Chapter VII

Generalized Lambert Summability of Orthogonal Series

7.1 Introduction

Let $\{\varphi_n(\theta)\}, n = 0,1, \ldots$ be an orthonormal system defined in an interval (a, b).

Let

$$\sum_{n=0}^{\infty} c_n \varphi_n(\theta) \tag{7-1}$$

be an orthogonal series, where $\{c_n\}$ be a sequence of real numbers.

Let $f(\theta) \in L^2(a, b)$. The orthogonal expansion of $f(\theta)$ is

$$f \sim \sum_{n=0}^{\infty} c_n \varphi_n\,(\theta) \tag{7-2}$$

where, c_n is determined by

$$c_n = \int_a^b f(\theta)\varphi_n(\theta)d\theta;$$

Let

$$S_n(\theta) = \sum_{n=1}^{\infty} c_n\,\varphi_n(\theta)$$

be a sequence of partial sums of series (7-1).

The $(L, 1)$ sum of the series (7-1) is given by

$$\lim_{x\to 1}(1 - x) \sum_{n=1}^{\infty} \frac{n c_n \varphi_n}{1 - x^n} x^n$$

(See Bellman, R. 1943)

The (L, α) sum of the series (7-1) is

$$\lim_{x\to 1}(1 - x) \sum_{n=1}^{\infty} c_n\,\varphi_n(\theta) \left(\frac{n(1 - x)}{1 - x^n}\right)^{\alpha} x^n$$

where, α is any real number and $0 < x < 1$.

(See Zhogin, I. 1969)

7.2 Generalized Lambert summability of Orthogonal series

Richard Bellman (Bellman, R. 1943) has proved the following theorem:

Theorem 7.1

Lambert summability of an orthogonal expansion (7-2) implies the convergence of partial sums $S_{2^n}(\theta)$ of orthogonal expansion (7-2).

We would like to generalize the Theorem 7.1 for generalized Lambert summability of an orthogonal expansion.

Our theorem is as follows:

Theorem 7A

Generalized Lambert summability of an orthogonal expansion (7-2) implies the convergence of partial sum $S_{2^n}(\theta)$ of an orthogonal expansion (7-2).

7.3 Proof of Theorems

Proof of Theorem 7A

Let $x = 1 - \frac{1}{2^n}$

Define

$$U_n(\theta) = \sum_{k=1}^{\infty} c_k\, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - S_2 n(\theta)$$

Hence,

$$U_n(\theta) = \sum_{k=1}^{2^n} c_k\, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - \sum_{k=1}^{2^n} c_k\, \varphi_k(\theta) + \sum_{k=2^n+1}^{\infty} c_k\, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k$$

$$= \sum_{k=1}^{2^n} c_k\, \varphi_k(\theta) \left[\left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - 1\right] + \sum_{k=2^n+1}^{\infty} c_k\, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k$$

$$:= T_n(\theta) + V_n(\theta)$$

where,

$$T_n(\theta) = \sum_{k=1}^{2^n} c_k\, \varphi_k(\theta) \left[\left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - 1\right]$$

$$V_n(\theta) = \sum_{k=2^n+1}^{\infty} c_k\, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k$$

We may arrive at our conclusion if, we prove that

$$\lim_{n\to\infty} U_n(\theta) = 0$$

Now, we consider the series

$$\sum_{n=1}^{\infty}[U_n(\theta)]^2.$$

It is sufficient to prove

$$\sum_{n=1}^{\infty}\int_a^b [U_n(\theta)]^2 d\theta < \infty$$

for convergence almost everywhere in θ.

Now,

$$\sum_{n=1}^{\infty}\int_a^b [U_n(\theta)]^2 d\theta = \sum_{n=1}^{\infty}\int_a^b [\, T_n(\theta) + V_n(\theta)]^2 d\theta$$

$$\leq 2\sum_{n=1}^{\infty}\int_a^b [T_n(\theta)]^2 d\theta + 2\sum_{n=1}^{\infty}\int_a^b [V_n(\theta)]^2 d\theta$$

$$:= 2I_1 + 2I_2 \qquad \text{(7-3)}$$

Now, we shall show the convergence of I_1 .

