Chapter I

Introduction

1.1 History Related to Orthogonal Series

(Bhatnagar, S. 1973, Kantawala, P. 1986, Patel, C. 1966, Patel, D. 1990, Patel, R. 1975, Shah, B. 1993, Thangavelu, S. 1996)

A tremendous amount of work has been done in recent years in the field of summability, convergence and approximation problems of general orthogonal series. The theory of orthogonal series was originated during the discussion of the problem of vibrating string more than 200 years ago. The problem of vibrating string was considered by Bernoulli D. (1700-1782) around 1753.The problem was to find the solution of the following partial differential equation with given initial and boundary conditions:

$$u_{tt} = u_{xx}\,, \quad u(x,0) = f(x),\ u_t(x,0) = 0,\ u(0,t) = u(\pi,t) = 0$$

This is the well-known wave equation. The graph of $u(x,t)$ represents the shape of vibrating string at time t (Thangavelu, S. 1996). The equation has been studied by Euler L. (1707-1783) and D'Alembert J. L. R. (1717-1783) before Bernoulli D.. D'Alembert J. L. R. gave the solution of the form

$$u(x,t) = \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t)$$

We can say that $u(x,t)$ is a solution of the given problem if f is odd 2π periodic extension to the real line R of the given initial condition in the interval $[0,\pi]$.

Bernoulli D. suggested the solution of the form

$$u(x,t) = \sum_{k=0}^{\infty} a_k \cos(kx) b_k \sin(kx)$$

Based on this observation Bernoulli D. believed the possibility of expanding an arbitrary periodic function $f(x)$ with $f(0) = f(\pi) = 0$ in terms of $\sin(kx)$ but he didn't have a clue how to calculate Fourier coefficients.

Later on Euler L. and Lagrange J. L. (1736-1813) also worked on the same problem. They have given the possibility of representing an arbitrary function by trigonometric series. In 1807, while working on heat conduction Fourier J. (1768-1830) suggested the way for calculating Fourier coefficient and as a consequence a Fourier series of the function $f(x)$. This is how Fourier series was developed.

There has been a huge amount of development in the convergence, summability and approximation problems of Fourier series. However, less attention has been paid in the theory of orthogonal series.

Many leading mathematician like Alexits G., Andrienko V. A., Banach S., Bellman R., Bhatnagar S., Borwin D., Bor H., Bosanquet L. S., Fejér L., Hobson E. W., Hardy G. H., Hilbert D., Jaxfsbova M. A., Krasniqi Xh. Z., Kachhara D., Kantawala P. S., Kaczmarz S., Lebesgue H., Leindler L., Lorentz G., Menchoff D., Meder J., Misra U.

K., Móricz F., Okuyama Y., Patel C. M., Patel D. P., Patel R. K., Paikray S.K., Riesz F., Riesz M., Sahoo N. C., Shah B. M., Sönmez A., Tandori K., Tiwari S. K., Wiener N. and Weyl H. were working in the field of summability, convergence and approximation problems of general and particular orthogonal series.

In our thesis, we would like to discuss the recent trends in summability, convergence and approximation problems of general and particular orthogonal series. We shall start with number of definitions and concept related to later part of thesis.

1.2 Basic Definitions and Some Fundamentals

(Alexits, G. 1961, Bhatnagar, S. 1973, Kantawala, P. 1986, Patel, D. 1990, Patel, R. 1975, Shah, B. 1993)

The notion of orthogonality will be based on Stieltjes – Lebesgue integral. Let $\mu(x)$ be the positive, bounded and monotone increasing function in the closed interval $[a,b]$, whose derivative $\mu'(x) \geq 0$ vanishes almost in a set of measure zero (in the sense of Lebesgue).

1.2.1 Basic Definition

A function $g(x)$ is called L_μ-integrable, if it is μ measurable and if

$$\int_a^b |g(x)| d\mu(x) < \infty \tag{1-1}$$

If $\mu(x)$ is absolutely continuous and $\rho(x) = \mu'(x)$, then for any L_μ-integrable function $g(x)$, the relation

$$\int_a^b g(x) d\mu(x) = \int_a^b g(x)\rho(x) dx \tag{1-2}$$

is valid.

We shall call $\rho(x)$ the covering function or weight function.

If $\mu(x) = x$, the Stieltjes-Lebesgue integral reduces to an ordinary Lebesgue integral. In particular, if $\rho(x) = 1$, then we say that $g(x)$ is L- integrable.

A function $g(x)$ is called L_μ^2 or $L_{\rho(x)}^2$ integrable function, if it is L_μ or $L_{\rho(x)}$ integrable function respectively and in addition,

$$\int_a^b g^2(x) d\mu(x) < \infty \quad or \quad \int_a^b g^2(x)\rho(x) dx < \infty \tag{1-3}$$

holds respectively. We may talk about L^2 – integrable function, if $\rho(x) = 1$.

1.2.2 Orthogonality of System

A finite or denumerably infinite system $\{\varphi_n(x)\}$ of L_μ^2 integrable functions is said to be an orthogonal system with respect to a distribution $d\mu(x)$ in an interval (a, b), if

$$\int_a^b \varphi_m(x)\varphi_n(x)d\mu(x) = 0(m \neq n) \qquad (1\text{-}4)$$

holds and none of the functions $\{\varphi_n(x)\}$, vanishes almost everywhere.

1.2.3 Orthonormality of system

The system $\{\varphi_n(x)\}$ is said to be an orthonormal, if in addition to the condition (1-4), the condition

$$\int_a^b \varphi_n^2(x)d\mu(x) = 1, n = 0,1,2, \ldots \qquad (1\text{-}5)$$

is also satisfied.

Every orthogonal system $\psi_n(x)\}$ can be converted into an orthonormal system by means of multiplying every one of its member by a suitable chosen constant factor. Since none of factor $\psi_n(x)$ can vanish almost everywhere; the functions

$$\varphi_n(x) = \frac{\psi_n(x)}{\|\psi_n(x)\|} \qquad (1\text{-}6)$$

where,

$$\|\psi_n(x)\| = \left\{\int_a^b \psi_n^2(x)d\mu(x)\right\}^{\frac{1}{2}} \qquad (1\text{-}7)$$

If, $\mu(x) = x$ i.e. $\mu'(x) = \rho(x) = 1$, then $\{\varphi_n(x)\}$ is ONS in the ordinary sense.

1.2.4 Orthogonalization Process

A system of functions $\{g_n(x)\}$ is said to be linearly independent in an interval $[a, b]$, if

$$\sum_{j=0}^{n} a_j g_j(x) = 0$$

for μ – almost every $x \in [a, b]$ necessarily implies the relation

$a_0 = a_1 = a_2 = \ldots = a_n$= 0, for all $n \in N$.

Every orthogonal system $\{\varphi_n(x)\}$ is linearly independent (Alexits, G. 1961). Conversely; any linearly independent system of functions $\{g_n(x)\}$ can be converted into an ONS $\{\varphi_n(x)\}$ such that for each n, $\varphi_n(x)$ is linear combination of the functions $g_0(x), g_1(x), \ldots, g_n(x)$.The procedure of constructing an orthonormal system from a linearly independent system is known as Erhard Schmidt's (Schmidt, E. 1907) general process of orthogonalization.

1.2.5 Orthogonal Series and Orthogonal Expansion

A series of the form

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \qquad (1\text{-}8)$$

constructed from an orthogonal system $\{\varphi_n(x)\}$ and an arbitrary sequence of real number $\{c_n\}$ is called an orthogonal series.

