# CHAPTER 5

# CHARACTERIZATION OF OBJECTIVE FUNCTIONS OPTIMIZABLE THROUGH NETWORK FLOW TERMINOLOGY

## 5.1 INTRODUCTION

In this chapter, we present the characterization of objective functions which can be minimized through the network flow terminology. Although, our characterization is restricted to the class of objective functions involving variables with range comprising only two elements, it is quite general, as it deals with a wide range of objective functions those are minimized through network flow terminology. As we have seen in previous chapter, majority of graph cut model, in a single iteration, works with only two labels. Thus, they perform the task of assignment of two labels effectively to modify the initial labeling so that the penalty of the revised labeling assigned by objective function is minimized. Thus, the characterization is quite general in view of it's' application.

There are enormous amount of computer vision applications which can be efficiently dealt with network flow terminology. But, as presented in the earlier chapter, for every application, network flow specific to the problem is constructed and the minimization process is carried out. In some of the models, reserve vertices are employed to take care of complexities. Such non-uniform approach for construction of graph and assignment of weights to the edges leads to ambiguity and restricts the application of the approach to wide range of problems those can be effectively addressed by the technique. It has also been observed that, wide range of problems which can be efficiently solved by the network flow terminology are still not being exposed to the technique due to these limitations. The results presented in the chapter don't just characterize the class of objective functions with variables having range of size two which can be solved by Flow network terminology, but also leads to a standard protocol for construction of network flow which can be applied to all objective functions lying in the class uniformly irrespective of the special applications or vision problems they correspond to. In this chapter, we are going to talk about $\widehat{\mathrm{O}}$, the set of all objective functions of variables $y_1,....,y_n$ of range of size two. The entire work presented in this section is based on the characterization presented in **[120]** and **[121].**

## 5.2 IMPORTANCE OF VARIABLES HAVING RANGE OF SIZE TWO

The importance of such variables will be evident if we study the models presented in Chapter 3. Consider model with interchange move. The model deals with labeling involving large number of labels. However, in single iteration, it deals with only a pair of labels $\sigma_1$ and $\sigma_2$. Given an initial labeling X, it tries to refine it to $\mathrm{X}'$, so that, $\mathrm{O}(\mathrm{X}') < \mathrm{O}(\mathrm{X})$, where $\mathrm{X}'$ is within single interchange move from X and it is the best labeling with this property. The network flow involves only those pixels which are assigned label $\sigma_1$ or $\sigma_2$ by *X*. Mathematically, the vertex set of the network flow consists of the vertices of $V_1 \cup V_2$, where $V_1 = \{v \in V \mid X_v = \sigma_1\}$ and $V_2 = \{v \in V \mid X_v = \sigma_2\}$. The revised labeling $\mathrm{X}'$ can be encrypted by a variable *Y* having range $\{\sigma_1, \sigma_2\}$ of size two as, $Y = \{y_v \mid v \in V\}$, where $X'(v) = \sigma_1$ if $y_v =$

$\sigma_1$ and $X'(v) = \sigma_2$ if $y_v = \sigma_2$ . Note that, the variable Y takes only two values $\sigma_1$ or $\sigma_2$ and hence has range of size two.

Consider the model with growth move. The model deals with labeling involving large number of labels. However, in single iteration, it deals with only single label $\sigma$ . Given an initial labeling X, it tries to refine it to $\mathrm{X}'$ , so that, $\mathrm{O}(\mathrm{X}') < \mathrm{O}(\mathrm{X})$, where $\mathrm{X}'$ is within single growth move from X and it is the best labeling with this property. In this model, the network flow consists of non-terminal vertices corresponding to all the pixels. The labeling $\mathrm{X}'$ can be encrypted by a variable *Y*having range $\{\sigma, \sigma'\}$ of size two as, $Y = \{y_v \mid v \in V\}$, where $X'(v) = \sigma$ if $y_v = \sigma$ and $X'(v) = X(v)$ if $y_v = \sigma'$ .

Consider the model with shift move. The model deals with labeling involving large number of labels. However, in single iteration, it deals with only single label $\sigma$ . Given an initial labeling X, it tries to refine it to $\mathrm{X}'$ , so that, $\mathrm{O}(\mathrm{X}') < \mathrm{O}(\mathrm{X})$, where $\mathrm{X}'$ is within single shift move from X and it is the best labeling with this property. In this model, the network flow consists of non-terminal vertices corresponding to all the pixels. The labeling $\mathrm{X}'$ can be encrypted by a variable $Y$ having range $\{i, i'\}$ , of size two as, $Y = \{y_v \mid v \in V\}$, where $X'(v) = X(v) + k$ if $y_v = i$ and $X'(v) = X(v)$ if $y_v = i'$ . Note that, the model assigns the terminals $i$ and $j$ to the pixels, which finally leads to assignment of labels $X(v)$ and $X(v) + k$ to the pixels.

Thus, majority of the models we discussed deal with labeling corresponding to variables having range of size two. Thus, the characterization to be presented is quite useful in general and especially for computer vision applications.

## 5.3 GRAPHS REPRESENTING OBJECTIVE FUNCTIONS

In chapter 3, we constructed flow network for given objective function. For this Chapter, Let's try to review the notion with different perspective. Consider a network (G,*V, E*) with vertex set $= \{s, t, v_1, v_2, ...., v_n\}$ with terminal vertices*s* and *t,* and remaining non-terminal vertices. Then, every cut *C*on the network naturally leads to a function of *n* variables $\{y_1, y_2, ...., y_n\}$ as follows: If $v_i$ is connected to terminal vertex *s* in the induced graph G \ C, $y_i = \alpha$ ; If $v_i$ is connected to terminal vertex *t* in the induced graph G \ C, $y_i = \beta$ .

As G is a network flow, every edge of the graph has an associated weight, and thus, every cut *C* on *G* has a cost associated with it. Thus, there exists a real-valued function on the set of all cuts C on G. This function can also be viewed as a real valued function defined on $\{y_1, y_2, ...., y_n\}$, because every cut C naturally corresponds to a specific pattern or configuration of these variables. This function is an objective function defined on the set of variables of size two, as it assigns weight to every configuration of these n variables. Thus, every flow network with n non-terminal vertices constructed in similar fashion corresponds to an objective function of variables with range of size two. Let's think about the converse: Does every objective function corresponds to such flow network? If the answer is yes, then every objective function can be optimized by flow network terminology. Unfortunately, the answer is negative. There is a class of objective functions, for which the answer turns out to be yes. Our interest in this chapter lies in the characterization of this class.

We can generalize our notion as follows: Given a flow network (G,*V*, *E*), we can allow a proper subset of non-terminal vertices of *V* to correspond to the variables with range of size twoand remaining vertices of *V* can be made independent of any variables with range of size two. For example, if *V* contains *n* vertices, we can consider the flow network giving rise to objective function having less than *n* variables with range of size two. Let's consider that, two of the vertices $v_{n-1}$ and $v_n$ of *V* do not give rise to any variables with range of size two. i.e. they are auxiliary. Thus, the objective function we consider will have only *n-2*variables with range of size two, namely $y_1,....,y_{n-2}$ corresponding to vertices $v_1,....,v_{n-2}$. The only technical difficulty in this generalization is loss of one to one correspondence between the set of all possible configurations of the variables and the set of all cuts on the network that was established in the special case. There will be more than one cut leading to the same configuration, where these cuts will differ only in terms of edges corresponding to auxiliary vertices. As auxiliary vertices play no role in determination of the variables, all these cuts lead to the same configuration of *n-2* variables. This hurdle can be overcome by relating the configuration of these *n-2*variables under consideration to the cut with minimum cost among all the cuts (corresponding to the same configuration).

## DEFINITION 5.3.1

The objective function O of *n* variables $\{y_1, y_2,...., y_n\}$ is said to be ***Flow Network Optimizable*** or ***FNO-function*** if, there exists a network (G,*V, E*) with $|V \setminus \{s,t\}| \geq n$ such that, for any particular pattern of these *n* variables (i.e. any choice of specific values $y_i'$ from $\{\alpha, \beta\}$ assigned to these $y_i$ $(1 \leq i \leq n)$), the value of the objective function $O_1(y_1', y_2',...., y_n')$ and the cost of a minimum cut on the flow network differ by a constant. Note, that, a minimum cut is considered over all the cuts C on the flow network in which, $v_i$ is connected to terminal vertex *s* in the induced graph G \ C if $y_i' = \alpha$; $v_i$ is connected to terminal vertex *t* in the induced graph G \ C if $y_i' = \beta$.

If $|V \setminus \{s,t\}| = n$, the constant can be zero. i.e., if every non-terminal vertex of V corresponds to individual variable with range of size two, each configuration leads to unique cut. Thus, the objective function is said to be ***exact FNO-function***, if the constant is zero.

