Chapter 1

Introduction

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1.1 Introduction

Partial Differential Equations (PDE) have been widely used in the fields of technology and engineering and basic sciences. Almost all modeling phenomena could be visualized as PDE. This leads to discovery of many numerical algorithms and their modifications to tackle such solutions. PDE can be classified broadly as parabolic, elliptic and hyperbolic.

In a broader sense algorithms to solve them numerically has evolved from finite difference method(FDM), finite element method (FEM), finite volume method (FVM),

and discontinuous Galerikin approach. There are other approaches as well.
With the far reaching affects inclusions of wavelets into digital image processing [50] paved the way for combination and an insight into the involvement of wavelets with other methods to solve PDE.

The development of algorithms for the solution formulation of various differential equations using specific wavelets can be traced as follows:

- C.F.Chen and C.H.Hsiao [18] used Haar wavelets to analyze a lumped and distributed parameter dynamic system.
- Dempster [67] used Daubechies wavelet for solving PDE in management studies.
- Jean [48] used Morlet wavelet to solve parabolic equations.
- Hosseini [92] used Chebyshev wavelets to solve ordinary differential equations.
- Carlo [14] choose Shannon wavelets for solving fractional differential equations.

Numerical methods form the basic tool to solve/evolve solution to a mathematical model representing any physical phenomenon, according to the requirement of the process. A new approach using wavelet is utilized to improve accuracy and adaptability to various partial differential equations. The theory of wavelet is widely implemented in digital image processing, so far. The approach of using wavelet function as basis for solution of PDE reduces the computational efforts. The multi resolution property of wavelet is highly beneficial in the solution as it helps to increase the resolution of the function. The basic concept of wavelet can be summarized as a family of functions constructed from translation and dilation of a single function called mother wavelet.

Wavelets are of a variety of types namely Haar wavelet, Daubechies wavelet, Coiflets wavelet, Meyer wavelet, Symlets wavelet etc. Among the various types of continuous and discrete wavelets, Haar wavelet is the one which is discrete, simple and orthonormal. It was proposed in 1909 by Alfred Haar. In numerical analysis, wavelet based algorithms have become an important tool because of its important properties of localization. The recent development in this direction is the use of wavelets in solving PDEs combined with already known algorithms in literature.

Paulo Cruz, Mendes and Fernao [76] combined FDM with wavelets using Daubechies wavelet. Dempster [67] and Mani [69] utilized FEM approach combined with wavelet utilizing Daubechies wavelet. Kessarwani and Liang [51] combined discontinuous Galerikin approach with wavelet using Haar wavelets. Vuik [107] combined discontinuous Galerikin approach which itself is a combination of FVM and FEM using multiwavelets. Dilshad [28] combined FVM with wavelet using Haar wavelet for Shallow water problems. Our study aims to combine FVM with wavelet approach using Daubechies, Coiflet and Symlet wavelet families.

Our study combines the properties of multi resolution to various well known approaches with an aim to improve the computation in terms of accuracy and convergence. Finally in this work we have used such improved algorithms to solve PDEs or BVPs representing some real world phenomenon.

1.2 Organization of Thesis

In this thesis our motive is to combine the wavelet approximation with various algorithms utilizing finite difference approach and finite volume approaches. Here the main concept is to improve the experimental order of convergence which is attained satisfactorily.

Our study deals with combining the concepts of multi resolution analysis to algorithms for solving ordinary as well as partial differential equations.

The Thesis is divided into six chapters.

1.2.1 Chapter 1

This chapter contains motivation, concepts and problems discussed in this Thesis. It also includes short description of contents of other chapters, definitions and symbolic notations.

1.2.2 Chapter 2

Chapter 2 is divided into three sections. First is introductory section 2.1 which includes wavelet transform classification, techniques of multiresolution and discusses various wavelet families. Wavelet families including Haar wavelet, Morlet wavelet, Symlet wavelet, Coiflet wavelet, Harmonic wavelet, Shannon wavelet, Complex gaussian wavelet, Mexican hat wavelet, Daubechies wavelet, Discrete Meyer wavelet, etc are discussed. Graphical representation and functional form are stated.

