Chapter III

Absolute Indexed Generalized Nörlund Summability of Orthogonal Series

3.1 Introduction

Let $\{\varphi_n(x)\}, (n = 0,1,2 \ldots)$ be an orthonormal system (ONS) of L^2 integrable functions defined in closed interval $[a, b]$. We consider the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{3-1}$$

with real coefficients $\{c_n\}$. We also assume that $f(x)$ belong to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n\, \varphi_n(x) \tag{3-2}$$

represents an orthogonal expansion of $f(x)$, where

$$c_n = \int_b^a f(x)\varphi_n(x)dx, \;\; n = 0,1,2, \ldots$$

We may denote the partial sum s_n, generalized Nörlund mean i.e. (N, p_n, q_n) mean and generalized $(\bar{N}, p_n, q_n)$ mean by

$$s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$$

$$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k\,(x)$$

and

$$\bar{t}_n^{p,q} = \frac{1}{\bar{R}_n} \sum_{k=0}^{n} p_k q_k s_k\,(x)$$

respectively.

Here, R_n and $\bar{R}_n$ are as follows:

$$R_n = \sum_{k=0}^{n} p_{n-k} q_k = \sum_{k=0}^{n} p_k q_{n-k} = (p * q)_n;\; R_n \neq 0 \tag{3-3}$$

for all n, and

$$\bar{R}_n = \sum_{k=1}^{n} p_k q_k;\; \bar{R}_n \neq 0 \tag{3-4}$$

for all n.

We may write,

$$R_n^j = \sum_{v=j}^{n} p_{n-v} q_v \tag{3-5}$$

$$\bar{R}_n^j = \sum_{v=j}^{n} p_v q_v \qquad (3\text{-}6)$$

and

$$R_n^{n+1} = 0, R_n^0 = R_n$$

$$\bar{R}_n^{n+1} = 0, \bar{R}_n^0 = \bar{R}_n$$

Also, we may consider

$$P_n := (p * 1)_n = \sum_{v=0}^{n} p_v$$

$$Q_n := (1 * q)_n = \sum_{v=0}^{n} q_v \, .$$

The series (3.1) is $|\bar{N}, p, q|_k$; $k \geq 1$ summable if

$$\sum_{n-1}^{n} n^{k-1} \left|\bar{t}_n^{p,q} - \bar{t}_{n-1}^{p,q}\right|^k < \infty$$

We may refer to equations (1-19) and (1-22) for more details.

3.2 Absolute Indexed Generalized Nörlund Summability of Orthogonal Series

Krasniqi, Xh. Z. 2010 has established the following theorem on the absolute indexed generalized Nörlund summability $|N, p, q|_k, k \geq 1$ of (3-1):

Theorem 3.1

If, for $1 \leq k \leq 2$, the series

$$\sum_{n=0}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{k}{2}} \qquad (3\text{-}7)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|_{k,}$ almost everywhere.

In this chapter, we have extended the Theorem 3.1 to $|\bar{N}, p, q|_k, k \geq 1$ summability of orthogonal series.

Our theorem is as follows:

Theorem 3A

Let $1 \leq k \leq 2$ and if the series

$$\sum_{n=0}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{\bar{R}_n^{j}}{\bar{R}_n}-\frac{\bar{R}_{n-1}^{j}}{\bar{R}_{n-1}}\right)^2 |c_j|^2\right\}^{\frac{k}{2}} < \infty$$

then, the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|\bar{N}, p, q|_k$ summable almost everywhere.

Corollary 3A1

If the series

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{\bar{R}_n^{j}}{\bar{R}_n}-\frac{\bar{R}_{n-1}^{j}}{\bar{R}_{n-1}}\right)^2 |c_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|\bar{N}, p, q|$ almost everywhere.

Corollary 3A2

Let $1 \leq k \leq 2$ and the series

$$\sum_{n=0}^{\infty}\left(\frac{n^{1-\frac{1}{k}}p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} p_{j-1}^2 |c_j|^2\right\}^{\frac{k}{2}} < \infty \tag{3-8}$$

then, orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|\bar{N}, p|_k$ almost everywhere.

Corollary 3A3

Let $1 \leq k \leq 2$ and the series

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{j=1}^{n} p_{j-1}^2 \left|c_j\right|^2 \right\}^{\frac{k}{2}}$$

converges, then orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|\bar{N}, p|$ almost everywhere.

3.3 Proof of Theorems

Proof of Theorem 3A

Let $\bar{t}_n^{p,q}$ be the $(\bar{N}, p, q)$ mean of the orthogonal series (3-1).

Now,

$$\begin{aligned}
\bar{t}_n^{p,q} &= \frac{1}{\bar{R}_n} \sum_{k=0}^{n} p_k q_k \, s_k(x) \\
&= \frac{1}{\bar{R}_n} \sum_{k=0}^{n} p_k q_k \sum_{j=0}^{k} c_j \varphi_j(x) \\
&= \frac{1}{\bar{R}_n} \sum_{j=0}^{n} c_j \varphi_j(x) \sum_{k=j}^{n} p_k q_k \\
&= \frac{1}{\bar{R}_n} \sum_{j=0}^{n} c_j \varphi_j(x) \, \bar{R}_n^j
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta \bar{t}_n^{p,q}(x) &= \bar{t}_n^{p,q}(x) - \bar{t}_{n-1}^{p,q}(x) \\
&= \frac{1}{\bar{R}_n} \sum_{j=0}^{n} c_j \varphi_j(x) \, \bar{R}_n^j - \frac{1}{\bar{R}_{n-1}} \sum_{j=0}^{n} c_j \varphi_j(x) \, \bar{R}_{n-1}^j \\
&= \sum_{j=0}^{n} \left(\frac{\bar{R}_n^j}{\bar{R}_n} - \frac{\bar{R}_{n-1}^j}{\bar{R}_{n-1}} \right) c_j \varphi_j(x) \qquad \text{(3-9)}
\end{aligned}$$