Let's start with a simple but important observation

## LEMMA 5.3.2

*Let* $O_1$ *and* $O_2$ *be two objective functions of n variables* $y_1, y_2,...., y_n$ *with range of size two and both the functions differ by a constant.* $O_1$ *is an* $FNO-function$ *iff* $O_2$ *is an* $FNO-function$.

Proof: Let's assume that, $O_1$ is an FNO – function. By Definition 5.3.1, there must exist a flow network G, such that, for any configuration $y_1', y_2', ...., y_n' \left( y_i' \in \{\alpha, \beta\}, \forall i \in \{1, 2, ...n\} \right)$ of variables $y_1, y_2, ...., y_n$ the difference between $O_1(y_1', y_2', ...., y_n')$ and the cost of a minimum cut (considered over all the cuts C on the flow network in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C) on G differs by a constant. As $O_1$ and $O_2$ differ by a constant, $O_1(y_1', y_2', ...., y_n')$ and the cost of a minimum cut (considered over all the cuts C on the flow network in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C) on G also differs by a constant. This proves that, $O_2$ is also an FNO – function.

The converse can also be proved with similar argument.

### 5.3.1 CLASS OF FUNCTIONS INVOLVING CLIQUES OF SIZE AT MOST TWO

We initially confine our attention on characterization of a particular subclass of objective functions for FNO property. We call this subclass $O^2$ .First, let's define the class. $O^2$ is class of all objective functions those can be represented as sum of terms involving clique of size at most two. To be more specific, every member of this class can be expressed as

$$O(y_1, y_2, ....., y_n) = \sum_{i=1}^{n} O_i(y_i) + \sum_{1<i<j<n} O_{ij}(y_i, y_j) \tag{5.1}$$

If follows that, this class of objective functions is FNO iff the member functions satisfy the following property.

$$O_{ij}(\alpha, \beta) + O_{ij}(\beta, \alpha) - O_{ij}(\alpha, \alpha) - O_{ij}(\beta, \beta) \geq 0 \tag{5.2}$$

Now onwards, we will refer to all the objective functions of class $O^2$ satisfying (5.2) as R – function.

### THEOREM 5.3.1.1

*Every R-function of $O^2$ is an FNO – function.*

***Proof:*** Let O be a R – function of $O^2$. We prove that, O is an FNO – function by constructing a flow network satisfying the property that, the difference between $O_1(y_1', y_2', ...., y_n')$ and the cost of a minimum cut (considered over all the cuts C on the flow network in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C) on G differs by a constant, where $y'_1, y'_2, ...., y'_n$ is an arbitrary configuration of the variables $y_1, y_2, ...., y_n$. Construct the graph as follows:

The network will contain vertex set $\{s, t, v_1, v_2, ...., v_n\}$ containing $n$ non-terminal vertices $v_1, v_2, ...., v_n$ and two terminal vertices $s$ and $t$.

If $w_{si} = O_i(\beta) - O_i(\alpha) > 0$, add a terminal edge $e^s_{v_i}$ of weight $w_{si}$ connecting non-terminal vertex $v_i$ and terminal vertex $s$.

If $w_{si} \le 0$, add a terminal edge $e^t_{v_i}$ of weight (- $w_{si}$ ) connecting non-terminal vertex $v_i$ and terminal vertex $t$.

Now, let's consider the terms involving clique of size two. i.e. term $O_{ij}$ involving neighbor interaction of size two. Note that,

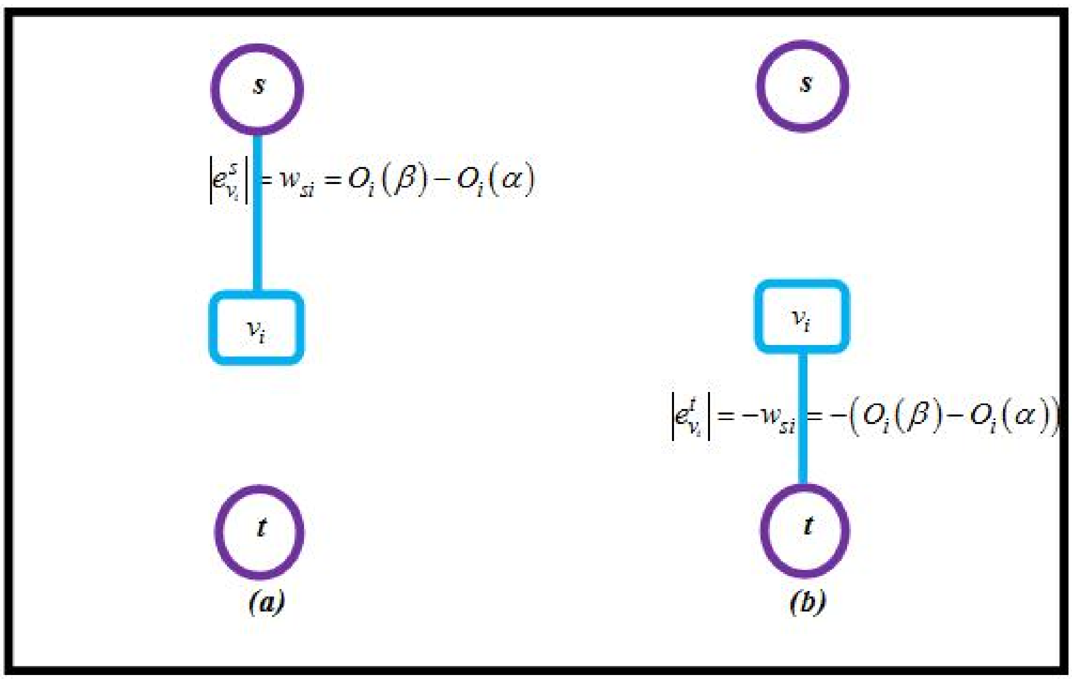


*Figure 5.1: (a) edges in flow network corresponding to $O_i$ when (a) $w_{si} = O_i(\beta) - O_i(\alpha) > 0$ (b) $w_{si} = O_i(\beta) - O_i(\alpha) < 0$*

$O_{ij} =$

| | |
|---|---|
| $O_{ij}(\alpha,\alpha)$ | $O_{ij}(\alpha,\beta)$ |
| $O_{ij}(\beta,\alpha)$ | $O_{ij}(\beta,\beta)$ |

$=$ $O_{ij}(\alpha,\alpha) +$

| | |
|---|---|
| 0 | 0 |
| $O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha)$ | $O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha)$ |

| | |
|---|---|
| 0 | $O_{ij}(\beta,\beta)-O_{ij}(\beta,\alpha)$ |
| 0 | $O_{ij}(\beta,\beta)-O_{ij}(\beta,\alpha)$ |

+

| | |
|---|---|
| 0 | $O_{ij}(\alpha,\beta)+O_{ij}(\beta,\alpha)-O_{ij}(\beta,\beta)-O_{ij}(\alpha,\alpha)$ |
| 0 | 0 |

(5.3)

The term first term $O_{ij}(\alpha,\alpha)$ of (5.3) being a constant term doesn't require any edge in the graph representing it. The second and third terms in form of tables in (5.3) are expressed in terms of $O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha)$ and $O_{ij}(\beta,\beta)$ - $O_{ij}(\beta,\alpha)$, which are functions of single variable $y_i$ and $y_j$ respectively.

We will add a terminal edge $e^s_{v_i}$ of weight $O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha)$ connecting non-terminal vertex $v_i$ and terminal vertex *s* provided $O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha) > 0$, otherwise, we will add a terminal edge $e^t_{v_i}$ of weight -($O_{ij}(\beta,\alpha)$ - $O_{ij}(\alpha,\alpha)$) connecting non-terminal vertex $v_i$ and terminal vertex *t*.(Refer to Figure 5.1 (a) and 5.1 (b))

Similarly, We will add a terminal edge $e^s_{v_j}$ of weight $O_{ij}(\beta,\beta)$ - $O_{ij}(\beta,\alpha)$ connecting non-terminal vertex $v_j$ and terminal vertex *s* provided $O_{ij}(\beta,\beta)$ - $O_{ij}(\beta,\alpha) > 0$, otherwise, we will add a terminal edge $e^s_{v_j}$ of weight -($O_{ij}(\beta,\beta)$ - $O_{ij}(\beta,\alpha)$) connecting non-terminal vertex $v_j$ and terminal vertex *t*.

