Chapter 2

Physico-Mathematical Background

Contents

This chapter discusses in detail concept of the Physico-Mathematical essentials to comprehend the subsequent chapters. The concepts pertaining to the subject are taken from various sources [1-33].

2.1 Basic Definitions

In this article various definitions, which are required for the subsequent study, are discussed.

Definition 2.1.1 Fluid

A fluid is a substance that deforms continuously when subjected to a shear stress, no matter how small that shear stress may be. This continuous deformation under the action of forces compels the fluid to flow and this tendency of fluid is called ***fluidity***.

Definition 2.1.2 Surface Tension

It is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension. The unit of surface tension is Nm^{-2}.

Definition 2.1.3 Viscosity

Viscosity is derived from the word viscous, which means sticky, adhesive, or tenacious. Viscosity is the property of a fluid by virtue of which it offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. The unit of viscosity is Nsm^{-2}.

Definition2.1.4 Kinematic Viscosity

It is defined as the ratio of the dynamic viscosity μ to the mass density ρ of fluid. It is denoted and defined as

$$\nu = \frac{\mu}{\rho}.$$

The unit of kinetic viscosity is m^2s^{-1}.

Definition 2.1.5 Density

The density of a fluid is the ratio of the mass of fluid in a fluid element to its volume. It is generally denoted by ρ. The unit of density is $\mathrm{kg\,m^{-3}}$.

Definition 2.1.6 Compressible Fluid

Compressible fluid is that type of fluid in which the density of the fluid changes from point to point.

Definition 2.1.7 Incompressible Fluid

Incompressible fluid is that type of fluid in which the density of the fluid is constant.

Definition 2.1.8 Ideal Fluid

An ideal fluid is one that has no viscosity. Since there is no viscosity, there is no shear stress between adjacent fluid layers, and that between the fluid layers and the boundary. Only normal stresses can exist in an ideal fluid flow.

Definition 2.1.9 Real Fluid

A real fluid is one that possesses viscosity, so shear stress comes into play in real fluid flow. Thus, a real fluid is characterized by its frictional resistance when it is in motion.

Definition 2.1.10 Newtonian Fluid

The Newtonian fluid is the fluid in which the shear stress is directly proportional to the rate of shear strain or velocity gradient.

For example, Glycerin, light-hydrocarbon oils, silicone oils, air, gases, etc.

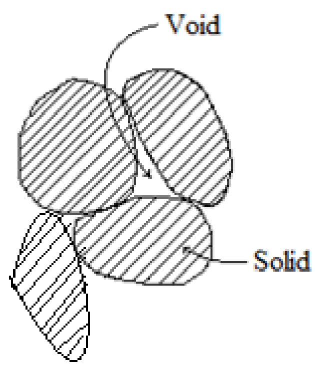
Definition 2.1.11 Non-Newtonian Fluid

The Non-Newtonian fluid is the fluid in which the shear stress is not directly proportional to the rate of shear strain or velocity gradient.

For example, slurries, tooth paste, gel, etc.

Definition 2.1.12 Porous Medium

A medium is considered porous when it is made up of loosely arranged solid particles with void space between them. A good example is the natural soil or sand.



Definition 2.1.13 Porosity

Porosity is measure of the void spaces in a material, and is defined as fraction of the volume of void spaces over the total volume of the material.

Definition 2.1.14 Permeability

Permeability is a measure of the ease with which a fluid can flow through the porous material. The SI unit for permeability is m^2. A practical unit for permeability is the Darcy (D).

Definition 2.1.15 Darcy's Law

The governing equation for fluid motion in a vertical porous column was first introduced by Darcy in 1856. Accordingly, the law is given by

$$V = -\frac{\phi}{\eta}\nabla P,$$

where V is the space averaged velocity (Darcian velocity), ϕ is the permeability of the porous region, η is the coefficient of viscosity and P is the pressure in the porous region. There is a negative sign as the flow is in the direction of decreasing pressure.

Definition 2.1.16 Random Variable

Intuitively, by a random variable (*r. v.)* we mean a real number X associated with the outcomes of a random experiment. It can take any one of the various possible values each with a definite probability.

For example, in a throw of a die if X denotes the number obtained then X is a random variable which can take anyone of the values 1, 2, 3, 4, 5 or 6, each with equal probability 1/6. In other words, random variable is a function which takes real values which are determined by the outcomes of the random experiments.

Definition 2.1.17 Probability Density Function

In case of a continuous random variable, we do not talk of probability at a particular point (which is always zero) but we always talk of probability in an interval. If $p(x)\, dx$ is the probability that the random variable X takes the value in a small interval of magnitude dx, e.g., $(x, x+dx)$ or $\left(x-\frac{dx}{2}, x+\frac{dx}{2}\right)$, then $p(x)$ is called the probability density function of the random variable X.