Now,

$$\int_a^b \left| \Delta \bar{t}_n^{p,q}(x) \right|^k dx = \int_a^b 1 \cdot \left| \Delta \bar{t}_n^{p,q}(x) \right|^k dx$$

Using Hölder's inequality

$$\int_a^b \left|\Delta\bar{t}_n^{p,q}(x)\right|^k dx \le \left(\int_a^b (1)^{\frac{2}{2-k}} dx\right)^{1-\frac{k}{2}} \left\{\int_a^b \left(\left|\Delta\bar{t}_n^{p,q}(x)\right|^k\right)^{\frac{2}{k}} dx\right\}^{\frac{k}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \left\{\int_a^b \left(\left|\Delta\bar{t}_n^{p,q}(x)\right|\right)^2 dx\right\}^{\frac{k}{2}}$$

Hence,by orthonormality relation, we have

$$= (b-a)^{1-\frac{k}{2}} \left\{\sum_{j=0}^{n} \left(\frac{\overline{R}_n^j}{\overline{R}_n} - \frac{\overline{R}_{n-1}^j}{\overline{R}_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

Hence,

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b \left|\Delta\bar{t}_n^{p,q}(x)\right|^k dx \le \sum_{n=1}^{\infty} n^{k-1} (b-a)^{1-\frac{k}{2}} \left\{\sum_{j=0}^{n} \left(\frac{\overline{R}_n^j}{\overline{R}_n} - \frac{\overline{R}_{n-1}^j}{\overline{R}_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} n^{k-1} \left\{\sum_{j=1}^{n} \left(\frac{\overline{R}_n^j}{\overline{R}_n} - \frac{\overline{R}_{n-1}^j}{\overline{R}_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left\{n^{\left(2-\frac{2}{k}\right)} \sum_{j=1}^{n} \left(\frac{\overline{R}_n^j}{\overline{R}_n} - \frac{\overline{R}_{n-1}^j}{\overline{R}_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

$$= M_1 \sum_{n=1}^{\infty} \left\{n^{\left(2-\frac{2}{k}\right)} \sum_{j=1}^{n} \left(\frac{\overline{R}_n^j}{\overline{R}_n} - \frac{\overline{R}_{n-1}^j}{\overline{R}_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

where $M_1 := (b-a)^{1-\frac{k}{2}}$

Using condition (3-7), we have

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b \left|\Delta\bar{t}_n^{p,q}(x)\right|^k dx < \infty$$

Therefore, according to Beppo Levi's theorem

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta \bar{t}_n^{p,q}(x)\right|^k < \infty$$

Hence (3-1) is summable $|\bar{N},p,q|_{k,}\ \ 1 \leq k \leq 2$.

3.4 Proof of Corollaries

Proof of Corollary 3A1

If we put $k = 1$ in Theorem 3A, we obtain the corollary 3A1 immediately.

Proof of Corollary 3A2

We have

$$\frac{\bar{R}_n^j}{\bar{R}_n} - \frac{\bar{R}_{n-1}^j}{\bar{R}_{n-1}} = \frac{P_{j-1}p_n}{P_n P_{n-1}}$$

Now, we shall take $q_n = 1$ in Theorem 3A. Hence,

$$\int_a^b \left|\Delta \bar{t}_n^{p,q}(x)\right|^k dx \ \leq (b-a)^{\frac{k}{2}} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} P_{j-1}^2 \left|c_j\right|^2\right\}^{1/2}$$

So,

$$\int_a^b \left|\Delta \bar{t}_n^{p,q}(x)\right|^k dx \ \leq (b-a)^{\frac{k}{2}} \left\{\sum_{j=1}^{n} \left(\frac{P_{j-1}p_n}{P_n P_{n-1}}\right)^2 \left|c_j\right|^2\right\}^{\frac{k}{2}}$$

Hence,

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b \left|\Delta \bar{t}_n^{p,q}(x)\right|^k dx \leq (b-a)^{\frac{k}{2}} \sum_{n=1}^{\infty} \left(\frac{n^{1-\frac{1}{k}} p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} P_{j-1}^2 \left|c_j\right|^2\right\}^{k/2}$$

So,

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b \left|\Delta \bar{t}_n^{p,q}(x)\right|^k dx \leq M_1 \sum_{n=1}^{\infty} \left(\frac{n^{1-\frac{1}{k}} p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} P_{j-1}^2 \left|c_j\right|^2\right\}^{k/2}$$

Using (3-8), we have

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b \left|\Delta \bar{t}_n^{p,q}(x)\right|^k < \infty$$

Hence, according to Beppo Levi's theorem,

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta \bar{t}_n^{p,q}(x)\right|^k < \infty$$

Hence, the proof follows.

Proof of Corollary 3A3

If we put $k = 1$ in corollary 3A2, our result follows immediately.