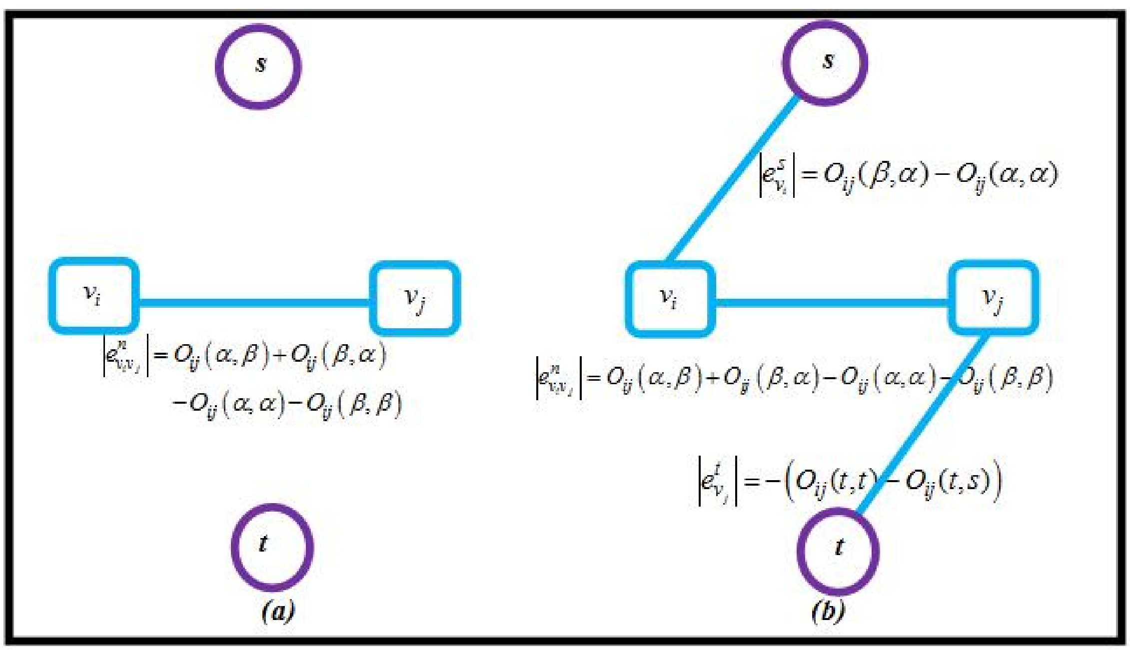


*Figure 5.2 (a) non-terminal edge $e^n_{v_iv_j}$ with weight $O_{ij}(s,t)+O_{ij}(t,s)$ - $O_{ij}(s,s)$ - $O_{ij}(t,t)$ due to term $O_{ij}$ (b) Complete network flow due to $O_{ij}$*

In case of parallel edges, we can simply replace them by a single edge whose weight is sum of weights of all the parallel edges. (We will prove justifiability of this in Theorem 5.3.2.15).

The last term of (5.3) in the tabular form is expressed in terms of $O_{ij}(\alpha,\beta)+O_{ij}(\beta,\alpha)$- $O_{ij}(\alpha,\alpha)$- $O_{ij}(\beta,\beta)$, which is function of both variables $y_i$ and $y_j$. We take care of this term by adding a non-terminal edge $e^n_{v_i v_j}$ connecting $v_i$ and $v_j$ with weight $O_{ij}(\alpha,\beta)+O_{ij}(\beta,\alpha)$- $O_{ij}(\alpha,\alpha)$- $O_{ij}(\beta,\beta)$, Note that the weight is positive because O is an R-function.(Refer to Figure 5.2 (a) and (b))

It can be easily checked that, the flow network satisfies the property required in the definition of FNO – function and hence proves that, O is an FNO – function.

The converse of Theorem 5.3.1.1 is true. i.e. Being R – function is also a necessary condition for a class $O^2$. The proof of the converse is presented in Theorem 5.3.2.13.

Now, let's try to analyze how this result facilitated the application of the technique in some known models. Consider the growth move model discussed in 3.2.2. In the model, a complicated graph involving reserve vertices was considered. Using the graph construction shown in Theorem 5.3, the network flow will be much simpler and won't involve any additional vertex. Thus, the computational complexity can be reduced. The model was theoretically proved to achieve local minimum up to some scalar multiple of global minimum provided the term $\varphi_v$ has non-negative range for all non-terminal vertices and the term $\psi_{u,v}$ is a metric. The theoretical success of the model is shown to be independent of the representation of $\varphi_v$ but dependent only on the fact that $\psi_{u,v}$. According to Theorem 5.3, the objective function is FNO provided it is an R – function. To show the equivalence of both the terminology, we need to show that, $\psi_{u,v}$ is a metric iff Objective function to be minimized using growth move is an R – function. Note that,

$$\psi_{u,v}\left(X(u),X(v)\right)=O_{ij}\left(\alpha,\alpha\right),\ \psi_{u,v}\left(X(u),\sigma\right)=O_{ij}\left(\alpha,\beta\right)$$
$$\psi_{u,v}\left(\sigma,X(v)\right)=O_{ij}\left(\beta,\alpha\right),\qquad \psi_{u,v}\left(\sigma,\sigma\right)=O_{ij}\left(\beta,\beta\right)$$

Assume that, $\psi$ is a metric, we have to prove that, O is an R – function. If $\psi$ is a metric, $\psi_{u,v}\left(\sigma,\sigma\right)=0$ and $\psi_{u,v}\left(X(u),X(v)\right)\le\psi_{u,v}\left(X(u),\sigma\right)+\psi_{u,v}\left(\sigma,X(v)\right)$. These together imply that, $\psi_{u,v}\left(\sigma,\sigma\right)+\psi_{u,v}\left(X(u),X(v)\right)\le\psi_{u,v}\left(X(u),\sigma\right)+\psi_{u,v}\left(\sigma,X(v)\right)$, which is equivalent to

$O_{ij}(\alpha,\beta)+O_{ij}(\beta,\alpha)-O_{ij}(\alpha,\alpha)-O_{ij}(\beta,\beta)\ge 0$. This proves that, the objective function is an R – function.

## THEOREM 5.3.1.2

*Minimizing arbitrary function of* $O^2$ *which is not an R-function is NP – hard.*

*Proof:* The result is proved in much general form in Theorem 5.3.2.14.

## 5.3.2 CLASS OF FUNCTIONS INVOLVING CLIQUES OF SIZE ATMOST THREE

In image processing the constraint of structural interdependence is being taken care of by $\psi$ . In order to keep the objective function simple, the term $\psi$ is defined in such a way that it takes into consideration the dependence of pairs of neighbouring pixels. Actually, larger groups of pixels situated geometrically in nearby region have dependence on one another. Thus, if we consider the structural interdependence of cliques of higher order, it more precisely encodes the constraint of structural interdependence. But, higher the order of the clique size, harder it is to handle such objective functions. Thus, moderate order clique size is being preferred to balance between the quality of encoding of the constraint in the objective function and computational complexity of the objective function. In this sub-section, we will consider class $O^3$ of all objective functions of order three and try to characterize it's sub-class that is Flow Network Optimizable (FNO).

Before defining the class $O^3$, let's define the notion of projection of objective function of variables of range of size two. If some of the variables of the given objective function are kept fixed (with fixed values from the range set), the resulting objective function is called projection of the given objective function.

### DEFINITION 5.3.2.1

Let O be an objective function of *n* variables $y_1,....,y_n$ of range of size two. If the first *k* variables $y_1,....,y_k$ are fixed, i.e. we assume that, $y_1 = y_1', y_2 = y_2'....,y_k = y_k'$, where $y_1',....,y_k' \in \{\alpha,\beta\}$ are some fixed values of the variables $y_1,....,y_k$, then, the function O turns out to be function of remaining *n–k* variables $y_{n-k+1},....,y_n$. Thus, projection of O on $y_1 = y_1', y_2 = y_2'....,y_k = y_k'$, denoted by $proj[\mathrm{O}(y_1 = y_1', y_2 = y_2'....,y_k = y_k')]$ is defined as, $proj[\mathrm{O}(y_1 = y_1', y_2 = y_2'....,y_k = y_k')]= O\left(y_1',....,y_k',y_{n-k+1},...y_n\right)$, where $y_1',....,y_k' \in \{\alpha,\beta\}$ are some fixed values.

For objective functions involving more than two variables of size two, the notion of R – function can be generalized as follows:

### DEFINITION 5.3.2.2

An objective function O of variables of range of size two is said to be an *R – function*, if every possible projection of two variables of the objective function O is an R – function.

