-1}}\right)c_{\upsilon\mu}\,\varphi_{\upsilon\mu}(x)$$

$$=\sum_{\upsilon=1}^{m}\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right)c_{\upsilon 0}\varphi_{\upsilon 0}(x)+\sum_{\mu=1}^{n}\Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right)c_{0\mu}\varphi_{0\mu}(x)$$

$$+\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left(\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right)c_{\upsilon\mu}\varphi_{\upsilon\mu}(x)$$

$$\int_a^b \left|\Delta_{11}\bar{t}_{m,n}^{p,q}(x)\right|^k dx$$

$$= \int_a^b \left|\sum_{\upsilon=1}^{m} \Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right) c_{\upsilon 0}\varphi_{\upsilon 0}(x) + \sum_{\mu=1}^{n} \Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right) c_{0\mu}\varphi_{0\mu}(x) + \sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left(\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right) c_{\upsilon\mu}\varphi_{\upsilon\mu}(x)\right|^k dx$$

Now, we shall apply $|\alpha+\beta|^s \leq 2^s(|\alpha|^s + |\beta|^s) \quad for \;\; s \geq 1$

$$\leq 2^k \int_a^b \left|\sum_{\upsilon=1}^{m} \Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right) c_{\upsilon 0}\varphi_{\upsilon 0}(x)\right|^k dx$$

$$+2^{2k} \int_a^b \left|\sum_{\mu=1}^{n} \Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right) c_{0\mu}\varphi_{0\mu}(x)\right|^k dx$$

$$+2^{2k} \int_a^b \left|\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n} \Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right) c_{\upsilon\mu}\varphi_{\upsilon\mu}(x)\right|^k dx$$

Applying Hölder's inequality, with $p = \frac{2}{2-k}$ and $q = \frac{2}{k}$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

we have,

$$\leq 2^k (b-a)^{\left(\frac{2-k}{2}\right)} \left\{\int_a^b \left|\sum_{\upsilon=1}^{m} \Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right) c_{\upsilon 0}\varphi_{\upsilon 0}(x)\right|^2 dx\right\}^{\frac{k}{2}}$$

$$+2^{2k} (b-a)^{\left(\frac{2-k}{2}\right)} \left\{\int_a^b \left|\sum_{\mu=1}^{n} \Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right) c_{0\mu}\varphi_{0\mu}(x)\right|^2 dx\right\}^{\frac{k}{2}}$$

$$+2^{2k} (b-a)^{\left(\frac{2-k}{2}\right)} \left\{\int_a^b \left|\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n} \Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right) c_{\upsilon\mu}\varphi_{\upsilon\mu}(x)\right|^2 dx\right\}^{\frac{k}{2}}$$

Now, by applying orthonormality, we have

$$= 2^k(b-a)^{\left(\frac{2-k}{2}\right)}\left\{\sum_{\upsilon=1}^{m}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right)\right]^2|c_{\upsilon 0}|^2\right\}^{\frac{k}{2}}$$

$$+2^{2k}(b-a)^{\left(\frac{2-k}{2}\right)}\left\{\sum_{\mu=1}^{n}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right)\right]^2|c_{0\mu}|^2\right\}^{\frac{k}{2}}$$

$$+2^{2k}(b-a)^{\left(\frac{2-k}{2}\right)}\left\{\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right]^2|c_{\upsilon\mu}|^2\right\}^{\frac{k}{2}}$$

Hence,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\int_a^b\left|\Delta_{11}\bar{t}_{m,n}^{p,q}(x)\right|^k dx$$

$$\leq 2^k(b-a)^{\left(\frac{2-k}{2}\right)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right)\right]^2|c_{\upsilon 0}|^2\right]^{\frac{k}{2}}$$

$$+2^{2k}(b-a)^{\left(\frac{2-k}{2}\right)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\mu=1}^{n}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right)\right]^2|c_{0\mu}|^2\right]^{\frac{k}{2}}$$

$$+2^{2k}(b-a)^{\left(\frac{2-k}{2}\right)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left[\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right]^2|c_{\upsilon\mu}|^2\right]^{\frac{k}{2}}$$

By applying the conditions (6-6), (6-7), and (6-8)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\int_a^b\left|\Delta_{11}\bar{t}_{m,n}^{p,q}\right|^k dx<\infty.$$

Hence, by Beppo Levi's theorem

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\Delta_{11}\bar{t}_{m,n}^{p,q}\right|^k$$

converges almost everywhere.

For $k=1$ and $k=2$ we may apply Schwarz's inequality and applying the same argument as above, our result follows immediately.

Hence the proof is completed for all $1\leq k\leq 2$.