However, if the coefficients $\{c_n\}$ in the series (1-8) are of the form:

$$c_n = \frac{1}{\|\varphi_n(x)\|^2} \int_a^b f(x)\varphi_n(x)d\mu(x)\,, n = 0,1,2 \ldots \qquad (1\text{-}9)$$

for some function $f(x)$, then (1-8) is called the orthogonal expansion of function $f(x)$. We shall express this relation by the formula

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x) \qquad (1\text{-}10)$$

In this case, we shall call the numbers c_0 , c_1 , c_2 ,..., the expansion coefficients of function $f(x)$.

1.2.6 Double Orthogonal Series

Let $\{\varphi_{mn}(x,y)\}; m,n = 0,1,2, \ldots$ be a double sequence of functions in the rectangle

$R = \{(x,y)\ / a \le x \le b, c \le y \le d\}$ such that

$$\iint_R \varphi_{mn}\,(x,y)\varphi_{kl}\,(xy)\,dxdy = \begin{cases} 0\,, m \neq k, n \neq l \\ 1, m = k, n = l. \end{cases}$$

The series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\,\varphi_{mn}(x,y) \qquad (1\text{-}11)$$

where $\{c_{mn}\}$ is an arbitrary sequence of real numbers is called a double orthogonal series.

The series (1-11) is called double orthogonal expansion of the function

$$f\,(x,y) \in L^2(a \le x \le b, c \le y \le d)$$

with respect to an orthonormal system of the function $\{\varphi_{mn}\,(x,y)\}$ if the coefficient c_{mn} is given by

$$c_{mn} = \iint_R \varphi_{mn}\,(x,y) f(x,y) dxdy$$

and is denoted by

$$f(x,y) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\, \varphi_{mn}(x,y)$$

1.3 Different summabilities

(Alexits, G. 1961, Bhatnagar, S. 1973, Datta, H. et al. 2016, Kantawala, P. 1986, Misra, U. K. et al. 2002, Paikray, S. et al. 2012, Patel, D. 1990, Patel, R. 1975, Shah, B. 1993)

Now, we would like to define different summability methods which will be used in later part of our thesis.

In each summability methods, we consider an infinite series of the form

$$\sum_{n=1}^{\infty} u_n \tag{1-12}$$

where, $\{s_n\}$ be the sequence of partial sums of (1-12).

1.3.1 Banach Summability

(Datta, H. et al. 2016, Paikray, S. et al. 2012)

Let ω and l_∞ be the linear spaces of all sequences and bounded sequences respectively on R. A linear functional defined on l and defined on l_∞ is called a limit functional if and only if l satisfies:

(i) For $e = (1,1,1 \ldots)$

$l(e) = 1;$

(ii) For every $x \geq 0$, that is to say,

$x_n \geq 0, \forall\, n \in N, x \in l_\infty\,,\ l(x) \geq 0;$

(iii) For every $x = \{x_n\} \in l_\infty$

$l(x) = l(\tau(x))$

where τ is the shift operator on l_∞ such that $\tau(x_n) = (x_n + 1)$.

Let $x \in l_\infty$ and l be the functional on l_∞, then $l(x)$ is called the "Banach limit" of x. (Banach, S. 1932)

A sequence $x \in l_\infty$ is said to be Banach summable if all the Banach limits of x are the same.

Similarly, a series (1-12) with the sequence of partial sums $\{s_n\}$ is said to be Banach summable if and only if $\{s_n\}$ is Banach summable.

Let the sequence $\{t_k^*(n)\}$ be defined by

$$t_k^*(n) = \sum_{v=0}^{k-1} s_{n+v}, k \in N \tag{1–13}$$

Then $t_k^*(n)$ is said to be the k^{th} element of the Banach transformed sequence.

If

$$\lim_{k\to\infty} t_k^*(n) = s$$

a finite number, uniformly for all $n \in N$, then (1-12) is said to be Banach summable to s.

Thus, if

$$\sup_n |t_k^*(n) - s| \to 0, as\ k \to \infty \tag{1-14}$$

then, (1-12) is Banach summable to s.

1.3.2 Absolute Banach Summability

If

$$\sum_{k=1}^{\infty} |t_k^*(n) - t_{k+1}^*(n)| < \infty,$$

uniformly for all $n \in N$, then the series (1-12) is called absolutely Banach summable (Lorentz, G. 1948) or $|B|$-summable, where $t_k^*(n)$ is defined according to (1-13),

1.3.3 Cesàro Summability

(Cesàro, E. 1890, Chapman, S. 1910, Chapman, S. et al. 1911, Knopp, K. 1907,)

Let $\alpha > -1$

Suppose A_n^α denote

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \quad \text{or} \quad \binom{n+\alpha}{n}.$$

The sequence σ_n^α defined by sequence-to-sequence transformation

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1-15}$$

is called Cesàro mean or (C, α) mean of the series (1-12)

The series (1-12) is said to be summable by the Cesàro method of order α or summable (C, α) to sum s if

$$\lim_{n\to\infty} \sigma_n^\alpha = s$$

where, s is a finite number.

1.3.4 Absolute Cesàro Summability

If

$$\sum_{n=1}^{\infty} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

then, the series (1-12) is said to be absolutely (C, α) summable or $|C, \alpha|$ summable, where, $\{\sigma_n^\alpha\}$ is according to (1-15).

1.3.5 Euler Summability

(Hardy, G. H. 1949, Bhatnagar, S. C. 1973)

The n^{th} Euler mean of order q of the series (1-12) is given by

$$T_n^q = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \tag{1-16}$$

The series (1-12) is said to be (E, q) summable to the sum s or Euler summable to the sum s
If

$$\lim_{n\to\infty} T_n^q = s$$

where, s is a finite number.

In particular, if we take $q = 1$ then (E, q) summablity reduces to $(E, 1)$ summability.

Hence, the series (1-12) is said to be $(E, 1)$ summable to the sum s, if

$$\lim_{n\to\infty} T_n^q = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k = s \tag{1-17}$$

where s is a finite number.

1.3.6 Absolute Euler Summability

If

$$\sum_{n=1}^{\infty} \left|T_n^q - T_{n-1}^q\right| < \infty,$$

then, the series (1-12) is said to be absolutely (E, q) summable or $|E, q|$ summable, where $\{T_n^q\}$ is according to (1-17).

1.3.7 Nörlund Summability

(Hille, E. et al. 1932, Nörlund, N. 1919, Woroni, G. 1901)

Let $\{p_n\}$ be a sequence of non-negative real numbers. A sequence-to-sequence transformation given by

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}s_v \tag{1-18}$$

with $p_0 > 0, p_n \geq 0,$ and $P_n = p_0 + p_1 + p_2 + \cdots p_n$; $n \in N$

defines the Nörlund mean of the series (1-12) generated by the sequence of constants $\{p_n\}$. It is symbolically represented by (N, p_n) mean.

The series (1-12) is said to be Nörlund summable or (N, p_n) summable to the sum s if ,

$$\lim_{n\to\infty} t_n = s$$

where, s is finite number.

The regularity of Nörlund method is presented by

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \tag{1-19}$$

1.3.8 Absolute Nörlund Summability

(Mears, F. et al. 1937)

The series (1-12) is said to be absolutely Nörlund summable or $|N,p_n|$ summable if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty,$$

where $\{t_n\}$ is according to (1-18).

1.3.9 ($\bar{N}$,p_n) Summability or Riesz Summability or (R, p_n) Summability

(Hardy, G. 1949)

Let $\{p_n\}$ be a sequence of non-negative real numbers. A sequence-to-sequence transformation given by

$$\bar{t}_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1-20}$$

with $p_0 > 0, p_n \geq 0,$ and $P_n = p_0 + p_1 + p_2 + \cdots p_n$; $n \in N$

define the $(\bar{N}, p_n)$ mean of the series (1-12) generated by the sequence of constants $\{p_n\}$.