Here,

$$I_1 = \sum_{n=1}^{\infty}\int_a^b [T_n(\theta)]^2 d\theta$$

$$= \sum_{n=1}^{\infty}\int_a^b \left[\sum_{k=1}^{2^n} c_k\, \varphi_k(\theta) \left\{\left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - 1\right\}\right]^2 d\theta$$

Hence, by orthonormality, we have

$$I_1 \leq \sum_{n=1}^{\infty}\left[\sum_{k=1}^{2^n} c_k^2 \left\{\left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k - 1\right\}^2\right]$$

Now, for $0 \leq x \leq 1$

$$\frac{1-x^k}{1-x} \leq k$$

So,

$$1 - x^k \leq k(1-x),$$

So,

$$1 - x^k \geq 1 - \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k \geq 0,$$

Hence,

$$I_1 = \sum_{n=1}^{\infty} \int_a^b [T_n(\theta)]^2 d\theta \ \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_k^2 \, (1 - x^k)^2$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} k^2 c_k^2 \, (1-x)^2$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} k^2 c_k^2 \left(1 - \left(1 - \frac{1}{2^n}\right)\right)^2$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \sum_{k=1}^{2^n} k^2 c_k^2$$

$$\leq \sum_{k=1}^{2^n} k^2 c_k^2 \sum_{n \geq \log_2 k}^{\infty} 2^{-2n}$$

$$= O(1) \sum_{k=1}^{2^n} c_k^2$$

Since, $f \in L^2(a, b)$, we have

$$\sum_{k=1}^{2^n} c_k^2 < \infty$$

Hence, $I_1 < \infty$.

Now, we shall show convergence of I_2.

$$\sum_{n=1}^{\infty} \int_a^b [\mathrm{V}_n(\theta)]^2 d\theta < \infty$$

$$I_2 = \sum_{n=1}^{\infty} \int_a^b [\mathrm{V}_n(\theta)]^2 d\theta$$

$$= \sum_{n=1}^{\infty} \int_a^b \left\{ \sum_{k=2^n+1}^{\infty} c_k \, \varphi_k(\theta) \left(\frac{k(1-x)}{1-x^k}\right)^{\alpha} x^k \right\}^2 d\theta$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 \frac{(1-x)^{2\alpha}}{(1-x^k)^{2\alpha}} x^{2k}$$

Since $(1-2^{-n})^k$ is decreasing function of k,

$$\sum_{n=1}^{\infty} \frac{1}{[1-(1-2^{-n})^{2^n}]^{\alpha}} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (1-x)^{2\alpha} x^{2k}$$

$$= O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (1-x)^{2\alpha} x^{2k}$$

$$= O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 \left(\frac{1}{2^n}\right)^{2\alpha} \left(1-\frac{1}{2^n}\right)^{2k}$$

$$= O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (2^{-n})^{2\alpha} (1-2^{-n})^{2k}$$

We can majorize

$$k^{2\alpha} \sum_{n=1}^{\infty} 2^{-2n\alpha} (1-2^{-n})^{2k}$$

by integral,

$$k^{2\alpha} \int_0^{\infty} 2^{-2x\alpha} (1-2^{-x})^{-2k} \, dx$$

Hence,

$$I_2 = O(1) \sum_{k=2^n+1}^{\infty} c_k^2 \int_0^{\infty} 2^{-2\alpha x} (1-2^x)^{2k} k^{2\alpha} dx$$

$$= O(1) \sum_{k=2^n+1}^{\infty} c_k^2 \frac{k^{2\alpha}}{(2k+1)(2k+2)\ldots(2k+\alpha)}$$

Since $\frac{k^{2\alpha}}{(2k+1)(2k+2)\ldots(2k+\alpha)}$ is bounded,

$$I_2 = O(1) \sum_{k=2^n+1}^{\infty} c_k^2$$

But $f \in L^2(a,b)$,So,

$$\sum_{k=2^n+1}^{\infty} c_k^2 < \infty$$

Hence,

$$I_2 < \infty$$

Hence, we have proven the convergence of I_1 and I_2 separately.

Hence by (7-3) we have

$$\sum_{n=1}^{\infty} \int_a^b [U_n(\theta)]^2 d\theta < \infty$$

Hence,

$$\sum_{n=1}^{\infty} [U_n(\theta)]^2 < \infty$$

Therefore,

$$\lim_{n \to \infty} U_n(\theta) = 0$$

Hence, the proof.