Second section describes multiresolution in the context of orthogonal wavelets on real line. Function representation as inverse discrete wavelet transform is discussed. As a specific illustration we express sin function using the Haar wavelet in section 2.2.

Third section 2.3 gives literature survey of application of wavelets for solving ODE, PDE and its current status.

1.2.3 Chapter 3

Chapter 3 describes the implementation of the wavelet based approach for finite difference method for ordinary differential equations with numerical examples. Here the solution of BVP is obtained using an approach which is different than the traditional shooting approach.

In shooting methods by Keinert [63] the procedure followed is to write the BVP in vector form and begin the solution at one end of the BVP, and attempt by a shoot to the other end with an initial value solver until the boundary condition at the other end converges to its correct value.

With reference to a second order boundary value problem defined as,

$$y"(x) = f(x), \quad y(0) = 0, \quad y(1) = 1.$$

the main steps of the shooting approach [96] involves the conversion of BVP to initial value problem (IVP). The solution of IVP is obtained by Taylor's series or Runge Kutta method. For solving the IVP, $y'(0)$ is needed. A true value of $y'(0)$ is assumed. Starting the procedure with two initial guess for $y'(0)$ the corresponding value of $y(1)$ is computed using initial value method. Let the two initial guesses be t_0 and t_1.

$y(1)$ is obtained by the chosen method as $y(t_0; 1)$ and $y(t_1; 1)$ respectively. By linear approximation a better approximation t_2 is obtained as,

$$\frac{t_2 - t_0}{y(1) - y(t_0; 1)} = \frac{t_1 - t_0}{y(t_1; 1) - y(t_0; 1)}$$

which gives,

$$t_2 = t_0 + (t_1 - t_0)\frac{y(1) - y(t_0; 1)}{y(t_1; 1) - y(t_0; 1)}.$$

Finally we solve the initial value problem,

$$y"(x) = f(x), \quad y(0) = 0 \quad and \quad y'(0) = t_2.$$

The speed of convergence depends on the initially assumed guess, the method is tedious to apply for higher order BVP and in case of nonlinear problems. To avoid these drawbacks we implement the proposed approach. In this chapter various IVP and BVPs are solved and analysed using Haar wavelet based methods. A generalized unified approach is proposed for arbitrary interval implementation using transformation by K. P. Mredula and D.C. Vakaskar [53], [56].

In all Chapter 3 is divided into eight sections. It includes introductory section 3.1, wavelet approximation for ODE section 3.2, Haar approximation section 3.3, Algorithm for solving initial value problem (IVP) section article 3.4.1, Algorithm for solving boundary value problem (BVP) section article 3.5.1, Numerical examples section 3.6, Generalization proposed for the algorithm 3.7. Extension of the region of solution section 3.8.

1.2.4 Chapter 4

We have solved elliptic, parabolic and nonlinear PDE [52] in this chapter without converting the PDE to system of ODE as an extension to Fazal [95]. The complexity of converting the governing equation to abstract problem [108] is also avoided. Lepik's [102] concept of Haar approximation is reflected in the solution of parabolic equation in this chapter, then it is extended to both space and time variables in the solution of elliptic equation.

We investigated the implementation of wavelet based collocation approach on, parabolic

PDE in one dimension and extended it to elliptic PDE for two dimensions [58]. By considering the Haar approximation of a function, the convergence of the approximation is obtained as in Haar wavelet convergence analysis.
Overall Chapter 4 consist of eight sections. It involves introduction section 4.1, in section 4.2 solution process for parabolic partial differential equation, numerical examples in section 4.3, convergence of Haar wavelet series and observations in section 4.4 and 4.5 respectively.
Section 4.6 states solution of elliptic partial differential equation, its algorithm EPDE and section 4.7 gives convergence analysis for two dimensional wavelet and its observations in section 4.8.