The series (1-12) is said to be $(\bar{N}, p_n)$ summable to the sum s if ,

$$\lim_{n\to\infty} \bar{t}_n = s$$

where, s is finite number.

1.3.10 Absolute $(\bar{N},p_n)$ Summability

The series (1-12) is said to be absolutely $(\bar{N},p_n)$ summable or $|\bar{N},p_n|$ summable if

$$\sum_{n=1}^{\infty} |\bar{t}_n - \bar{t}_{n-1}| < \infty,$$

where, $\{\bar{t}_n\}$ is according to equation (1-20).

1.3.11 Generalized Nörlund Summability or (N, p, q) Summability

(Borwin, D. et al. 1968)

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0, q_n \geq 0$ for all $n \in N$ and $\mathrm{P}_n = p_0 + p_1 + p_2 + \cdots p_n$, $\mathrm{Q}_n = q_0 + q_1 + q_2 + \cdots q_n$; $n \in N$.

A sequence-to-sequence transformation given by

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \qquad (1\text{-}21)$$

where,

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v$$

defines the (N, p, q) mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

The series (1-12) is said to be generalized Nörlund summable or (N, p, q) summable If

$$\lim_{n\to\infty} t_n^{p,q} = s,$$

where, s is a finite number and $t_n^{p,q}$ is according to (1-21).
If we take $p_n = 1$ for all n, then (N, p, q) summability reduces to (R, q_n) or $(\bar{N}, q_n)$ summability.

If we take $q_n = 1$ for all n, then (N, p, q) summability reduces to (N, p_n) summability.

1.3.12 Absolute Generalized Nörlund Summability or Absolute (N, p, q) Summability or $|N, p, q|$ summability

The absolute generalized Nörlund Summability was introduced by Tanaka, M. (1978).

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0, q_n \geq 0$ for all $n \in N$ and $\mathrm{P}_n = p_0 + p_1 + p_2 + \cdots p_n,\ \mathrm{Q}_n = q_0 + q_1 + q_2 + \cdots q_n\,; n \in N$.

A sequence-to-sequence transformation given by

$$t_n^{p,q} = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v} q_v s_v \tag{1-22}$$

where,

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v$$

defines the (N, p, q) mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

If

$$\sum_{n=1}^{\infty} \left| t_n^{p,q} - t_{n-1}^{p,q} \right| < \infty$$

then the series (1-12) is said to be absolutely generalized Nörlund summable or absolutely (N, p, q) summable or $|N, p, q|$ summable, where, $t_n^{p,q}$ is according to (1-22).

1.3.13 $(\bar{N}, p, q)$ Summability

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0, q_n \geq 0$ for all $n \in N$ and $\mathrm{P}_n = p_0 + p_1 + p_2 + \cdots p_n,\ \mathrm{Q}_n = q_0 + q_1 + q_2 + \cdots q_n\,; n \in N$.

A sequence-to-sequence transformation given by

$$\bar{t}_n^{p,q} = \frac{1}{\bar{R}_n}\sum_{v=0}^{n} p_v q_v s_v \tag{1-23}$$

where,

$$\bar{R}_n = \sum_{v=0}^{n} p_v q_v$$

defines the $(\bar{N}, p, q)$ mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

The series (1-12) is said to be $(\bar{N}, p, q)$ summable if

$$\lim_{n\to\infty} \bar{t}_n^{p,q} = s$$

where, s is a finite number and $\bar{t}_n^{p,q}$ is according to (1-23).

If we take $p_n = 1$ for all n, then $(\bar{N}, p, q)$ summability reduces to $(\bar{N}, q_n)$ summability.

If we take $q_n = 1$ for all n, then $(\bar{N}, p, q)$ summability reduces to $(\bar{N}, p_n)$ summability.

1.3.14 Absolute $(\bar{N}, p, q)$ Summability or $|\bar{N}, p, q|$ Summability

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0, q_n \geq 0$ for all $n \in N$ and $P_n = p_0 + p_1 + p_2 + \cdots p_n,\ Q_n = q_0 + q_1 + q_2 + \cdots q_n\ ; n \in N$.

A sequence-to-sequence transformation given by

$$\bar{t}_n^{p,q} = \frac{1}{\bar{R}_n} \sum_{v=0}^{n} p_v q_v s_v \tag{1-24}$$

where,

$$\bar{R}_n = \sum_{v=0}^{n} p_v q_v$$

defines the $(\bar{N}, p, q)$ mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

If

$$\sum_{n=1}^{\infty} \left|\bar{t}_n^{p,q} - \bar{t}_{n-1}^{p,q}\right| < \infty$$

then the series (1-12) is said to be absolutely $(\bar{N}, p, q)$ summable or $|\bar{N}, p, q|$ summable, where, $\bar{t}_n^{p,q}$ is according to (1-24).

1.3.15 Indexed Summability methods

1.3.15.1 $|N, p_n|_k$ Summability

Let $\{p_n\}$ be sequence of non-negative real numbers, with $p_0 > 0, p_n \geq 0$

$$P_n = p_0 + p_1 + \cdots + p_n\ ; n \in N$$

If

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^{k-1} < \infty$$

where, $\{t_n\}$ is according to (1-18), then the series (1-12) said absolute Nörlund summable with index $k \geq 1$ or $|N, p_n|_k$.

If $k = 1$, $|N, p_n|_k$ summability reduces to $|N, p_n|$.

1.3.15.2 $|\bar{N}, p_n|_k$ Summability

Let $\{p_n\}$ be sequence of non-negative real numbers, with $p_0 > 0, p_n \geq 0$

$$P_n = p_0 + p_1 + \cdots + p_n \; ; n \in N$$

If

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{t}_n - \bar{t}_{n-1}|^{k-1} < \infty$$

when $\{\bar{t}_n\}$ is according to (1-20), then the series (1-12) is said to be absolutely $(\bar{N}, p_n)$ summable with index $k \geq 1$ or $|\bar{N}, p_n|_k$.

If $k = 1$, then $|\bar{N}, p_n|_k$ summability reduces to $|\bar{N}, p_n|$.

1.3.15.3 $|N, p_{n,} q_n|_k$ Summability

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0, q_n \geq 0$ for all $n \in N$ and $\mathrm{P}_n = p_0 + p_1 + p_2 + \cdots p_n$, $\mathrm{Q}_n = q_0 + q_1 + q_2 + \cdots q_n \; ; n \in N$.

A sequence-to-sequence transformation given by

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \qquad (1\text{-}25)$$

where,

$$R_n = \sum_{v=0}^{n} p_{n-v} q_v$$

defines the (N, p, q) mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

If

$$\sum_{n=1}^{\infty} n^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k < \infty$$

then the series (1-12) is summable to be $|N, p_{n,} q_n|_k$ for $k \geq 1$,where $t_n^{p,q}$ according to (1-25).

If we take $p_n = 1$ for all n, then $|N, p, q|_k$ summability reduces to $|R, q_n|_k$ or $|\bar{N}, q_n|_k$ summability.

If we take $q_n = 1$ for all n, then $|N, p, q|_k$ method reduces to $|N, p_n|_k$ summability.

1.3.15.4 $|\bar{\mathbf{N}}, \mathbf{p_{n,}} \mathbf{q_n}|_{\mathbf{k}}$ Summability

Let $\{p_n\}$ and $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0,\ p_n \geq 0,\ q_0 > 0,\ q_n \geq 0$ for all $n \in N$ and $\mathrm{P}_n = p_0 + p_1 + p_2 + \cdots p_n$, $\mathrm{Q}_n = q_0 + q_1 + q_2 + \cdots q_n\,; n \in N$.