2.2 Various Types of Flows

Definition 2.2.1 Laminar Flow

The laminar flow is also called the streamline or viscous flow. This type of flow is characterized by a smooth flow of one lamina of fluid over another. This type of flow occurs, generally, in smooth pipes when the velocity of flow is low and also in liquids having a high viscosity.

Definition 2.2.2 Turbulent Flow

In turbulent flows, the fluid elements move in erratic and unpredictable paths. The random eddying motion is called turbulence. This type of flow generally prevails in rivers, canals, and in atmosphere.

Definition 2.2.3 Rotational Flow

A flow is said to be rotational, if the fluid particles while moving in the direction of flow rotate about their mass center.

Definition 2.2.4 Irrotational Flow

The flow in which the fluid particles do not rotate about its own axis is called irrotational flow. Only in the case of ideal fluid flow, irrotational flow exists, whereas, in real fluid flow one may assume fluid flow as irrotational, if the viscosity of the fluid has little significance.

Definition 2.2.5 Steady Flow

It is defined as that type of flow in which the fluid characteristics like pressure, velocity, density etc. at a point do not change with respect to time. Thus, for steady flow, mathematically, we have

$$\left(\frac{\partial \mathbf{q}}{\partial t}\right)_{x_0,y_0,z_0} = 0, \quad \left(\frac{\partial p}{\partial t}\right)_{x_0,y_0,z_0} = 0, \quad \left(\frac{\partial \rho}{\partial t}\right)_{x_0,y_0,z_0} = 0,$$

where (x_0, y_0, z_0) is a fixed point in flow field.

Definition 2.2.6 Unsteady Flow

It is defined as the flow in which the fluid characteristics like pressure, velocity, density etc. at a point changes with respect to time. Thus, mathematically, for unsteady flow,

$$\left(\frac{\partial \mathbf{q}}{\partial t}\right)_{x_0,y_0,z_0} \neq 0, \quad \left(\frac{\partial p}{\partial t}\right)_{x_0,y_0,z_0} \neq 0, \quad \left(\frac{\partial \rho}{\partial t}\right)_{x_0,y_0,z_0} \neq 0,$$

where (x_0, y_0, z_0) is a fixed point in flow field.

Definition 2.2.7 Uniform Flow

Uniform flow is defined as those types of flow in which the flow parameters like pressure, velocity, density etc. at given time do not change with respect to space (length of direction of flow). Mathematically,

$$\left(\frac{\partial p}{\partial s}\right)_{t=\text{constant}} = 0, \quad \left(\frac{\partial \mathbf{q}}{\partial s}\right)_{t=\text{constant}} = 0, \quad \left(\frac{\partial \rho}{\partial s}\right)_{t=\text{constant}} = 0,$$

etc.

Definition 2.2.8 Non-uniform Flow

Non-uniform flow in which the flow parameters like pressure, velocity, density etc. at a given time change with respect to space (length of direction of flow).

Mathematically,

$$\left(\frac{\partial p}{\partial s}\right)_{t=\text{constant}} \neq 0, \quad \left(\frac{\partial \mathbf{q}}{\partial s}\right)_{t=\text{constant}} \neq 0, \quad \left(\frac{\partial \rho}{\partial s}\right)_{t=\text{constant}} \neq 0.$$

Definition 2.2.9 Compressible Flow

When the density changes are appreciable, the flow is called compressible.

Definition 2.2.10 Incompressible Flow

Flow is incompressible, if the density changes due to pressure and temperature variations are insignificant in the flow field.

Definition 2.2.11 One, Two and Three-dimensional flows

One dimensional flow is that type of flow in which the flow parameters such as pressure, velocity are the function of time and one space coordinate only.

Two dimensional flow in which the flow parameters are the function of time and two space coordinate.

Three dimensional flows is that type of flow in which the flow parameters vary in all the three directions.

2.3 Magnetic parameters

The magnetic property of materials depends on the degree of magnetization. The parameters such as magnetization, magnetic susceptibility and magnetic permeability are used to characterize the magnetic materials. The important magnetic parameters which are used to characterize the magnetic materials are as follows.

Definition 2.3.1 Magnetic Dipole

The magnetic dipoles, generally known as north and south poles, commonly exist in magnetic materials. The magnetic dipoles are not separate poles unlike an electric dipole. It is found that magnetic poles always occur in pairs and cannot be isolated.