Note that, definition 5.3.2.2 implies the following:

(i) If O is an objective function of single variable of range of size two, O is an *R – function* as it does not have any projection of two variables which is not regular.
(ii) If O is an objective function of two variables of range of size two, O itself is the projection of two variables and it is the only such projection and it is an R – function if, $O_{ij}(\alpha,\beta)+O_{ij}(\beta,\alpha)-O_{ij}(\alpha,\alpha)-O_{ij}(\beta,\beta)\geq 0$. This proves that, both the definitions

of R-functions are equivalent in the common domain (i.e. for functions with one and two variables)

Now, let's define the class $O^3$. The class is a collection of all objective functions of variables of range of size two which can be expressed as sum of terms of clique size at most three. Mathematically, $O^3$ is collection of all objective functions O of variables $y_1,....,y_n$ with range of size two which can be expressed as,

$$O\left(y_1,y_2,.....,y_n\right)=\sum_{i=1}^{n}O_i\left(y_i\right)+\sum_{1<i<j<n}O_{ij}\left(y_i,y_j\right)+\sum_{1<i<j<k<n}O_{ijk}\left(y_i,y_j,y_k\right) \tag{5.4}$$

## NOTES

***(i)*** If objective function O expressed in (5.4) is an R – function, it does not necessarily imply that all functions involved in the summation on the right hand side of (5.4) are R – functions.

***(ii)*** If O given by (5.4) is an R – function, there exist it's at least one representation of O in the form given by (5.4), where each function involved in the summation of the expression is an R – function.

## DEFINITION 5.3.2.3

$\theta:\hat{\mathrm{O}}\rightarrow\mathbb{R}$ is a function defined on $\hat{\mathrm{O}}$, the set of all objective functions of variables $y_1,....,y_n$ of range of size two as follows:

$$\theta\left(O\right)=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')O\left(y_1',y_2',....,y_n'\right),\forall O\in\hat{\mathrm{O}} \tag{5.5}$$

Where, $y_1',y_2',....,y_n'$ is a configuration of the $n$ variables with $\sum_{i=1}^{n}y_i'=n_1\alpha+n_2\beta\left(n_1,n_2\in\mathbb{Z}^+\right)$

and $\gamma(y_1',y_2',....,y_n')$ is defined as,

$$\gamma(y_1',y_2',....,y_n')=\begin{cases}-1, if\left(n_2,2\right)=1\\ \quad 1,\text{otherwise}\end{cases}$$

Note that $\left(n_2,2\right)=1$ denotes that, the pair of integers $n_2$ and 2 are relatively prime.

In simple words, the functional $\theta$ is the difference of the sums of values of objective functions under all possible configurations of n binary variables with even number of variables with value $\beta$ and that of odd number of variables with value $\beta$.

## LEMMA 5.3.2.4

*$\theta$ is a linear functional.*

Proof: To prove that, $\theta$ is linear, we need to prove that,

(i) $\theta(O_1+O_2)=\theta(O_1)+\theta(O_2)$ and

(ii) $\theta(a.O)=a.\theta(O)$

where $O_1$ and $O_2$ are arbitrary objective functions of n variables $y_1,....,y_n$ of range of size two and $a$ is a constant.

First, let's consider (i).

$$\theta(O_1+O_2)=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')(O_1+O_2)(y_1',y_2',....,y_n')$$

$$=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')\left(O_1(y_1',y_2',....,y_n')+O_2(y_1',y_2',....,y_n')\right)$$

$$=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\left(\gamma(y_1',y_2',....,y_n').O_1(y_1',y_2',....,y_n')+\gamma(y_1',y_2',....,y_n')O_2(y_1',y_2',....,y_n')\right)$$

($\because \gamma(y_1',y_2',....,y_n')$ is a constant independent of $O_1$ and $O_2$ and is dependent only on the configuration $y_1',y_2',....,y_n'$ of $y_1,....,y_n$)

$$=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n').O_1(y_1',y_2',....,y_n')+\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n').O_2(y_1',y_2',....,y_n')$$

Thus, $\theta(O_1+O_2)=\theta(O_1)+\theta(O_2)$

This proves (i).

Now, $$\theta(a.O)=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')(a.O)(y_1',y_2',....,y_n')$$

$$=\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')a.\left(O(y_1',y_2',....,y_n')\right)$$

$$=a.\left(\sum_{\substack{y_i'\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y_1',y_2',....,y_n')O(y_1',y_2',....,y_n')\right)$$

($\because \gamma(y_1',y_2',....,y_n')$ is a constant independent of $O$ and is dependent only on the configuration $y_1',y_2',....,y_n'$ of $y_1,....,y_n$).

Thus, $\theta(a.O) = a.(\theta(O))$, which proves (ii).

This proves that, $\theta$ is linear functional.

## LEMMA 5.3.2.5

*Let O be an objective function of n variables of n variables* $y_1,\ldots.,y_n$ *of range of size two. If there exist k* $(1 \le k \le n)$ *such that, O is independent of* $y_k$ *,* $\theta(\mathrm{O})$ *is zero.*

***Proof:*** Given that, O is independent of $k^{th}$ binary variable where for some fixed k with $1 \le k \le n$.

i.e. $\theta(y_1', y_2', .., y_k = \alpha, .., y_n') = \theta(y_1', y_2', .., y_k = \beta, .., y_n')$ for every configuration $y_1', \ldots, y_{k-1}', y_{k+1}', \ldots, y_n'$ of the remaining k-1 variables $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$.

$$\theta(O) = \sum_{\substack{y_i' \in \{\alpha,\beta\} \\ 1 \le i \le n}} \gamma(y_1', y_2', \ldots., y_n') O(y_1', y_2', \ldots., y_n') \quad (5.6)$$

In the expression (5.6), the sum is considered over all possible configurations $y_1', y_2', \ldots., y_n'$ of $y_1, \ldots., y_n$. Note that, for every configuration $y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n'$, there is a corresponding configuration $y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n'$, where both the configurations agree in terms of values of all variables except that of $y_k$. Thus, (5.6) can be rewritten as,

$$\theta(O) = \sum_{\substack{y_i' \in \{\alpha,\beta\} \\ 1 \le i \le n \\ i \ne k}} \gamma(y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n') O(y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n')$$

$$+ \sum_{\substack{y_i' \in \{\alpha,\beta\} \\ 1 \le i \le n \\ i \ne k}} \gamma(y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n') O(y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n') \quad (5.7)$$

Note that, for pair of each corresponding configurations $y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n'$ and $y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n'$ of variables $y_1, \ldots., y_n$,

$$\gamma(y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n') = -\gamma(y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n')$$
(5.8)

If $\sum_{i=1}^{n} y_i' = n_1\alpha + n_2\beta \left(n_1, n_2 \in \mathbb{Z}^+\right)$ for the configuration $y_1', \ldots, y_{k-1}', y_k' = \alpha,\ y_{k+1}',\ \ldots, y_n'$ of $y_1, \ldots., y_n$ with $n_2$ being an odd number, similar representation for configuration $y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n'$ must have $n_2$ as an even number, because both the configurations differ only in terms of the variable $y_k$. Thus, if $\gamma(y_1', \ldots, y_{k-1}', y_k' = \alpha, y_{k+1}', \ldots, y_n')$ = -1, we must have $\gamma(y_1', \ldots, y_{k-1}', y_k' = \beta, y_{k+1}', \ldots, y_n')$ =1.

Similarly, if $\gamma(y'_1,...,y'_{k-1},y'_k=\alpha,y'_{k+1},...,y'_n)=$ 1, we must have $\gamma(y'_1,...,y'_{k-1},y'_k=\beta,y'_{k+1},...,y'_n)=$ -1. This proves (5.8).

From (5.8), it follows that, corresponding to each term of the first sum of (5.7), there is a term in the second sum of (5.7), which cancels it. This proves that, (5.7) sums up to zero.

Thus, we have proved that, if an objective function is independent of at least one of its variable, its value under $\theta$ .

## LEMMA 5.3.2.6

*Let $O_{ij}$ be an objective function of two variables with range of size two. If $O_{ij}$ is R-function, $\theta\left(O_{ij}\right)<0$ .*

*Proof:* Given that $O_{ij}$ is R – function.

$$\theta\left(O\right)=\sum_{\substack{y'_i\in\{\alpha,\beta\}\\1\le i\le n}}\gamma(y'_1,y'_2)O\left(y'_1,y'_2\right)$$

$$=\gamma(\alpha,\alpha)O\left(\alpha,\alpha\right)+\gamma(\alpha,\beta)O\left(\alpha,\beta\right)+\gamma(\beta,\alpha)O\left(\beta,\alpha\right)+\gamma(\beta,\beta)O\left(\beta,\beta\right)$$

$$=O\left(\alpha,\alpha\right)-O\left(\alpha,\beta\right)-O\left(\beta,\alpha\right)+O\left(\beta,\beta\right)$$

$$\left(\because\gamma\left(\alpha,\alpha\right)=\gamma\left(\beta,\beta\right)=1,\gamma\left(\alpha,\beta\right)=\gamma\left(\beta,\alpha\right)=-1\right)$$

$< 0$ ($\because O$ is R- function)

This proves the result.

This Lemma means that, for every objective function of two variables of range of size two, it is regular only if $\theta\left(O_{ij}\right)<0$ .