A sequence-to-sequence transformation given by

$$\bar{t}_n^{p,q} = \frac{1}{\bar{R}_n} \sum_{v=0}^{n} p_v q_v s_v \tag{1-26}$$

where,

$$\bar{R}_n = \sum_{v=0}^{n} p_v q_v$$

defines the $(\bar{N}, p, q)$ mean of the series (1-12) generated by sequence of coefficients $\{p_n\}$ and $\{q_n\}$.

If

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{t}_n^{p,q} - \bar{t}_{n-1}^{p,q}|^k < \infty$$

then the infinite series (1-12) is said to be $|\bar{N}, p_{n,} q_n|_k$ for ≥ 1 , where $\bar{t}_n^{p,q}$ according to (1-26).

If we take $p_n = 1$ for all n, then $|\bar{N}, p, q|_k$ summability reduces to $|\bar{N}, q_n|_k$ summability. If we take $q_n = 1$ for all n, then $|\bar{N}, p, q|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

1.3.15.5 $(\boldsymbol{N}, \boldsymbol{p_n^{\alpha}})$ Summability

We shall restrict ourselves to Nörlund method (N, p_n) for which $p_0 > 0\ ;\ p_n \geq 0$.

Let,

$$\varepsilon_0^{\alpha} = 1,\ \ \varepsilon_n^{\alpha} = \binom{n+\alpha}{n};$$

Given any sequence $\{v_n\}$ we use the following notation:

(i) $\sum_{r=0}^{n} \varepsilon_r^{\alpha-1} v_{n-r} = v_n^{\alpha}$

(ii) $\Delta v_n = \frac{1}{v_n}$

The following identities are immediate:

$$\sum_{r=0}^{n} \varepsilon_r^{\beta-1} v_{n-r}^{\alpha} = v_n^{\alpha+\beta}$$

$$P_n^{\alpha} = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^{\alpha}.$$

Now, we shall consider (N, p_n^{α}) summability for $\alpha > -1$, and, when $p_n \neq 0$ for all values of n, we shall allow for $\alpha = -1$.

When $p_0 = 1, p_n = 0$ for $n > o$; $p_n^{\alpha} = \varepsilon_n^{\alpha-1}$,so that (N, p_n^{α}) method is (c, α) mean.

We say that (1-12) is said to be (N, p_n^{α}) summable if

$$\lim_{n\to\infty} t_n^{\alpha} = s$$

where,

$$t_n^{\alpha} = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} p_{n-v}^{\alpha} s_v \tag{1-27}$$

1.3.15.6 $|N, p_n^{\alpha}|$ Summability

We shall restrict ourselves to Nörlund method (N, p_n) for which $p_0 > 0$; $p_n \geq 0$.

Let,

$$\varepsilon_0^{\alpha} = 1, \;\; \varepsilon_n^{\alpha} = \binom{n+\alpha}{n};$$

Give any sequence $\{v_n\}$ we use the following notation:

(i) $\sum_{r=0}^{n} \varepsilon_r^{\alpha-1} v_{n-r} = v_n^{\alpha}$

(ii) $\Delta v_n = \frac{1}{v_n}$

The following identities are immediate:

$$\sum_{r=0}^{n} \varepsilon_r^{\beta-1} v_{n-r}^{\alpha} = v_n^{\alpha+\beta}$$

$$P_n^{\alpha} = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^{\alpha}.$$

Now, we shall consider (N, p_n^{α}) summability for $\alpha > -1$, and, when $p_n \neq 0$ for all values of n, we shall allow for $\alpha = -1$.

When $p_0 = 1, p_n = 0$ for $n > o$; $p_n^{\alpha} = \varepsilon_n^{\alpha-1}$ so that (N, p_n^{α}) method is (C, α) mean.

We say that (1-12) is said to be absolutely (N, p_n^α) or $|N, p_n^\alpha|$ summable if

$$\sum_{n=1}^{\infty} |t_n^\alpha - t_{n-1}^\alpha| < \infty$$

where, $\{t_n^\alpha\}$ is according to (1-27).

1.3.15.7 $(N, p_n^\alpha, q_n^\alpha)$ Summability

We shall restrict ourselves to

Nörlund method (N, p_n, q_n) for which $p_0 > 0$; $p_n \geq 0$, $q_0 > 0$; $q_n \geq 0$

Let,

$$\varepsilon_0^\alpha = 1, \;\; \varepsilon_n^\alpha = \binom{n+\alpha}{n};$$

Give any sequence $\{v_n\}$ we use the following notation:

(i) $\sum_{r=0}^{n} \varepsilon_r^{\alpha-1} v_{n-r} = v_n^\alpha$

(ii) $\Delta v_n = \frac{1}{v_n}$

The following identities are immediate:

$$\sum_{r=0}^{n} \varepsilon_r^{\beta-1} \, v_{n-r}^\alpha = v_n^{\alpha+\beta}$$

$$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^\alpha .$$

$$Q_n^\alpha = q_n^{\alpha+1} = \sum_{r=0}^{n} q_r^\alpha .$$

Now, we shall consider $(N, p_n^\alpha, q_n^\alpha)$ summability for $\alpha > -1$, and, when $p_n \neq 0$ for all values of n, we shall allow for $\alpha = -1$

We say that (1-12) is $(N, p_n^\alpha, q_n^\alpha)$ summable if $t_n^{p^\alpha, q^\alpha} \to s \;\; as \; n \to \infty$

$$t_n^{p^\alpha, q^\alpha} = \frac{1}{R_n^\alpha} \sum_{r=0}^{n} p_{n-r}^\alpha \, q_r^\alpha s_r \qquad (1\text{-}28)$$

where,

$$R_n^\alpha = \sum_{r=0}^{n} p_{n-r}^\alpha \, q_r^\alpha$$

1.3.15.8 Absolute $(\mathbf{N}, \mathbf{p_n^\alpha}, \mathbf{q_n^\alpha})$ Summability

We shall restrict ourselves to Nörlund method (N, p_n, q_n) for which $p_0 > 0$; $p_n \geq 0$, $q_0 > 0$; $q_n \geq 0$

Let,

$$\varepsilon_0^\alpha = 1, \ \ \varepsilon_n^\alpha = \binom{n+\alpha}{n};$$

Given any sequence $\{v_n\}$ we use the following notation:

(i) $\sum_{r=0}^{n} \varepsilon_r^{\alpha-1} v_{n-r} = v_n^\alpha$

(ii) $\Delta v_n = \frac{1}{v_n}$

The following identities are immediate:

$$\sum_{r=0}^{n} \varepsilon_r^{\beta-1} v_{n-r}^\alpha = v_n^{\alpha+\beta}$$

$$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=0}^{n} p_r^\alpha.$$

$$Q_n^\alpha = q_n^{\alpha+1} = \sum_{r=0}^{n} q_r^\alpha.$$

Now, we shall consider $(N, p_n^\alpha, q_n^\alpha)$ summability for $\alpha > -1$, and, when $p_n \neq 0$, $q_n \neq 0$ for all values of n, we shall allow for $\alpha = -1$

We say that (1-12) is summable $|N, p_n^\alpha, q_n^\alpha|$ if

$$\sum_{n=1}^{\infty} \left| t_n^{p^\alpha, q^\alpha} - t_{n-1}^{p^\alpha, q^\alpha} \right| < \infty$$

where

$$t_n^{p^\alpha, q^\alpha} = \frac{1}{R_n^\alpha} \sum_{r=0}^{n} p_{n-r}^\alpha \, q_r^\alpha s_r$$

$$R_n^\alpha = \sum_{r=0}^{n} p_{n-r}^\alpha \, q_r^\alpha$$

1.3.15.9 $|\mathbf{A}|_{\mathbf{k}}\,;\mathbf{k}\geq 1$ Summability

Let $A=(A_{nv})$ be a normal matrix. i.e. lower triangular matrix of non zero diagonal entries. Than A defines the sequence-to-sequence transformation, mapping to a sequence $s=\{s_n\}$ to $As=\{A_n(s)\}$ where

$$A_n(s)=\sum_{v=0}^{n} A_{nv}s_v\,;\ n=0,1,2,\ldots \tag{1-29}$$

The series (1-12) is said to be summable $|A|_k\,;\ k\geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}\,|\bar{\Delta}A_n(s)|^k<\infty$$

$$\bar{\Delta}\mathrm{A}_\mathrm{n}(\mathrm{s})=\mathrm{A}_\mathrm{n}(\mathrm{s})-\mathrm{A}_{\mathrm{n}-1}(\mathrm{s})$$