Definition 2.3.2 Magnetic Field

The space around a magnet where its magnetic influence is experienced is known as the magnetic field.

Definition 2.3.3 Magnetic Field Strength

The magnetic field strength H at any point in a magnetic field is the force experienced by a unit north pole placed at that point. The unit of magnetic field strength is Am^{-1}.

Definition 2.3.4 Magnetic Dipole Moment

The magnetic dipole moment is equal to the product of the magnetic pole strength and the length of the magnet.

Definition 2.3.5 Magnetization (Intensity of Magnetization)

Magnetization, or intensity of magnetization M is the measure of magnetism of magnetic materials and is defined as the magnetic moment per unit volume. The unit of magnetization is Am^{-1}.

Definition 2.3.6 Magnetic Susceptibility

Magnetic susceptibility $\overline{\mu}$ is used to explain the magnetization of material. It is defined as the ratio of magnetization M to the magnetic field strength H .

Magnetic susceptibility

$$\overline{\mu} = \frac{M}{H}$$

It is dimensionless quantity.

Definition 2.3.7 Magnetic Permeability

The magnetic permeability μ is defined as the ratio of amount of magnetic density B to the applied magnetic field intensity H. It is used to measure the magnetic lines of forces penetrating through a material. That is,

Magnetic permeability

$$\mu = \frac{B}{H}.$$

Definition 2.3.8 Magnetic Induction or Flux Density

The magnetic induction or flux density B in any material is defined as the number of lines of force through a unit area of cross-section perpendicularly. Therefore, magnetic induction

$$B = \frac{\phi}{A},$$

where A is the area of cross-section and ϕ, the magnetic force.

The unit of Magnetic Induction is Tesla [Tesla= Wbm^{-2}].

Definition 2.3.9 Permeability of Free Space

The permeability of free space or vacuum is denoted by μ_0.

The permeability of free space is:

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7} \quad \text{Hm}^{-1} \\ &= 1.257 \times 10^{-6} \quad \text{henry/mete r} \end{aligned}$$

The constant μ_0 appears in Maxwell's equation

$$H = \frac{B}{\mu_0} - M,$$

where M is magnetization density.

In vacuum,

$$M = 0,$$

Mathematically,

$$\mu_0 = \frac{B}{H}$$

The permeability of free space μ_0 is defined as the ratio of magnetic induction B to the strength of magnetization H.

2.4 Fundamental Equations from Fluid Dynamics

2.4.1 Equation of Continuity

According to the principle of mass conservation 'matter can be neither created nor destroyed'. This principle can be applied to a flowing flow.

Consider the flow of the fluid through a control volume shown in the Figure 2.1 having length *dx*, *dy* and *dz in x, y* and *z* –directions, respectively. A point *P* in a flowing fluid at which fluid velocity is given by

$$\mathbf{q}(x, y, z, t) = u\,\mathbf{i} + v\,\mathbf{j} + w\mathbf{k}$$

and density of fluid is ρ, where **i**, **j**, **k** are unit vectors along the coordinate directions of Cartesian axes and t is time.

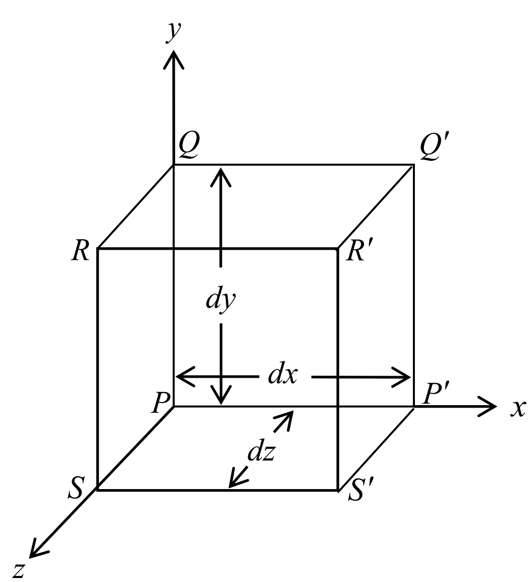


Figure 2.1

Construct a rectangular parallelepiped $PQRS\,P'Q'R'S'$, with edges parallel to the coordinate axes and of lengths *dx*, *dy* and *dz*, respectively.