## NOTATION

The total number of projections $p$ of objective function $O_{ijk}$ of three variables (with range of size two) with $\theta(p)>0$ is denoted by $n_p(O_{ijk})$ .

## LEMMA 5.3.2.7

*Let $O_{ijk}$ be an objective function of three variables with range of size two. Then, $O_{ijk}$ is R-function iff $n_p(O_{ijk})$ =0.*

***Proof:*** Let's assume that, $O_{ijk}$ is regular. By Definition 5.3.2.2, all projections $O_{ijk}$ of two variables with range of size two are R - functions. Hence, by Lemma 5.3.2.6, for every projection p of two variables of the objective function $O_{ijk}$ , $\theta(p)<0$. This proves that, there does not exist any projection p of two variables of the objective function $O_{ijk}$ for which $\theta(p)>0$ .

Thus, $n_p(O_{ijk})$ , the number of projections of two variables of $O_{ijk}$ with $\theta(p)>0$ is zero. This proves the first part.

Now assume that, $n_p(O_{ijk})$ =0. This proves that, $O_{ijk}$ does not have any projections of two variables with $\theta(p) > 0$, which means that, all projections of $O_{ijk}$ with two variables have non-negative value under $\theta$. This proves that, $O_{ijk}$ is an R-function.

## LEMMA 5.3.2.8

*Let O be an R-function of* $O^3$*, such that,* $O(y_1, y_2, ....., y_n) = \sum_{1<i<j<k<n} O_{ijk}(y_i, y_j, y_k)$.

*Then, there exists another representation of O given by,* $O(y_1, y_2, ....., y_n) = \sum_{1<i<j<k<n} O'_{ijk}(y_i, y_j, y_k)$ *such that,* $\sum_{1<i<j<k<n} n_p(O_{ijk}) > \sum_{1<i<j<k<n} n_p(O'_{ijk})$.

***Proof:*** As $\sum_{1<i<j<k<n} n_p(O_{ijk}) > 0$, there must exist at least one function $O_{ijk}$ of three variables $y_i, y_j$ and $y_k$ with range of size two with positive value under $n_p$. Without loss of generality, let's assume that, the function is $O_{123}$ with variables $y_1, y_2$ and $y_3$ such that, $n_p(O_{123}) > 0$. Thus, there must exist at least one projection of $O_{123}$ with two variables, whose value under $\theta$ is positive. Thus, either of the projections $proj[O_{123}(y_1 = y_1')]$, $proj[O_{123}(y_2 = y_2')]$ or $proj[O_{123}(y_3 = y_3')]$ may have positive value under $\theta$. Let's assume that, $\theta\left(proj[O_{123}(y_3 = y_3')]\right) > 0$ for some value of $y_3'$. Note that, if $\theta\left(proj[O_{123}(y_3 = y_3')]\right) \le 0$, we can rename the indices *i, j* and *k* in such a way that, $\theta\left(proj[O_{123}(y_3 = y_3')]\right) > 0$. Thus, either $\theta(proj[O_{123}(y_3 = \alpha)]) > 0$ or $\theta(proj[O_{123}(y_3 = \beta)]) > 0$. Let's define,

$$\eta_k = \max_{y_k' \in \{\alpha, \beta\}} \theta(proj[O_{12k}(y_k = y_k')]), k \in \mathbb{N} \setminus \{1,2,3\} \text{ and } \eta_3 = \sum_{k=4}^{n} (-\eta_k).$$

Now, define $O'_{ijk}$ as follows:

$$O'_{ijk} = \begin{cases} O_{ijk}, \text{ if } i \ne 1, j \ne 2 \text{ and } k \in \{1,2\} \\ O_{12k} - \hat{\eta}_k, \text{otherwise} \end{cases}$$

(5.9)

Where, $\hat{\eta}_k$ is function of two variables $y_1$ and $y_2$ defined as follows:

$$\hat{\eta}_k(\alpha, \alpha) = 0, \hat{\eta}_k(\alpha, \beta) = 0, \hat{\eta}_k(\beta, \alpha) = 0, \hat{\eta}_k(\beta, \beta) = \eta_k$$

(5.10)

Now, $\sum_{1<i<j<k<n} O'_{ijk}(y_i, y_j, y_k) = \sum_{\substack{1<i<j<k<n \\ i\ne1, j\ne2, k\in\{1,2\}}} O'_{ijk}(y_i, y_j, y_k) + \sum_{2<k<n} O'_{12k}(y_i, y_j, y_k)$

$$\therefore \sum_{1<i<j<k<n} O'_{ijk}(y_i, y_j, y_k) = \sum_{\substack{1<i<j<k<n \\ i\ne1, j\ne2, k\in\{1,2\}}} O_{ijk}(y_i, y_j, y_k) + \sum_{2<k<n} (O_{12k} - \hat{\eta}_k)(y_1, y_2, y_k)$$

$$\therefore \sum_{1<i<j<k<n} O'_{ijk}(y_i, y_j, y_k) = \sum_{\substack{1<i<j<k<n \\ i\ne1, j\ne2, k\in\{1,2\}}} O_{ijk}(y_i, y_j, y_k) + \sum_{2<k<n} O_{12k}(y_1, y_2, y_k)$$

$$- \sum_{2<k<n} \hat{\eta}_k \left( y_1, y_2, y_k \right)$$

$$= \sum_{1<i<j<k<n} O_{ijk} \left( y_i, y_j, y_k \right) - \sum_{2<k<n} \hat{\eta}_k \left( y_1, y_2, y_k \right)$$

$$= \sum_{1<i<j<k<n} O_{ijk} \left( y_i, y_j, y_k \right) - \hat{\eta}_3 \left( y_1, y_2, y_k \right) - \sum_{3<k<n} \hat{\eta}_k \left( y_1, y_2, y_k \right)$$

$$= \sum_{1<i<j<k<n} O_{ijk} \left( y_i, y_j, y_k \right) - \left( \sum_{k=4}^{n} \left( -\eta_k \left( y_1, y_2, y_k \right) \right) \right) - \sum_{3<k<n} \hat{\eta}_k \left( y_1, y_2, y_k \right)$$

$$\left( \because \eta_3 = \sum_{k=4}^{n} (-\eta_k) \right)$$

$$= \sum_{1<i<j<k<n} O_{ijk} \left( y_i, y_j, y_k \right) + \left( \sum_{k=4}^{n} \left( \eta_k \left( y_1, y_2, y_k \right) \right) \right) - \sum_{3<k<n} \hat{\eta}_k \left( y_1, y_2, y_k \right)$$

$$= \sum_{1<i<j<k<n} O_{ijk} \left( y_i, y_j, y_k \right)$$

Thus, $O\left(y_1, y_2, \ldots\ldots, y_n\right) = \sum_{1<i<j<k<n} O_{ijk}\left(y_i, y_j, y_k\right) = \sum_{1<i<j<k<n} O'_{ijk}\left(y_i, y_j, y_k\right)$. This proves that, we have obtained an alternative expression for $O\left(y_1, y_2, \ldots\ldots, y_n\right)$. Now, the only point remains to be proved is that, this new expression of *O* has less no. of positive projections of two variables than the older one.

It should be noted that, $\theta\left(\hat{\eta}_k\right) = \hat{\eta}_k\left(\alpha,\alpha\right) + \hat{\eta}_k\left(\beta,\beta\right) - \hat{\eta}_k\left(\alpha,\beta\right) - \hat{\eta}_k\left(\beta,\alpha\right)$

$\therefore \theta\left(\hat{\eta}_k\right) = \eta_k$ $\quad \left(\because \hat{\eta}_k\left(\alpha,\alpha\right) = 0, \hat{\eta}_k\left(\alpha,\beta\right) = 0, \hat{\eta}_k\left(\beta,\alpha\right) = 0, \hat{\eta}_k\left(\beta,\beta\right) = \eta_k\right)$

(5.11)

From (5.9), it's clear that, $O'_{ijk}$ and $O_{ijk}$ are different only when $i = 1, j = 2$ and $3 \le k \le n$. Thus, for all $O'_{ijk}$ not lying in this group, $\theta\left( proj[O'_{ijk}(y_k = y_k')] \right) = \theta\left( proj[O_{ijk}(y_k = y_k')] \right)$. We need to prove that, there is at least one term $O'_{12k}$ $(3 \le k \le n)$, such that, $\theta\left( proj[O'_{12k}(y_k = y_k')] \right) < 0$.