1.3.15.10 $|\mathbf{A}\,;\ \boldsymbol{\delta}|_{\mathbf{k}}, \boldsymbol{k}\geq \mathbf{1}, \boldsymbol{\delta}\geq \mathbf{0}$ Summability

We say that series (1-12) is $|A\,,\delta|_k$, summable, where $k\geq 1$,$\delta\geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}\ \ |\bar{\Delta}A_n(s)|^k<\infty$$

1.3.15.11 $\boldsymbol{\Phi}-|\boldsymbol{A}\,;\ \boldsymbol{\delta}|_{\boldsymbol{k}}, \boldsymbol{k}\geq \mathbf{1}$ Summability

Let $\{\Phi_n\}$ be sequence of positive real numbers. We say that series (1-12) is $\Phi-|A\,,\delta|_k$, summable, where $k\geq 1$,$\delta\geq 0$, if

$$\sum_{n=1}^{\infty} \Phi_n^{\delta k+k-1}\ \ |\bar{\Delta}A_n(s)|^k<\infty$$

1.3.15.12 The product summability: $|(\boldsymbol{N},\boldsymbol{p_n},\boldsymbol{q_n})(\boldsymbol{N},\boldsymbol{q_n},\boldsymbol{p_n})|_{\boldsymbol{k}}, \boldsymbol{k}\geq \mathbf{1},$

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of real numbers and let

$$P_n=p_0+p_1+\cdots+p_n=\sum_{v=0}^{n} p_v$$

$$Q_n=q_0+q_1+\cdots\ +q_n=\sum_{v=0}^{n} q_v$$

Let p and q represents two sequences $\{p_n\}$ and $\{q_n\}$ respectively.
The convolution between p and q is denoted by $(p*q)_n$ and is defined by

$$R_n\coloneqq(p*q)_n=\sum_{v=0}^{n} p_{n-v}q_v=\sum_{v=0}^{n} p_v q_{n-v}$$

Define

$$R_n^j := \sum_{v=j}^{n} p_{n-v} q_v$$

The generalized Nörlund mean of series (1-12) is defined as follows and is denoted by $t_n^{p,q}(x)$.

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v\,(x) \tag{1-30}$$

where, $R_n \neq 0$ for all n.

The series (1-12) is said to be absolutely summable (N, p, q) i.e.$|N, p, q|$ summable, if the series

$$\sum_{n=1}^{\infty} \left| t_n^{p,q} - t_{n-1}^{p,q} \right| < \infty$$

If we take $p_n = 1$ for all n then, the sequence-to-sequence transformation $t_n^{p,q}$ reduces to $(\bar{N}, q_n)$ transformation

$$\bar{t}_n := \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v$$

If we take $q_n = 1$ for all n then, the sequence- to- sequence transformation $t_n^{p,q}$ reduces to $(\bar{N}, p_n)$ transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

(See Krasniqi, Xh. Z. 2013(2))

Das, G. 1969 defined the following transformation

$$U_n := \frac{1}{P_n} \sum_{v=0}^{n} \frac{p_{n-v}}{Q_v} \sum_{j=0}^{v} q_{v-j} s_j \tag{1-31}$$

The infinite series (1-12) is said to be summable$|(N, p)(N, q)|$, if the series

$$\sum_{n=1}^{\infty} |U_n - U_{n-1}| < \infty$$

Later on, Sulaiman, W. 2008 considered the following transformation:

$$V_n := \frac{1}{Q_n} \sum_{v=0}^{n} \frac{q_v}{P_v} \sum_{j=0}^{v} p_j\, s_j \tag{1-32}$$

The infinite series (1-12) is said to be summable$|(\bar{N}, q_n)(\bar{N}, p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k < \infty$$

Krasniqi, Xh. Z. 2013(2) have defined the transformation which is as follows:

$$D_n \coloneqq \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{j=0}^{v}p_j q_{v-j}s_j \ , \tag{1-33}$$

The infinite series (1-12) is said to be $|(N,p_n,q_n)(N,q_n,p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|D_n - D_{n-1}|^k < \infty$$

We have defined the transformation as follows:

$$E_n \coloneqq \frac{1}{\bar{R}_n}\sum_{v=0}^{n}\frac{p_v q_v}{\bar{R}_v}\sum_{j=0}^{v}p_j q_j s_j, \tag{1-34}$$

1.3.15.13 The product summability $|(\bar{N},p_n,q_n)(\bar{N},q_n,p_n)|_k\,, k \geq 1$

The infinite series (1-12) is said to be summable $|(\bar{N},p_n,q_n)(\bar{N},q_n,p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|E_n - E_{n-1}|^k < \infty$$

1.3.16 Lambert Summability

The series (1-12) is said to be Lambert summable to s

if

$$\lim_{x\to 1_-}(1-x)\sum_{n=1}^{\infty}\frac{nu_n x^n}{1-x^n} = s$$

1.3.17 Generalized Lambert Summability

The series (1-12) is said to be generalized Lambert summable to s

if

$$\lim_{x\to 1_-} u_n\left(\frac{n(1-x)}{1-x^n}\right)^{\alpha} x^n = s$$

1.4 Absolute Summability of Double Orthogonal Series

Consider

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} \tag{1-35}$$

be a given double infinite series. Suppose $\{s_{mn}\}$ be a sequence of partial sums of the series (1-35). Suppose the sequence $\{p_n\}$ and $\{q_n\}$ are denoted by p and q respectively. Then the convolution of p and q denoted by $(p*q)_n$ and defined as follows:

$$R_{mn} \coloneqq (p * q)_n = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m-i,n-j}q_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}q_{m-i,n-j}$$

The following notations were used by Krasniqi, Xh. Z. 2011(2) while estimating the Norlund summability of double orthogonal series:

$$R_{mn}^{\upsilon\mu} = \sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n} p_{m-i,n-j}q_{ij}\,;$$

$$R_{mn}^{00} = R_{mn};$$

$$R_{m,n-1}^{\upsilon n} = R_{m-1,n-1}^{\upsilon n} = 0\,; 0 \le \upsilon \le m;$$

$$R_{m,n-1}^{m\mu} = R_{m-1,n-1}^{m\mu} = 0\,; 0 \le \mu \le n;$$

$$\bar{\Delta}_{11}\left(\frac{R_{mn}^{\upsilon\mu}}{R_{mn}}\right) = \frac{R_{m,n}^{\upsilon\mu}}{R_{m,n}} - \frac{R_{m-1,n}^{\upsilon\mu}}{R_{m-1,n}} - \frac{R_{m,n-1}^{\upsilon\mu}}{R_{m,n-1}} + \frac{R_{m-1,n-1}^{\upsilon\mu}}{R_{m-1,n-1}}$$

The generalized (N, p_n, q_n) transform of the sequence $\{s_{mn}\}$ is t_{mn}^{pq} and is defined by

$$t_{mn}^{pq} = \frac{1}{R_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m-i,n-j}q_{ij}s_{ij} \tag{1-36}$$

We define the following

$$\bar{R}_{mn} \coloneqq \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}q_{ij}$$