Mass of the fluid entering through the face *PQRS* is given by

$$\text{density} \times \text{velocity in } x\text{-direction} \times \text{area of } PQRS = \rho u\, dy\, dz.$$

Then the mass of the fluid leaving the face $P'Q'R'S'$ is given by

$$\rho u\, dy\, dz + \frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx$$

Therefore, rate at which fluid accumulates due to flow in x -direction through the faces *PQRS* and $P'Q'R'S'$ is given by

$$\text{mass through the face } PQRS - \text{mass through the face } P'Q'R'S'$$

$$= \rho u\, dy\, dz - \rho u\, dy\, dz - \frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz.$$

Similarly, the net inflow in y -direction is given by

$$-\frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz,$$

and mass gain in z -direction is given by

$$-\frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz,$$

Thus, the net gain in the fluid flowing into the parallelepiped through the six faces is

$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]dx\,dy\,dz.$$

(2.1)

Also, rate of increase of the mass within the control volume per unit time is

$$\frac{\partial}{\partial t}(\rho \times \text{volume}) = \frac{\partial \rho}{\partial t}dx\,dy\,dz.$$

(2.2)

According to principle of conservation of mass, equating equation (2.1) and equation (2.2), we get

$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]dx\,dy\,dz = \frac{\partial \rho}{\partial t}dx\,dy\,dz.$$

Dividing by $dx\,dy\,dz$, we get the equation of continuity in the Cartesian coordinates at the point P as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0.$$

(2.3)

Equation (2.3) is the most general form of **Equation of Continuity in Cartesian Coordinates** which is applicable for steady as well as unsteady flow, uniform as well as non-uniform flow, and compressible as well as incompressible fluids.

- For the steady fluid flow

$$\frac{\partial \rho}{\partial t}=0,$$

Hence the equation (2.3) of continuity becomes

$$\frac{\partial}{\partial x}(\rho \mathrm{u})+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

(2.4)

- If the fluid is incompressible, so that the mass density ρ, does not changes with *x*, *y*, *z* and *t*. Therefore, the equation (2.3) of continuity becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

(2.5)

- **Vector form of Continuity Equation**

The vector form of equation (2.3) can be written as

$$\frac{\partial \rho}{\partial t}+\nabla \bullet(\rho \mathbf{q})=0.$$

(2.6)

- **Cylindrical form of Continuity Equation**

The equation of continuity for an incompressible steady flow (2.5) presented in cylindrical form as

$$\frac{1}{r}\frac{\partial}{\partial r}(r u)+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0,$$

(2.7)

Where r, θ, z are cylindrical polar coordinates.

2.4.2 Navier-Stokes Equation (Momentum Equation)

It is based on the principle of conservation of momentum. Considering an infinitely small mass of the fluid enclosed in an elementary parallelepiped shown in the Figure 2.2 of the sides *dx*, *dy* and *dz*.

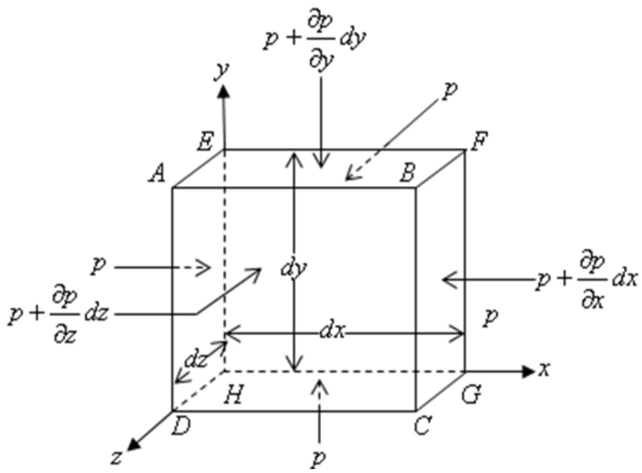


Figure 2.2

The motion of the fluid element is influenced by the following forces:

(i) Normal forces due to pressure :

The net pressure force in the x -direction is given by

$$pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz$$

$$= -\frac{\partial p}{\partial x}dxdydz.$$

(ii) Body or gravity force

Let g_x, g_y and g_z are components of gravitational force or body force $\mathbf{g}$ in the x, y and z-directions, respectively. Then body force per unit mass of the fluid in the x -direction is

$$m\,g_x = g_x\,\rho\,dxdydz.$$

(iii) Inertia forces :

Inertia force on the fluid mass along x -direction is given by

$$\text{mass} \times \text{acceleration}$$

$$= \rho\, dxdydz\, \frac{du}{dt}.$$

(iv) Shear forces :

The components of shear force per unit mass by the viscous forces be S_x, S_y and S_z in the x, y and z -directions, respectively.