Let's first consider $O'_{12k}$ $(4 \le k \le n)$.

$$\theta\left(proj[O'_{12k}(y_k = y_k')]\right) = \theta\left(proj[O_{12k}(y_k = y_k')]\right) - \theta\left(proj[\hat{\eta}_k(y_k = y_k')]\right)$$

$$= \theta\left(proj[O_{12k}(y_k = y_k')]\right) - \theta\left(proj[\hat{\eta}_k(y_k = y_k')]\right)$$

$$\leq \left(\max_{y_k' \in \{\alpha,\beta\}} \left(\theta\left(proj[O_{12k}(y_k = y_k')]\right)\right)\right) - \theta\left(proj[\hat{\eta}_k(y_k = y_k')]\right)$$

$$\leq \eta_k - \eta_k$$

$$= 0$$

Thus, all of these projections are R – functions.

Let's consider the last and final modified function, i.e., $O'_{123}$.

$$\theta\left(proj[O'_{123}(y_3 = y_3')]\right) = \theta\left(proj[O_{123}(y_3 = y_3')]\right) - \theta\left(proj[\hat{\eta}_3(y_3 = y_3')]\right) \ (\because \text{By (5.9)})$$

$$= \theta\left(proj[O_{123}(y_3 = y_3')]\right) - \eta_3 \ (\because \text{By (5.11)})$$

$$= \theta\left(proj[O_{123}(y_3 = y_3')]\right) - \left(\sum_{k=4}^{n}(-\eta_k)\right) \left(\because \eta_3 = \sum_{k=4}^{n}(-\eta_k)\right)$$

$$= \theta\left(proj[O_{123}(y_3 = y_3')]\right) + \left(\sum_{k=4}^{n}\eta_k\right)$$

$$= \sum_{k=4}^{n}\theta\left(proj[O_{12k}(y_k = y_k')]\right),$$

where $y_k'$ is the configuration of $y_k$ that leads to $\eta_k \ (4 \leq k \leq n)$.

This proves that, $\theta\left(proj[O'_{123}(y_3 = y_3')]\right) = \theta\left(proj[O(y_3 = y_3', y_4 = y_4', \ldots\ldots, y_n = y_n')]\right)$ as $\sum_{k=4}^{n}\theta\left(proj[O_{12k}(y_k = y_k')]\right) = \theta\left(proj[O(y_3 = y_3', y_4 = y_4', \ldots\ldots, y_n = y_n')]\right)$. This also proves that, $\theta\left(proj[O'_{123}(y_3 = y_3')]\right) \leq 0$ as $O$ is given to be an R-function.

Thus, $\sum_{1<i<j<k<n} n_p(O_{ijk}) > \sum_{1<i<j<k<n} n_p(O'_{ijk})$. This proves the result.

## THEOREM 5.3.2.9

*Let $O$ be a function of $O^3$. Then, $O$ is an FNO- function if it is an $R$ – function.*

***Proof:*** Let's assume that, O is an FNO – function. We want to prove that, it is an R – function. As O is a function of $O^3$, it can be expressed as

$$O(y_1, y_2, \ldots\ldots, y_n) = \sum_{i=1}^{n} O_i(y_i) + \sum_{1<i<j<n} O_{ij}(y_i, y_j) + \sum_{1<i<j<k<n} O_{ijk}(y_i, y_j, y_k)$$

(5.12)

It is not necessary that, all the terms of (5.12) are R – functions. But, by Lemma 5.3.2.8, we can rewrite O in the form of (5.12) where function of each term of (5.12) is an R – function. Hence, without loss of generality, we assume that, each term of (5.12) represents an R- function. As O is function of *n* variables, we will construct a network with *n* non-terminal vertices and two terminals *s* and *t* and a few reserve vertices, in case if they are required.

We will construct sub-graph or sub-network corresponding to each term of (5.12) and finally merge them. This can be justified by (Theorem 5.3.15 ). Note that, the first summation and second summation of (5.12) involves functions of single and two variables respectively. In Theorem 5.3.1.1, we have derived a protocol of constructing networks/ graphs for functions of one and two variables of range of size two. We can use them to handle the terms of first and second summations of (5.12).

The third summation of (5.12) involves functions of three variables of range of size two. Let's try to develop a protocol for constructing a network or graph corresponding to this type of functions. For that, consider a function $O_{ijk}(y_i, y_j, y_k)$ of three variables $y_i, y_j$ and $y_k$. There are eight different configurations possible of the three variables. They are $(\alpha,\alpha,\alpha)$, $(\alpha,\alpha,\beta)$, $(\alpha,\beta,\alpha)$, $(\beta,\alpha,\alpha)$, $(\beta,\beta,\alpha)$, $(\alpha,\beta,\beta)$, $(\beta,\alpha,\beta)$ and $(\beta,\beta,\beta)$. Let's try to consider them in the form of a table.

$O_{ijk}$ =

| | |
|---|---|
| $O_{ij}(\alpha,\alpha,\alpha)$ | $O_{ij}(\alpha,\alpha,\beta)$ |
| $O_{ij}(\alpha,\beta,\alpha)$ | $O_{ij}(\alpha,\beta,\beta)$ |
| $O_{ij}(\beta,\alpha,\alpha)$ | $O_{ij}(\beta,\alpha,\beta)$ |
| $O_{ij}(\beta,\beta,\alpha)$ | $O_{ij}(\beta,\beta,\beta)$ |

Let $g = \left(O_{ij}(\alpha,\alpha,\alpha) + O_{ij}(\alpha,\beta,\beta) + O_{ij}(\beta,\alpha,\beta) + O_{ij}(\beta,\beta,\alpha)\right)$

$$- \left(O_{ij}(\alpha,\alpha,\beta) + O_{ij}(\alpha,\beta,\alpha) + O_{ij}(\beta,\alpha,\alpha) + O_{ij}(\beta,\beta,\beta)\right).$$

Here, there are two cases based on the sign of *g*.

***Case (i)****: g is non-negative*

Note that,

$$O_{ijk} = O_{ij}\left(\alpha,\alpha,\alpha\right) + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline a & a \\ \hline a & a \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline b & b \\ \hline 0 & 0 \\ \hline b & b \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & c \\ \hline 0 & c \\ \hline 0 & c \\ \hline 0 & c \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & d \\ \hline 0 & 0 \\ \hline 0 & d \\ \hline 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline e & 0 \\ \hline e & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline f & f \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & -g \\ \hline \end{array} \qquad (5.13)$$

Where,

$$a = O_{ij}\left(\beta,\alpha,\beta\right) - O_{ij}\left(\alpha,\alpha,\beta\right) = -\theta\left(proj[O_{ijk}(y_j = \alpha, y_k = \beta)]\right)$$

$$b = O_{ij}\left(\beta,\beta,\alpha\right) - O_{ij}\left(\beta,\alpha,\alpha\right) = -\theta\left(proj[O_{ijk}(y_i = \beta, y_k = \alpha)]\right)$$

$$c = O_{ij}\left(\alpha,\beta,\beta\right) - O_{ij}\left(\alpha,\beta,\alpha\right) = -\theta\left(proj[O_{ijk}(y_i = \alpha, y_j = \beta)]\right)$$

$$d = O_{ij}\left(\alpha,\alpha,\beta\right) + O_{ij}\left(\alpha,\beta,\alpha\right) - O_{ij}\left(\alpha,\alpha,\alpha\right) - O_{ij}\left(\alpha,\beta,\beta\right) = -\theta\left(proj[O_{ijk}(y_i = \alpha)]\right)$$

$$e = O_{ij}\left(\alpha,\alpha,\beta\right) + O_{ij}\left(\beta,\alpha,\alpha\right) - O_{ij}\left(\alpha,\alpha,\alpha\right) - O_{ij}\left(\beta,\alpha,\beta\right) = -\theta\left(proj[O_{ijk}(y_j = \alpha)]\right)$$

$$f = O_{ij}\left(\alpha,\beta,\alpha\right) + O_{ij}\left(\beta,\alpha,\alpha\right) - O_{ij}\left(\alpha,\alpha,\alpha\right) - O_{ij}\left(\beta,\beta,\alpha\right) = -\theta\left(proj[O_{ijk}(y_k = \alpha)]\right)$$

$$g = \left(O_{ij}\left(\alpha,\alpha,\alpha\right) + O_{ij}\left(\alpha,\beta,\beta\right) + O_{ij}\left(\beta,\alpha,\beta\right) + O_{ij}\left(\beta,\beta,\alpha\right)\right)$$

$$- \quad \left(O_{ij}\left(\alpha,\alpha,\beta\right) + O_{ij}\left(\alpha,\beta,\alpha\right) + O_{ij}\left(\beta,\alpha,\alpha\right) + O_{ij}\left(\beta,\beta,\beta\right)\right)$$

Note that, the first term in (5.12) is $O_{ij}\left(\alpha,\alpha,\alpha\right)$, which is a constant and hence, that does not contribute in the graph or network. The second term of (5.12) is in the tabular form and is in terms of $a = -\theta\left(proj[O_{ijk}(y_j = \alpha, y_k = \beta)]\right)$, which is projection of function $O_{ijk}$ under $\theta$ with negative sign, precisely, $-\theta\left(proj[O_{ijk}(y_j = \alpha, y_k = \beta)]\right)$, which is function of only one free variable $y_i$. Similarly, third and fourth terms of (5.12), appearing in tabular form, are

$-\theta\left(proj[O_{ijk}(y_i=\beta,y_k=\alpha)]\right)$ and $-\theta\left(proj[O_{ijk}(y_i=\alpha,y_j=\beta)]\right)$, which are functions of single variables $y_j$ and $y_k$ respectively. We have devised a mechanism of constructing graph for function of one variable in Theorem 5.3.1.1. Using that, we can handle these three terms.