We use the following notations:

$$\bar{R}_{mn}^{\upsilon\mu} = \sum_{i=\upsilon}^{m}\sum_{j=\mu}^{n} p_{ij}q_{ij}\,;$$

$$\bar{R}_{mn}^{00} = \bar{R}_{mn};$$

$$\bar{R}_{m,n-1}^{\upsilon n} = \bar{R}_{m-1,n-1}^{\upsilon n} = 0\,; 0 \le \upsilon \le m;$$

$$\bar{R}_{m,n-1}^{m\mu} = \bar{R}_{m-1,n-1}^{m\mu} = 0\,; 0 \le \mu \le n;$$

$$\bar{\Delta}_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right) = \frac{\bar{R}_{m,n}^{\upsilon\mu}}{\bar{R}_{m,n}} - \frac{\bar{R}_{m-1,n}^{\upsilon\mu}}{\bar{R}_{m-1,n}} - \frac{\bar{R}_{m,n-1}^{\upsilon\mu}}{\bar{R}_{m,n-1}} + \frac{\bar{R}_{m-1,n-1}^{\upsilon\mu}}{\bar{R}_{m-1,n-1}}$$

The generalized $(\bar{N}, p_n, q_n)$ transform of the sequence $\{s_{mn}\}$ is $\bar{t}_{mn}^{pq}$ and is defined by

$$\bar{t}_{mn}^{pq} = \frac{1}{\bar{R}_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}q_{ij}s_{ij} \tag{1-37}$$

1.4.1 $\left|N^{(2)}, p, q\right|_k$ for $k \ge 1$ Summability

The series (1-35) is $\left|N^{(2)}, p, q\right|_k$ for $k \geq 1$, if the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|t_{m,n}^{p,q} - t_{m-1,n}^{p,q} - t_{m,n-1}^{p,q} + t_{m-1,n-1}^{p,q}\right|^k < \infty$$

with the condition

$$t_{m,-1}^{p,q} = t_{-1,n}^{p,q} = t_{-1,-1}^{p,q} = 0, \quad m, n = 0,1, \ldots$$

1.4.2 $\left|\bar{N}^{(2)}, p, q\right|_k$ for $k \geq 1$Summability

The series (1-35) is $\left|\bar{N}^{(2)}, p, q\right|_k$ for $k \geq 1$, if the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\bar{t}_{m,n}^{p,q} - \bar{t}_{m-1,n}^{p,q} - \bar{t}_{m,n-1}^{p,q} + \bar{t}_{m-1,n-1}^{p,q}\right|^k < \infty$$

with the condition

$$\bar{t}_{m,-1}^{p,q} = \bar{t}_{-1,n}^{p,q} = \bar{t}_{-1,-1}^{p,q} = 0, \quad m, n = 0,1, \ldots$$

1.5 History related to Convergence of an Orthogonal Series

The thesis focuses on convergence and summability of general orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1-38}$$

where, $\{c_n\}$ is any arbitrary sequence of real numbers. It can be seen that

$$\sum_{n=0}^{\infty} |c_n| < \infty \tag{1-39}$$

implies the absolute convergence of series (1-38) almost everywhere in the interval of orthogonality. On the other hand, it has been shown through example of a series of Rademacher functions that the condition.

$$\sum_{n=0}^{\infty} c_n^2 < \infty \tag{1-40}$$

is necessary condition for the convergence of series (1-38) almost everywhere in the interval of orthogonality. It is reasonable to say that the useful condition for the convergence of series (1-38) lies between (1-39) and (1-40).

The question of convergence of orthogonal series was originally started by Jerosch, F. et al. 1909, Weyl, H. 1909 who pointed out that the condition:

$$c_n = O\left(n^{-\frac{3}{4}-\epsilon}\right), \epsilon > 0$$

is sufficient for the convergence of series (1-38).

Further, Weyl, H. 1909, has improved the condition by showing that the condition

$$\sum_{n=1}^{\infty} c_n^2\sqrt{n} < \infty$$

is sufficient for convergence of series (1-28). Later on Hobson, E. 1913, modified the Weyl's condition which is of the form

$$\sum_{n=1}^{\infty} c_n^2 n^{\in} < \infty; \in > 0$$

and Plancherel, N. 1910 has also modified the condition

$$\sum_{n=2}^{\infty} c_n^2 log^3 n < \infty.$$

In this direction, many attempts have been made to improve the condition of convergence of series (1-38). Finally, important contribution was put forward by Rademacher, H. 1922 and by Menchoff, D. 1923, simultaneously and independently of one another for convergence of an orthogonal series (1-38). They have shown that the series (1-38) is convergent almost everywhere in the interval of orthogonality, if

$$\sum_{n=1}^{\infty} c_n^2 log^2 n < \infty$$

is satisfied.

Later on Gapoškin, V. 1964, Salem, R. 1940, Talalyan, A. 1956, and Walfisz, A. 1940 generalized the above theorem.

The theorem of Rademacher, H. and Manchoff, D. is the best of its kind which is obvious from the below theorem of convergence theory given by Manchoff, D..

If $w(n)$ is an arbitrary positive monotone increasing sequence of numbers with $w(n) = o(\log|n|)$, then there exist an everywhere divergent orthogonal series,

$$\sum_{n=0}^{\infty} c_n \Psi_n(x)$$

whose coefficients satisfy the condition

$$\sum_{n=1}^{\infty} c_n^2 w_n^2 < \infty$$

Tandori, K. 1975 proved that if $\{c_n\}$ is positive monotone decreasing sequence of number for which,

$$\sum_{n=1}^{\infty} c_n^2 log^2 n = \infty$$

holds true then there exists in (a,b) an orthonormal system, $\{\psi_n(x)\}$ dependent on c_n such that the orthogonal series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

is convergent almost everywhere.

1.6 Summability of Orthogonal Series

It was first shown by Kaczmarz, S. 1925 that under the condition

$$\sum_{n=0}^{\infty} c_n^2 < \infty$$

the necessary and sufficient condition for general orthogonal series (1-38) to be $(C, 1)$ summable almost everywhere is that there exist a sequence of partial sums

$\{S_{v_n}(x)\}$: 1≤ $q \leq \frac{v_{n+1}}{v_n} \leq r$ convergent everywhere in the interval of orthogonality. The same result was extended by Zygmund, A. 1927, Zygmund, A. 1959, for $(C, \alpha), \alpha > 0$ summability.

The classical result of H. Weyl, H. 1909 for $(C, 1)$ summability reads as follows;

The condition

$$\sum_{n=2}^{\infty} c_n^2 \log n < \infty$$

is sufficient for $(C, 1)$ summability of (1-38).

Again Borgen, S. 1928, Kaczmarz, S. 1927, Menchoff, D. 1925, and Menchoff, D. 1926, have refined the same condition and established an analogous of Rademacher-Menchoff theorem for (C, α), summability;

which shows that

$$\sum_{n=3}^{\infty} c_n^2 (log(logn))^2 < \infty$$

implies $(C, \alpha > 0)$ summability of series (1-38).

1.7 Absolute Summability of Orthogonal Series: Banach and Generalized Nörlund Summabilty

The absolute summability of an orthogonal series has been studied by many mathematicians like Bhatnagar, S. 1973, Grepacevskaja, L. 1964, Kantawala, P. 1986, Krasniqi, Xh. Z. 2010, Krasniqi, Xh. Z. 2011(1), Krasniqi, Xh. Z. 2011(2), Krasniqi, Xh. Z. 2011(3), Krasniqi, Xh. Z. 2012(1), Krasniqi, Xh. Z. 2012(2), Leindler, L. 1961, Leindler, L.1981, Leindler, L. 1983, Leindler, L.1995, Okuyama, Y. et. al. 1981, Patel, D. 1990, Patel, R. 1975, Spevekov, et al. 1977, Shah, B. 1993, Tandori, K. 1971.