Thus, shear force acting on the parallelepiped along x -direction is given by

$$S_x\, \rho\, dxdydz.$$

As per Newton's second law of the motion, the sum of all forces acting in the fluid element in any direction equals the resulting inertia forces in that direction, therefore along x -direction

$$-\frac{\partial p}{\partial x}\, dxdydz + g_x\, \rho\, dxdydz \; + S_x\, \rho\, dxdydz = \rho\, dxdydz\, \frac{du}{dt}$$

$$g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} - S_x. \tag{2.8}$$

Similarly,

$$g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{dv}{dt} - S_y, \tag{2.9}$$

and

$$g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - S_z .$$

(2.10)

Now, we find the values of S_x , S_y and S_z :

The shear force acting on the face *ADHE* is given by

$$-\eta \frac{\partial u}{\partial x} \, dydz$$

The shear force acting on the face *BCGF* is given by

$$\eta \frac{\partial}{\partial x}\left(\mathrm{u} + \frac{\partial u}{\partial x} dx \right) dydz$$

$$= \eta \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \right) dydz$$

The resultant force acting along the x -direction is given by

$$-\eta \frac{\partial u}{\partial x} \, dydz + \eta \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \right) dydz$$

$$= \eta \frac{\partial^2 u}{\partial x^2} dx \, dydz$$

(2.11)

Similarly, the x-component of resultant shear force acting on the faces *DCGH* and *ABFE* is equal to

$$\eta \frac{\partial^2 u}{\partial y^2} dx \, dydz$$

(2.12)

and, the x-component of resultant shear force acting on the faces $EFGH$ and $ABCD$ is given by

$$\eta \frac{\partial^2 u}{\partial z^2} dx\, dydz$$

(2.13)

Total force, parallel to x-axis, on all the six faces of the parallelepiped is given by the sum of forces defined in equations (2.11), (2.12) and (2.13) and is given by

$$\eta \frac{\partial^2 u}{\partial x^2} dx\, dydz + \eta \frac{\partial^2 u}{\partial y^2} dx\, dydz + \eta \frac{\partial^2 u}{\partial z^2} dx\, dydz$$

$$= \eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx\, dy\, dz$$

The shear (resistance) per unit mass is obtained by dividing above quantity by $\rho\, dxdydz$, we get

$$S_x = \frac{\eta}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

Similarly,

$$S_y = \frac{\eta}{\rho} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

and

$$S_z = \frac{\eta}{\rho} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right).$$

Putting these values of S_x, S_y and S_z in equations (2.8), (2.9) and (2.10), we get

$$g_x - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{du}{dt} - \frac{\eta}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

(2.14)

$$g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{dv}{dt} - \frac{\eta}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right),$$

(2.15)

and

$$g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{dw}{dt} - \frac{\eta}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

(2.16)

These equations (2.14), (2.15) and (2.16) are called **Navier-Stokes Equations** for viscous flow.

- **Vector form of the Navier-Stokes Equations**:

$$\rho\frac{D\mathbf{q}}{Dt} = -\nabla p + \rho\,\mathbf{g} + \eta\nabla^2\mathbf{q},$$

(2.17)

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \bullet \mathbf{q} \text{ and } \mathbf{q} = (u, v, w).$$

2.4.3 No Slip Condition of Viscous Fluids

When a viscous fluid flows over a solid surface, the fluid elements adjacent to the surface attain the velocity of the surface. In other words, the relative velocity between the solid surface and adjacent fluid particles is zero. This phenomenon is known as the no-slip conditions.

The physical reason for the non-slip condition is that the fluid molecules hitting the solid wall collide so frequently with the solid wall molecules that they have no average motion that is different from the wall molecules. The non-slip boundary condition applies to the three components of the fluid velocity **q** everywhere along the surface of a solid boundary.

2.5 Types of Lubrication

The following are the classification of different types of lubrication.

1. Hydrodynamic (or Fluid Film) Lubrication

If two mating surfaces during operating conditions are completely separated by lubricant film, such a type of lubrication is called fluid film lubrication. Bearings operating under this kind of lubrication are fluid film bearings. As metal to metal contact is completely avoided by this system of lubrication, it is sometimes known as perfect lubrication.

2. Boundary Lubrication

When the lubricating oil film becomes so thin that it cannot prevent the bearing surfaces asperities from striking each other and establishing occasional contact, it is called boundary lubrication. It takes place in a bearing when either the speed of the moving surfaces is too low, the load acting on the bearing is too high, or there is an insufficient supply of oil. In other words, it exists when the operating conditions make it impossible for a hydrodynamic oil film to be developed in the bearing.

3. Elasto-Hydrodynamic Lubrication

The process of hydrodynamic lubrication may be modified under conditions of extremely high contact pressures, e.g. between cams and followers, gear teeth and rolling bearings. Under such operating conditions the viscosity of the lubricant increases considerably and an elasto-hydrodynamic oil film may be generated between the interacting surfaces, as a result of their deforming elastically against the built up of oil pressure.