1.2.5 Chapter 5

In Chapter 5, we report the implementation of combined algorithm with finite volume and multi wavelet representation, to solve inviscid and viscous Burger equation. Finite volume approaches as described by Randall J Leveque [79] gives a clearer insight into the physical procedure and the mathematical techniques. It is based on the integral formulation of PDE and so is closer to the physics involved in the system.
Discontinuities lead to computational difficulties, we intent to solve this in a better manner. Classical finite difference methods, in which derivatives are approximated by finite differences can break down at the discontinuities in the solutions where the differential equation does not hold. Finite volume methods (FVM) divides the region into grid cells and approximates the total integral divided by the control volume of the cells, in contrast to the point wise approximation at grid points.
The values are then modified at each time step by the flux through the edges of the grid cells, the primary concern being the choice of correct numerical flux that

approximates the physical fluxes well, based on approximate cell averages (the only data available).

Our motivation to consider Burger equations as test problem comes from the fact that historically many of the fundamental ideas were first developed for the special case of gas dynamics (Euler equations), for applications in detonation waves, astrophysics and related fields where shock waves arise. The study of simpler equations such as the advection equations, Burger equations, and the shallow water equations has played an important role in the advancement of these methods. The hyperbolic PDE arise in a broad spectrum of disciplines where wave motion or advective transport is important such as acoustics, elastodynamics, optics, geophysics and biomechanics.

Chapter 5 consist of twelve sections. It includes description of a combined finite volume and wavelet approach, review of approaches used in the algorithm namely multiresolution, decomposition and reconstruction and finite volume Godunov approach in section 5.3, detailed algorithm of the proposed approach is given in section 5.4. It is followed by numerical test cases of inviscid Burger equations, convergence analysis for the proposed approach and observation in sections 5.5, 5.6 and 5.7 respectively.

In section 5.8 the solution formulation for viscous Burger equation using finite volume approach is discussed. Algorithm proposed for viscous Burger equation is given in section 5.9. It is followed by numerical experiments, convergence analysis for the approach and observations in sections 5.10, 5.11 and 5.12 respectively. Various test cases are considered and the results obtained by our approach are compared with the existing results in the literature.

1.2.6 Chapter 6

Chapter 6, discusses a real world application of the algorithm discussed in chapter 5. As a test case the modeled one dimensional nonlinear hyperbolic equation of yolk motion Bohun [10], which is the basis for development of embryo, is considered. Amphibian eggs provide several advantageous features as a model system for analyzing the effects of gravity on single cell. So the structure of oocyt is studied to obtain the governing equation to incorporate the physical process.
The numerical solution for the above modeled equation for settling of yolk platelets due to rotation of egg in the simplified form is solved. The wavelet based finite volume approach is utilized with different flux approximations such as Lax–Friedrichs, local Lax–Friedrichs and Roe, to depict the solution with more accuracy by K.P.Mredula, D.C. Vakaskar and Oleg V.K [55].
The numerical values are comparable with the solution obtained by Bohun [10] using the method of characteristics. The study indicates the correspondence between numerical and exact solution and the benefits of combined usage of the numerical approaches to get a better approximation. The error is analyzed in the sense of L^2 norm which is tabulated for increased grid values to capture the deviations. It also conclude with the overall conclusions and future scope of the work.
Chapter 6 is divided into seven sections. Section 6.1 states the problem background of the case discussed. The simplified mathematical model is given in section 6.2. Algorithm describing combined finite volume and wavelet approach is given in article 6.4.1 in section 6.4. Numerical simulation and observations are given in section 6.5 and 6.6 respectively. The overall conclusion and future scope is given in section 6.7.
Chapter 6 is followed by authors publications and references used during the course of research.