Paikray, S. et al. 2012 have proved the following theorem:

Theorem 1.1

Let

$$\Psi_\alpha(+0) = 0\,, \qquad 0 < \alpha < 1$$

and

$$\int_0^\pi \frac{d\Psi_\alpha(u)}{u^\alpha \log(n+U)} < \infty$$

then, the series

$$\sum_{n=1}^{\infty} \frac{B_n(t)}{log(n+1)}$$

is $|B|$ summable at $t = x$,

if

$$\sum_{k \leq \frac{1}{u}} log(n+U) k^{\alpha-1} = O(U^\alpha \, log(n+2)) \quad ; U = \left[\frac{1}{u}\right]$$

Tsuchikura, T. et al. 1953, have proved the following theorem on Cesàro summability of order α for orthogonal series.

Theorem 1.2

Let $\{\varphi_n(x)\}$ be orthonormal system defined in the interval (a, b) and let $\alpha > 0$. If the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \left[\sum_{k=1}^{n} k^2 (n-k+1)^{2(\alpha-1)} a_k{}^2 \right]^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

is summable $|C, \alpha|$ for almost every x.

In Chapter-II, we have extended the Theorem 1.2 of Tsuchikura, T. 1953 for the Banach summability. Our theorem is as follows:

Theorem 1A

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in (a, b).

If

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \left\{ \sum_{v=1}^{k} c_{n+v}^2 \right\}^{\frac{1}{2}} < \infty$$

for all n, then orthogonal series (1-28) is absolutely Banach summable i.e. $|B|$ summable for every x.

Tiwari, S. et al. 2011, obtained the following result on strong Nörlund summability of orthogonal series.

Theorem 1.3

If the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series (1-8) is summable $|N, p_n, q_n|$ almost everywhere.

Refer to equations (3-3) and (3-5) for R_n and R_n^j respectively.

In chapter-II, we have generalized the Theorem 1.3 for $|N, p_n^{\alpha}, q_n^{\alpha}|, \alpha > -1$ summability of an orthogonal series.

Our result is as follows:

Theorem 1B

If the series

$$\sum_{n=1}^{\infty}\left\{\sum_{\nu=1}^{n}\left(\frac{R_n^{\alpha\nu}}{R_n^{\alpha}}-\frac{R_{n-1}^{\alpha\nu}}{R_{n-1}^{\alpha}}\right)^2 \ |c_\nu|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal expansion

$$\sum_{\nu=0}^{\infty} c_\nu\, \varphi_\nu(x)$$

is summable $|N, p_n^\alpha, q_n^\alpha|$, $\alpha > -1$ almost everywhere.

Krasniqi, Xh. Z. 2010 has discussed some general theorem on the absolute indexed generalized $|N\ , p\ , q|_k\ , k \geq 1$ of

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

The theorem is as follows:

Theorem 1.4

If, for $1 \leq k \leq 2$, the series

$$\sum_{n=0}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2 |c_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|_k\ , k \geq 1$ almost everywhere.

In chapter-III, we have extended the Theorem 1.4 to $|\bar{N}, p, q|_k, k \geq 1$ summability of series (1-8), which as follows:

Theorem 1C

Let $1 \leq k \leq 2$ and if the series

$$\sum_{n=0}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{\bar{R}_n^j}{\bar{R}_n}-\frac{\bar{R}_{n-1}^j}{\bar{R}_{n-1}}\right)^2 |c_j|^2\right\}^{\frac{k}{2}} < \infty$$

then, the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|\bar{N}, p, q|_k, k \geq 1$ summable almost everywhere.

Chapter III also contains five important corollaries:

Corollary 3A says that for $k = 1$, our Theorem 1C reduces to $|\bar{N}, p, q|$ summability of (1-8)

Corollary 3B says that for $q_n = 1$, our Theorem 1C reduces to $|\bar{N}, p_n|_k$ summability of (1-8)

Corollary 3C says that for $p_n = 1$, our Theorem 1C reduces to $|\bar{N}, q_n|_k$ summability of (1-8)

Corollary 3D says that for $k = 1$, Corollary 3B reduces to $\left|\bar{N}, p_n\right|$ summability of (1-8)

Corollary 3E say that for $k = 1$, Corollary 3C reduces to $\left|\bar{N}, q_n\right|$ summability of (1-8)

1.8 Matrix Summability of an Orthogonal Series

Based on definition of Flett, T. et al. 1957, Krasniqi, Xh. Z. et. al. 2012 has proved the following theorems:

Theorem 1.5

If the series

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=0}^{n}\left|\hat{a}_{n,j}\right|^2\left|c_j\right|^2\right\}^{\frac{k}{2}}$$

converge for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

is $|A|_k$ summable almost everywhere.

Theorem 1.6

Let $1 \le \mathrm{k} \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges. If the following series

$$\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\omega^{(k)}(A;n)$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} c_n\varphi_n(x) \in |A|_k$$

almost everywhere.

In chapter IV, we have extended the two theorems of Krasniqi, Xh. Z. et. al. 2012, which are as follows:

Theorem 1D
If the series

$$\sum_{n=1}^{\infty}\left\{\Phi_n{}^{2\delta+2-\frac{2}{k}}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2\,|c_j|^2\right\}^{k/2}$$

converges for $1 \leq k \leq 2$,then orthogonal series

$$\sum_{n=0}^{\infty} c_n\,\varphi_n(x)$$

is $\Phi - |A\colon \delta|_k$ summable almost everywhere.

Theorem 1E
$1 \leq \mathrm{k} \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{\Phi_n}\right\}$ is non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{\Phi_n\Omega(n)}$$

converges.
If

$$\sum_{n=1}^{\infty}|c_n|^2(\Omega(\mathrm{n}))^{\frac{2}{k}-1}\,\omega^{(k)}(A,\delta\,;\,\Phi_n)$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} c_n\varphi_n(x)$$

is $\Phi - |A\,;\delta|_k$ summable almost everywhere, where $\omega^{(k)}(A,\delta\,;\,\Phi_n)$ is

$$\omega^{(k)}(A,\delta\,;\,\Phi_n) := \frac{1}{[\Phi_j]^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}\;[\Phi_n]^{2\left(\delta+\frac{1}{k}\right)}\left|\hat{a}_{n,j}\right|^2$$

1.9 Approximation by Nörlund Means of an Orthogonal Series

Móricz, F. et. al. 1992 have studied the rate of approximation by Nörlund means for Walsh-Fourier series. He has proved the following theorem:

Theorem 1.7

Let $f \in L^p, 1 \le p \le \infty$, let $n = 2^m + k$, $1 \le k \le 2^m, m \ge 1$ and let $\{q_k; k \ge 0\}$ be a sequence of non-negative numbers such that

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1)$$

for some $1 < \gamma \le 2$

If $\{q_k\}$ is non-decreasing, then

$$||t_n(f) - f||_p \le \frac{5}{2Q_n} \sum_{j=o}^{m-1} 2^j q_{n-2^j}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

If $\{q_k\}$ is non-increasing then

$$||t_n(f) - f||_p \le \frac{5}{2Q_n} \sum_{j=o}^{m-1} \left(Q_{n-2^j+1} - Q_{n-2^{j+1}+1}\right) \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

Moricz, F. et al. 1996 proved the following theorem:

Theorem 1.8

Let $f \in L^p, 1 \le p \le \infty$, let $n \coloneqq 2^m + k$, $1 \le k \le 2^m, m \ge 1$ and let $\{q_k; k \ge 0\}$ be a sequence of non-negative numbers.