4. Hydrostatic Lubrication

A condition of hydrostatic lubrication, termed a squeeze oil film, can exist in a bearing if the load reverses in direction and the speed of the moving surface is very low. Under these circumstances, the load-carrying surfaces of the bearing are initially separated by a relatively thick film of oil, which momentarily resists being instantaneously squeezed out from between the approaching surfaces, due to the viscous nature of the lubricant. Thus an oil film can be maintained between the bearing surfaces during the short interval of time till the load reverses its direction. The oil film may then be restored again to sufficient thickness, by forced oil feed system, before the next load reversal occurs. Thus, in bearings subjected to high reciprocating loads, the process of hydrostatic lubrication provides the cushioning effect, which is very necessary in efficient engine operation.

5. Solid Film Lubrication

This type of lubrication is used in a situation where a fluid lubrication is not desirable because there are certain industries like food and pharma where the risk of contamination of product by fluid lubrication is catastrophic. In such situation, solid or powder form of lubrication such as graphite, Teflon etc. are being used as lubricant.

2.5.1 Assumptions in Hydrodynamic Lubricated Bearings

The following are the basic assumptions used in the theory of hydrodynamic lubricated bearings in our present research work.

1. The lubricant obeys Newton's law of viscous flow.

2. The lubricant is assumed to be incompressible.
3. The fluid flow is laminar and viscous.
4. The gravity and inertia forces acting on the fluid are negligible compared with the viscous and pressure force.
5. The film thickness is very small compared with the bearing geometry.
6. The viscosity of the lubricant is assumed to be constant throughout the film.
7. The bearing surfaces are assumed to be perfectly rigid so that elastic deformations of the bearing surfaces may be disregarded.
8. Porous matrix of the bearing surface is assumed to be homogeneous.
9. Flow in the porous region follows Darcy's law.
10. Temperature changes in the lubricant are neglected.
11. There is no slip in the fluid-solid boundaries.

2.6 The Generalized Reynolds Equation

The generalized Reynolds equation, a differential equation in pressure, which is used frequently in the hydrodynamic theory of lubrication can be deduced from the Navier-Stokes equations along with the continuity equation under certain assumptions. The parameters involved in the Reynolds equation are viscosity, density and film thickness of lubricant. It was first derived by Osborne Reynolds in 1886.

The generalized Reynolds equation is [11]

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial z}\right)=\frac{\partial}{\partial x}\left[\frac{\rho(u_a+u_b)h}{2}\right]+\frac{\partial}{\partial z}\left[\frac{\rho(w_a+w_b)h}{2}\right]+\frac{\partial(\rho h)}{\partial t} \tag{2.18}$$

The velocity components u_a, v_a and w_a refer to the velocities of upper surface in *x, y* and *z* directions, and u_b, v_b and w_b refer to the velocities of the lower surface in *x, y* and *z* directions and *p is* the pressure in the fluid film.

The two terms in the left hand side of equation (2.18) describe the net flow rates due to pressure gradients, the first two terms of the right hand side of equation (2.18) describe the flow rate due to surface velocities. These are known as Poiseuille and Couette terms, respectively.

In practice all the velocity components are not present. In most of the cases we will be concerned with the following boundary velocities.

$$w_a = w_b = 0, \quad v_a = u_a \frac{\partial h}{\partial x} \tag{2.19}$$

Using equation (2.19) in equation (2.18), one obtains for constant velocities

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial z}\right)=U\frac{\partial}{\partial x}(\rho h)+\frac{\partial(\rho h)}{\partial t}, \tag{2.20}$$

where

$$U = \frac{u_a + u_b}{2}$$

For steady–state conditions, the generalized Reynolds Equation (2.20) becomes

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial z}\right)=U\frac{\partial}{\partial x}(\rho h). \tag{2.21}$$

If the fluid property ρ does not vary, as in the case of incompressible lubricant, equation (2.21) can be written as

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial z}\right)=U\frac{\partial h}{\partial x}$$

(2.22)

If one assumes the bearing to be infinitely long in the z-direction, there will be no variation of pressure in the z-direction or $\frac{\partial p}{\partial z}=0$. Equation (2.21) then becomes

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial x}\right)=12U\frac{\partial}{\partial x}(\rho h)$$

(2.23)

Equation (2.23) when integrated with respect to *x* yields

$$\frac{dp}{dx}=12\eta\, U\left[\frac{\rho h-(\rho h)_m}{\rho h^3}\right],$$

(2.24)

where subscript m refers to the condition at point where $\frac{dp}{dx}=0$.