Now, consider the fourth, fifth and sixth terms. They are $-\theta\left(proj[O_{ijk}(y_i=\alpha)]\right)$, $-\theta\left(proj[O_{ijk}(y_j=\alpha)]\right)$ and $-\theta\left(proj[O_{ijk}(y_k=\alpha)]\right)$ respectively. Thus, they are functions of pair of variables $\left(y_j,y_k\right),\left(y_i,y_j\right)$ and $\left(y_j,y_k\right)$ respectively. As O is an R –function, every projection of two variables has to be an R-function. This ensures that, $proj[O_{ijk}(y_i=\alpha)]$, $proj[O_{ijk}(y_j=\alpha)]$ and $proj[O_{ijk}(y_k=\alpha)]$ are R-functions and hence their values under $\theta$ are negative. Hence, these terms are non-negative ($\because (-\theta)$ is applied to each projection). These terms can be handled by procedure of graph construction for R-functions of two variables with range of size two developed in Theorem 5.3.1.1.(i.e. by adding three edges $e^n_{jk}, e^n_{ki}$ and $e^n_{ij}$ , each of weight *g*. Note that these weights are non-negative because of the assumption that g is non-negative) The only term of (5.12) remained to be addressed is the last one, which is in terms of *g*. As it is function of all the three variables $y_i$, $y_j$ and $y_k$, let's try to device a graph representing *g*.

For last term, we will construct a network with an additional (i.e. reserve) vertex $a_{ijk}$ and three non-terminal edges $e^n_{v_i a_{ijk}}, e^n_{v_j a_{ijk}}, e^n_{v_k a_{ijk}}$ and a terminal edge $e^t_{a_{ijk}}$ (Refer Figure 5.13(*a*)). We need to prove that, this sub-graph exactly encodes the last term of (5.12).

i.e.

| | |
|---|---|
| *g* | *g* |
| *g* | *g* |
| *g* | *g* |
| *g* | *0* |

= *g* +

| | |
|---|---|
| *0* | *0* |
| *0* | *0* |
| *0* | *0* |
| *0* | -*g* |

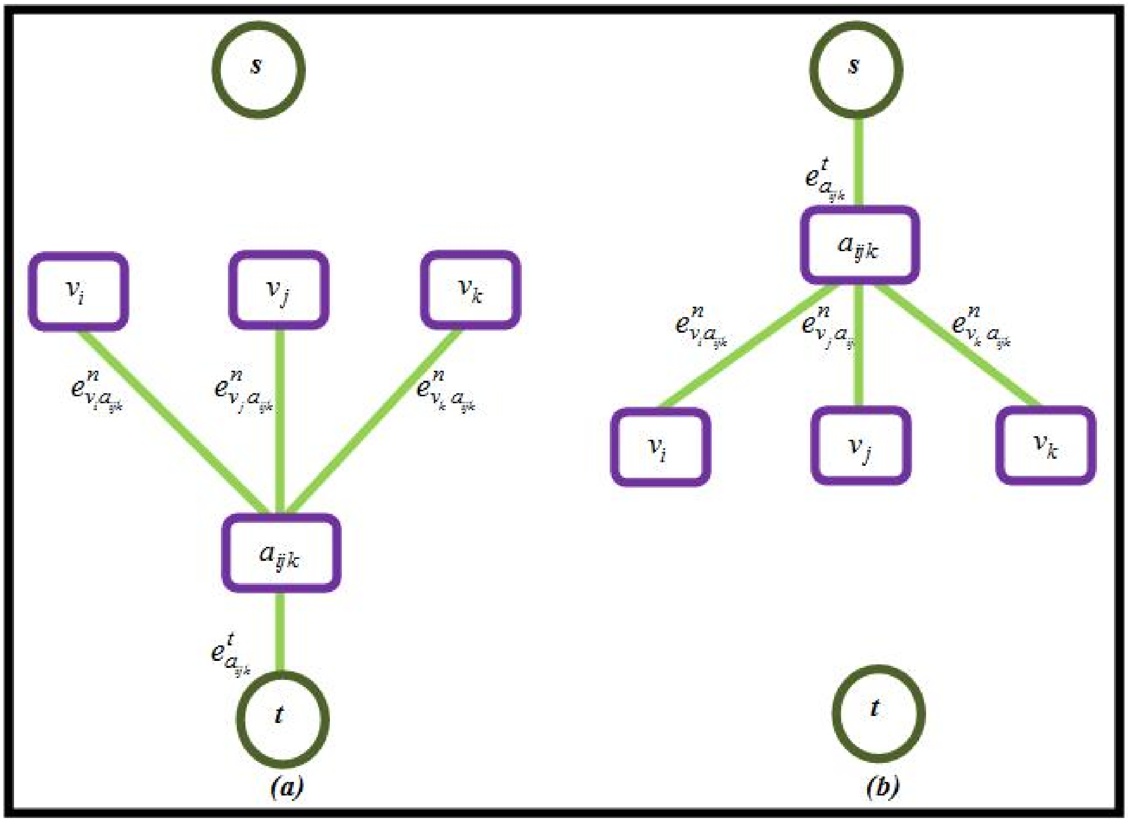


*Figure 5.3 (a) Part of network corresponding to g when g is non-negative (b) Part of network corresponding to g when g is negative*

Thus, we claim that, all the configurations except $(\beta, \beta, \beta)$ of the three variables leads to a minimum cut of the network (shown in Figure 5.3 (*a*)) of cost g and the configuration $(\beta, \beta, \beta)$ of the three variables $y_i$, $y_j$ and $y_k$ leads to a minimum cut $C$ of cost zero. Note that, if $y_i = \beta$, $y_j = \beta$ and $y_k = \beta$, then the induced graph G \ C should leave all the three vertices $v_i, v_j$ and $v_k$ disconnected from the terminal s. Note that, as the vertices $v_i, v_j$ and $v_k$ are already disconnected from the terminal vertex s, none of the edges $e^n_{v_i a_{ijk}}, e^n_{v_j a_{ijk}}, e^n_{v_k a_{ijk}}$ and $e^t_{a_{ijk}}$ are part of C. Thus, C = $\phi$. Thus, cost of the cut C is zero.

Now, it remains to prove that, for any configuration other than $(\beta, \beta, \beta)$ of the three variables $y_i$, $y_j$ and $y_k$, the cost of corresponding minimum cut is g. Let's consider that, $y_k \neq \beta$. i.e., $y_k = \alpha$. In this case the corresponding minimum cut C have two options: Either $e^n_{v_k a_{ijk}} \in C$ or $e^t_{a_{ijk}} \in C$. Note that, if $e^n_{v_k a_{ijk}} \in C$, then the cost of the cut C is at least $\left|e^n_{v_k a_{ijk}}\right| = g$. If $e^t_{a_{ijk}} \in C$, then, the cut will be independent of the configurations of remaining two variables and will have cost $|C| = \left|e^t_{a_{ijk}}\right| = g$. Refer to the table 5.4 for costs of the minimum cut corresponding to different configurations. This proves that, the cost of the minimum cut is zero for $(\beta, \beta, \beta)$ and $g$ otherwise. This proves the claim.

| Configuration | Corresponding minimum cut | Cost of the cut |
|---|---|---|
| $(\alpha,\alpha,\alpha)$ | $\{e^{t}_{a_{ijk}}\}$ | $g$ |
| $(\alpha,\alpha,\beta)$ | $\{e^{t}_{a_{ijk}}\}$ | $g$ |
| $(\alpha,\beta,\alpha)$ | $\{e^{t}_{a_{ijk}}\}$ | $g$ |
| $(\beta,\alpha,\alpha)$ | $\{e^{t}_{a_{ijk}}\}$ | $g$ |
| $(\beta,\beta,\alpha)$ | $\{e^{n}_{v_k a_{ijk}}\}$ | $g$ |
| $(\alpha,\beta,\beta)$ | $\{e^{n}_{v_i a_{ijk}}\}$ | $g$ |
| $(\beta,\alpha,\beta)$ | $\{e^{n}_{v_j a_{ijk}}\}$ | $g$ |
| $(\beta,\beta,\beta)$ | $\phi$ | 0 |