If $\{p_k\}$ is non-decreasing and satisfies the conditions

$$\frac{np_n}{P_n} = O(1)$$

then

$$||\bar{t}_n(f) - f||_p \le \frac{3}{P_n} \sum_{j=o}^{m-1} 2^j p_{2^{j+1}-1}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

If $\{p_k\}$ is non-increasing then

$$||\bar{t}_n(f) - f||_p \le \frac{3}{P_n} \sum_{j=o}^{m-1} 2^j p_{2^j}\, \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

We have generalized the result of Moricz, F. et al. 1992 and Moricz, F. et al. 1996 for $(E, 1)$ summability. Our result is as follows:

Theorem 1F

Let $f \in L^{p}$, $1 \leq p \leq \infty$, let $n := 2^{m} + k$, $1 \leq k \leq 2^{m}$, $m \geq 1$,then

$$||T_n(f) - f||_p \leq \frac{3}{2^n} \sum_{j=o}^{m-1} 2^j \binom{n}{2^{j+1} - 1} \omega_p(f, 2^{-j}) + O\{\omega_p(f, 2^{-m})\}$$

1.10 Absolute Generalized Nörlund Summability of Double Orthogonal Series

Krasniqi, Xh. Z 2011(2) has proved the theorem on absolute generalized Nörlund summability of double orthogonal series. Some of the important contributions for absolute summability of an orthogonal series are due to Fedulov, V. 1955, Mitchell, J. 1949, Patel, C.M. 1967 and Sapre, A. 1971.

Okuyama, Y. 2002 has developed the necessary and sufficient condition in which the double orthogonal series is $|N, p, q|$ summable almost everywhere.

Theorem 1.9

If the series

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 [c_j]^2 \right\}^{\frac{1}{2}}$$

converges then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|$ almost everywhere.

Krasniqi, Xh. Z. 2011(2) have proved the following theorem for absolute Nörlund summability with index for double orthogonal expansion.

Theorem 1.10

If

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{v=1}^{m} \left[\bar{\Delta}_{11} \left(\frac{R_{mn}^{v0}}{R_{mn}} \right) \right]^2 |a_{v0}|^2 \right\}^{\frac{k}{2}};$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left\{ \sum_{\mu=1}^{n} \left[\bar{\Delta}_{11} \left(\frac{R_{mn}^{0\mu}}{R_{mn}} \right) \right]^2 |a_{0\mu}|^2 \right\}^{\frac{k}{2}};$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left\{\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left[\bar{\Delta}_{11}\left(\frac{R_{mn}^{\upsilon\mu}}{R_{mn}}\right)\right]^{2}\left|a_{\upsilon\mu}\right|^{2}\right\}^{\frac{k}{2}}$$

converges for $1 \leq k \leq 2$, then the orthogonal series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\varphi_{mn}(x)$$

is $\left|N^{(2)},p,q\right|_{k}$ summable almost everywhere.

In Chapter VI, we have extended the theorem of Krasniqi Xh. Z. 2011 for $\left|\bar{N}^{(2)},p,q\right|_{k}$for $k \geq 1$which is as follows:

Theorem 1G
If

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon 0}}{\bar{R}_{mn}}\right)\right\}^{2}|c_{\upsilon 0}|^{2}\right]^{\frac{k}{2}};$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\mu=1}^{n}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{0\mu}}{\bar{R}_{mn}}\right)\right\}^{2}\left|c_{0\mu}\right|^{2}\right]^{\frac{k}{2}};$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left[\sum_{\upsilon=1}^{m}\sum_{\mu=1}^{n}\left\{\Delta_{11}\left(\frac{\bar{R}_{mn}^{\upsilon\mu}}{\bar{R}_{mn}}\right)\right\}^{2}\left|c_{\upsilon\mu}\right|^{2}\right]^{\frac{k}{2}}$$

converges for $1 \leq k \leq 2$, then the orthogonal series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}\,\varphi_{mn}(x)$$

is $\left|\bar{N}^{(2)},p,q\right|_{k}$ summable almost everywhere.

1.11 General Lambert Summability of Orthogonal Series

Let $\{\varphi_n(\theta)\}, n = 0,1,\dots$ be an orthonormal system defined in (a,b). Let

$$\sum_{n=0}^{\infty}c_n\varphi_n(\theta)$$

be an orthogonal series, where $\{\boldsymbol{c_n}\}$ be a sequence of real numbers.

Let $f(\theta) \in L^2(a,b)$; then

$$f \sim \sum_{n=0}^{\infty} c_n \varphi_n(\theta) \tag{1-41}$$

be an orthogonal expansion of $f(\theta)$, where

$$c_n = \int_a^b f(\theta)\varphi_n(\theta)d\theta;$$

Bellman, R. 1943 has proved that Lambert summability of an orthogonal expansion (1-41)

Theorem 1.11

Lambert summability of an orthogonal expansion (1-41) implies the convergence of partial sums $S_{2^n}(\theta)$ of orthogonal expansion (1-41).

In Chapter VII, we have generalized the Theorem 1.11 for generalized Lambert summability which is as follows:

Theorem 1H

Generalized Lambert summability of orthogonal expansion (1-41) implies the convergence of partial sums $S_{2^n}(\theta)$ of orthogonal expansion (1-41).

1.12 Generalized Product Summability of an Orthogonal Series

The product summability was introduced by Kransniqi, Xh. Z. 2013.

Okuyama, Y. 2002 has proved the following two theorems:

Theorem 1.12
If the series

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{1/2}$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j \varphi_j(x)$$

is summable $|N, p, q|$ almost everywhere.

Theorem 1.13
Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequences. If the series

$$\sum_{n=1}^{\infty}|c_n|^2\Omega(n)\omega^{(i)}(n)$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty}c_j\varphi_j(x)$$

is $|N,p,q|$ summable almost everywhere, where $\omega^{(i)}(n)$ is defined by

$$\omega^{(i)}(j) \coloneqq j^{-1}\ \sum_{n=j}^{\infty}n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2.$$

Krasniqi, Xh. Z. 2013(2) has proved the following theorems:

Theorem 1.14
If for $1\le k\le 2$, the series

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2\left|c_j\right|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\emptyset_n(x)$$

is summable $|(N,p_n,q_n)(N,q_n,p_n)|_k, k\ge 1$ almost everywhere.

Theorem 1.15
Let $1\le k\le 2$ and $\Omega(n)$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges,
Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequences.
If the series

$$\sum_{n=0}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\Re^{(k)}(n)$$

converges, then the orthogonal series,

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

is $|(N, p_n, q_n)(N, q_n, p_n)|_k$ summable almost everywhere, where

$$\mathfrak{R}^{(k)}(i) := \frac{1}{i^{\frac{2}{k}-1}} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left(\frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2$$

In this chapter, we prove and extend the result of Krasniqi, Xh. 2013 to $|\overline{N}, p_n, q_n|_k$; $k \geq 1$ summability.

Our theorems are as follows:

Theorem 1I

If for $1 \leq k \leq 2$, the series

$$\sum_{n=1}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left(\frac{\overline{R}_n^i \widetilde{\overline{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\overline{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{k}{2}} < \infty$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \emptyset_n(x)$$

is summable $|(\overline{N}, p_n, q_n)(\overline{N}, q_n, p_n)|_{k,}$ almost everywhere.

Theorem 1J

Let $1 \leq k \leq 2$ and $\Omega(n)$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges.

Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequences. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \widetilde{\mathfrak{R}}^{(k)}(n) < \infty$$

then the series,

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

is $|(\overline{N}, p_n, q_n)(\overline{N}, q_n, p_n)|_k$ summable almost everywhere, where $\widetilde{\mathfrak{R}}^{(k)}(n)$ is defined by

$$\widetilde{\mathfrak{R}}^{(k)}(i) := \frac{1}{i^{\frac{2}{k}-1}} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left(\frac{\overline{R}_n^i \widetilde{\overline{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\overline{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2$$