The Reynolds equation is solved with appropriate boundary conditions to get the expression for lubricant film pressure. From this film pressure, the load-carrying capacity of the bearing may be derived by integrating it over the entire surface.

2.7 Discussion on Different Types of Bearings

A Bearing is a system of machine elements whose function is to support an applied load by reducing friction between the relatively moving surfaces.

The bearings may be classified in many ways as follows.

Depending upon the direction of load to be supported, the bearings under this group are classified as

(a) Radial bearings

(b) Thrust bearings

(c) Conical bearings

In radial bearings, the load acts perpendicular to the direction of motion of the moving element. In thrust bearings, the load acts along the axis of rotation whereas conical bearings support both radial and axial load.

Depending upon the nature of contact, the bearings under this group are classified as

(a) Sliding contact bearings

(b) Rolling contact bearings

In sliding contact bearings, the sliding takes place along the surfaces of contact between the moving element and the fixed element. The sliding contact bearings are also known as plain bearings. In rolling contact bearings, the steel balls or rollers, are interposed between the moving and fixed elements. The balls offer rolling friction at two points for each ball or roller. Rolling element bearings have much wider use in industries since rolling friction is lower than the sliding friction.

2.8 Concept of Ferrofluid

A ferrofluid is composed of three components: magnetic nano particles, dispersion medium (also called carrier liquid) and a dispersant or surface active agent. A typical ferrofluid is comprised, by volume, of about 5% solid component, 85% liquid and 10% surface active agent. Ferrofluids are recognized as nano materials. Ferrofluids are man-made materials in which magnetism is found in a liquid state with a high magnetic susceptibility. When there is no magnet nearby, the magnetite particles in ferrofluid act like normal metal particles in suspension. But in the presence of a magnet,the particles are temporarily magnetized. Ferrofluid responds to an applied magnetic field as one homogeneous system. This enables the fluids location to be controlled through the application of a magnetic field. The particles are so small in ferrofluids that they will not settle over time, but remain in place as long as a magnetic field is present. Treatment of particles with surfactants prevents them from clumping up, causing the material to remain stable and act predictably. They act more like a solid. When the magnetic field is removed, the particles are demagnetized and ferrofluid acts like a liquid again. Thus, in summary, Ferrofluids [17] are stable colloidal suspensions containing fine ferromagnetic particles (~0.05-15 nm) dispersing in a liquid, called carrier liquid, in which a surfactant is added to generate a coating layer preventing the flocculation of the particles. When an external magnetic field $\mathbf{H}$ is applied, a ferrofluid experiences magnetic body force $(\mathbf{M} \bullet \nabla)\mathbf{H}$ which depends upon the magnetization vector $\mathbf{M}$ of ferromagnetic particles.

2.8.1 Equation of Motion for a Magnetic Fluid

In classical fluid dynamics there are only three forces viz.

(a) the pressure gradient

(b) gravity force

(c) the viscous force

which have been taken into account and accordingly the equation of motion is stated as the sum of the gradients of all those forces remains equal to rate of change of velocity multiplied by the density. If comparison is made between Momentum equation of classical fluid dynamics and ferrohydrodynamics the obvious difference between the two is the use of fourth force i.e. magnetic body force which has been used in ferrohydrodynamics. The momentum equation of ferrohydrodynamics takes the form

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\bullet\nabla)\mathbf{q}\right]=-\nabla p+\eta\nabla^{2}\mathbf{q}+\mu_{0}(\mathbf{M}\bullet\nabla)\mathbf{H}$$

Magnetic Body Force

In the presence of an external magnetic field, the ferromagnetic colloidal particles suspended in the carrier liquid of a ferrofluid become magnetized and produce attractive forces on each particle that produce a body force on the liquid. The magnetic (Kelvin) force **F** on ferrofluid per unit volume is given by $\mathbf{F}=\mu_0(\mathbf{M}\bullet\nabla)\mathbf{H}$, where μ_0 is magnetic permeability of free space, **M** is magnetization, **H** is magnetic field strength of the external magnetic field.

2.8.2 Ferrofluids Lubrication Equations Based on Neuringer- Rosensweig Model

The system of equations proposed by Neuringer and Rosensweig for governing the FFs are as follows.

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\bullet\nabla)\mathbf{q}\right]=-\nabla p+\eta\nabla^{2}\mathbf{q}+\mu_{0}(\mathbf{M}\bullet\nabla)\mathbf{H}$$

(2.25)

$$\nabla\bullet\mathbf{q}=0$$

(2.26)

$$\nabla\times\mathbf{H}=0$$

(2.27)

$$\mathbf{M}=\bar{\mu}\,\mathbf{H}$$

(2.28)

$$\nabla \bullet (\mathbf{H} + \mathbf{M}) = 0,$$

(2.29)

where $\rho, \mathrm{p}, \eta, \mathbf{q}, \mu_0, \mathbf{M}, \mathbf{H}, \bar{\mu}, \mathrm{t}$ are density of fluid, film pressure, fluid viscosity, fluid velocity, free space of permeability, the magnetization vector, magnetic field vector and magnetic susceptibility, time, respectively.

Using equation (2.27)

$$\mu_0 (\mathbf{M} \bullet \nabla) \mathbf{H} = \mu_0 \bar{\mu} (\mathbf{H} \bullet \nabla) \mathbf{H}$$

Using the vector identity

$$(\mathbf{H} \bullet \nabla) \mathbf{H} = \frac{1}{2} \nabla (\mathbf{H} \bullet \mathbf{H}) - \mathbf{H} \times (\nabla \times \mathbf{H})$$

and assuming the fluid is electrically non-conducting and that the displacement current is negligible so that $\nabla \times \mathbf{H} = 0$,we obtain

$$\mu_0 (\mathbf{M} \bullet \nabla) \mathbf{H} = \frac{1}{2} \mu_0 \, \bar{\mu} \, \nabla (\mathbf{H} \bullet \mathbf{H}) = \frac{1}{2} \mu_0 \, \bar{\mu} \, \nabla (\mathrm{H}^2)$$

equation (2.24) becomes

$$\rho \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \bullet \nabla) \mathbf{q} \right] = -\nabla \left(p - \frac{\mu_0 \bar{\mu} H^2}{2} \right) + \eta \nabla^2 \mathbf{q}.$$

(2.30)

2.9 Surface Roughness

No solid surface is perfectly smooth on atomic scale. In other words, all solid surfaces are rough to some extent. Roughness can be defined as a measure of random distribution of surface height about a carrier profile in the surfaces.

Previously rough surfaces were assumed to be represented by a single sine (cosine) wave or a series of sine (cosine) waves, which influences on the film thickness in the usual study of bearing characteristics. It has been pointed out that the load capacity, frictional forces etc. can differ from their values for a smooth surface and this difference depends mainly on the amplitudes and the wave lengths of the waves representing the roughness surface. This procedure is known as the deterministic approach.

In another method, called the stochastic approach, the surface roughness is assumed to be represented by a stochastic process and the usual procedure is followed by taking the statistical mean or the average of the basic governing equation. In fact, it has been demonstrated that the density distribution of roughness peaks in normal machined surfaces follows a nearly Gaussian distribution and the gap between two rough surfaces can be represented by a stochastic process [21].

The hydrodynamic lubrication of rough surfaces has been studied by many workers using both deterministic and stochastic approach [22-29]. A consistent hydrodynamic theory of lubrication of rough surfaces has been presented by Christensen and Tonder [30-31] and Christensen [26] by considering the film thickness as a stochastic process. With the aid of stochastic theory, a Reynolds-type equation in the mean pressure was developed. The above mentioned investigations are valid when the mean height of the surface asperities is much smaller than the minimum film thickness.

➢ Distribution of Roughness Heights

With film thickness being regarded as a random quantity because of roughness effect, a height distribution function must be associated. Therefore, a polynomial form, approximating the Gaussian is chosen. Such a probability density function of the stochastic film thickness, h_s, is taken as [26,32]

$$f(h_s) = \begin{cases} \dfrac{35}{32c^7}(c^2 - h_s^2)^3 \ ; & -c < h_s < c \\ 0 & ; \quad \text{elsewhere} \end{cases} \qquad , \tag{2.31}$$

where $c = \pm 3\sigma$, and σ is the standard deviation.

In the context of stochastic theory [26, 32], the analysis is usually done for two types of one-dimensional roughness patterns (viz. radial and circumferential) as follows.

Radial Roughness Pattern

In this model, the roughness is assumed to have the form of long, narrow ridges and valleys running in *r*-direction (that is, they are straight ridges and valleys passing through $z = 0$, $r = 0$ to form a star pattern). The film thickness in this case assumes the form

$$h = h_n + h_s(\theta, \xi).$$

Circumferential Roughness Pattern

In this model, the roughness is assumed to have the form of long, narrow ridges and valleys running in θ-direction. The film thickness in this case assumes the form

$$h = h_n + h_s(r, \xi).$$

2.10 References

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