*Table 5.4 Minimum cuts and their costs corresponding to different configurations of* $y_i$, $y_j$ *and* $y_k$

***Case (i)**: g is negative*

Note that, we can express $O_{ijk}$ as,

$$O_{ijk} = O_{ij}(\beta,\beta,\beta) + \begin{array}{|c|c|}\hline a' & a' \\ \hline a' & a' \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline b' & b' \\ \hline 0 & 0 \\ \hline b' & b' \\ \hline 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline c' & 0 \\ \hline c' & 0 \\ \hline c' & 0 \\ \hline c' & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline d' & 0 \\ \hline 0 & 0 \\ \hline d' & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & e' \\ \hline 0 & e' \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline f' & f' \\ \hline 0 & 0 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|}\hline g & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \qquad (5.14)$$

Where,

$$a' = O_{ij}(\alpha,\beta,\alpha) - O_{ij}(\beta,\beta,\alpha) = -\theta\left(proj[O_{ijk}(y_j = \beta, y_k = \alpha)]\right)$$

$$b' = O_{ij}(\alpha,\alpha,\beta) - O_{ij}(\alpha,\beta,\beta) = -\theta\left(proj[O_{ijk}(y_i = \alpha, y_k = \beta)]\right)$$

$$c' = O_{ij}(\beta,\alpha,\alpha) - O_{ij}(\beta,\alpha,\beta) = -\theta\left(proj[O_{ijk}(y_i = \beta, y_j = \alpha)]\right)$$

$$d' = O_{ij}(\beta,\alpha,\beta) + O_{ij}(\beta,\beta,\alpha) - O_{ij}(\beta,\alpha,\alpha) - O_{ij}(\beta,\beta,\beta) = -\theta\left(proj[O_{ijk}(y_i = \beta)]\right)$$

$$e' = O_{ij}(\alpha,\beta,\beta)+O_{ij}(\beta,\beta,\alpha)-O_{ij}(\alpha,\beta,\alpha)-O_{ij}(\beta,\beta,\beta) = -\theta\left(proj[O_{ijk}(y_j=\beta)]\right)$$

$$f' = O_{ij}(\alpha,\beta,\beta)+O_{ij}(\beta,\alpha,\beta)-O_{ij}(\alpha,\alpha,\beta)-O_{ij}(\beta,\beta,\beta) = -\theta\left(proj[O_{ijk}(y_k=\beta)]\right)$$

$$g = \left(O_{ij}(\alpha,\alpha,\alpha)+O_{ij}(\alpha,\beta,\beta)+O_{ij}(\beta,\alpha,\beta)+O_{ij}(\beta,\beta,\alpha)\right)$$

$$- \quad \left(O_{ij}(\alpha,\alpha,\beta)+O_{ij}(\alpha,\beta,\alpha)+O_{ij}(\beta,\alpha,\alpha)+O_{ij}(\beta,\beta,\beta)\right)$$

The first term of (5.14), $O_{ij}(\beta,\beta,\beta)$ is constant and hence does not require any vertex and edge in the graph to represent itself. The next three terms $-\theta\left(proj[O_{ijk}(y_j=\beta, y_k=\alpha)]\right)$, $-\theta\left(proj[O_{ijk}(y_i=\alpha, y_k=\beta)]\right)$ and $-\theta\left(proj[O_{ijk}(y_i=\beta, y_j=\alpha)]\right)$ are functions of only single variable (and are positive) and hence can be handled by the construction devised in Theorem 5.3.1.1. The next three terms of (5.14) are $-\theta\left(proj[O_{ijk}(y_i=\beta)]\right)$, $-\theta\left(proj[O_{ijk}(y_j=\beta)]\right)$ and $-\theta\left(proj[O_{ijk}(y_k=\beta)]\right)$, which are projections of two variables of R-function $O_{ijk}$ and hence are themselves R – functions. Thus, all these three terms are positive ($\because \theta$ of every R-function of two variables is negative). We can construct graph for these functions of two variables by the procedure developed in Theorem 5.3.1.1. For the last and final term, we need to construct graph exactly encoding it. For that, we can construct graph shown in Figure 5.3 (b) with three four non-terminal vertices $v_i, v_j, v_k$ and $a_{ijk}$ (reserve vertex) and two terminal vertices s and t. There are four edges $e^n_{v_i a_{ijk}}, e^n_{v_j a_{ijk}}, e^n_{v_k a_{ijk}}$ and $e^s_{a_{ijk}}$ in the graph, each with weight (-g). We can prove giving the argument similar to one given in case (i), that this sub-network exactly encodes the last term.

This proves that, O is FNO-function.

To facilitate the future discussion, let's define the graph corresponding to variables with range of size two with particular configuration as follows:

## DEFINITION 5.3.2.10

Let *G* (*V, E*) be a network. Let $y_1, y_2, \ldots\ldots, y_n$ be variables each with range $\{\alpha,\beta\}$. Then the network *G* for configuration $y_1 = y'_1, y_2 = y'_2, \ldots\ldots, y_n = y'_n$ is denoted by $G(y'_1, y'_2, \ldots\ldots, y'_n)$, where $G(y'_1, y'_2, \ldots\ldots, y'_n)$ is a network with vertex set $V$, where $\{v_1, v_2, \ldots\ldots, v_n\} \subset V$ and edge set $E \cup E'$, where $E'$ is the set of all edges $e_{y_i}$ corresponding to each variable $y_i$. $e_{y_i}$ is defined as follows:

$$e_{y_i} = \begin{cases} e^s_{y_i}, \text{if } y_i = \alpha \\ e^t_{y_i}, \text{if } y_i = \beta \end{cases}, \text{ where } \left|e^s_{y_i}\right| = \left|e^t_{y_i}\right| = \infty .$$

It's clear from the definition that, in case of $y_i = \alpha$, the edge $e^s_{y_i}$ (of infinite cost) can never be part of the minimum cut. Similarly, in case of $y_i = \beta$, the edge $e^t_{y_i}$ (of infinite cost) can never be part of the minimum cut. Thus, the graph forces the minimum cut to respect the configuration of the variable.

## DEFINITION 5.3.2.11

An objective function O of n variables $y_1, y_2, \ldots\ldots, y_n$ is said to be *exact FNO – function* if there exist a network G(V,E) with $G(y_1', y_2', \ldots\ldots, y_n') = O(y_1', y_2', \ldots\ldots, y_n')$ for every configuration $y_1', y_2', \ldots\ldots, y_n'$ of variables $y_1, y_2, \ldots\ldots, y_n$ .

## LEMMA 5.3.2.12

Let O be a FNO – function and p be its projection. Then, p is also FNO – function.

Proof: Let $p = proj[O(y_1 = y_1', y_2 = y_2', \ldots\ldots, y_k = y_k')]$ be an arbitrary projection of objective function O( $y_1, y_2, \ldots\ldots, y_n$). Given that, O is FNO – function. We need to prove that, *p* is an FNO – function.

As O is an FNO- function, there exists a network G(V,E) , which represents it. We claim that, G' = G( $y_1 = y_1', y_2 = y_2', \ldots\ldots, y_k = y_k'$) represents the projection *p* $= proj[O(y_1 = y_1', y_2 = y_2', \ldots\ldots, y_k = y_k')]$. If possible, assume that, there is some configuration $y_{k+1}', y_{k+2}', \ldots\ldots, y_n'$ of the *n-k* variables $y_{k+1}, y_{k+2}, \ldots\ldots, y_n$, for which $G'(y_{k+1}', y_{k+2}', \ldots\ldots, y_n') \neq p(y_{k+1}', y_{k+2}', \ldots\ldots, y_n')$. Then, this implies that, $G(y_1', y_2', \ldots\ldots, y_k', y_{k+1}', y_{k+2}', \ldots\ldots, y_n') \neq \mathrm{O}(y_1', y_2', \ldots\ldots, y_k', y_{k+1}', y_{k+2}', \ldots\ldots, y_n')$, which is a contradiction with the fact that, *G* is a network which exactly represents the objective function O( $y_1, y_2, \ldots\ldots, y_n$).

This proves that, the network $G' = G(y_1 = y_1', y_2 = y_2', \ldots\ldots, y_k = y_k')$ must represent the projection $p = proj[O(y_1 = y_1', y_2 = y_2', \ldots\ldots, y_k = y_k')]$ and hence *p* is an FNO – function. As p was an arbitrary projection of O, it proves that, every projection of an FNO – function is an FNO – function.

In Theorem 5.3.1.1 and Theorem 5.3.2.9, we have proved that, being R-function is sufficient condition for the function being FNO – function in case of functions of class $O^2$ and $O^3$ respectively. Now, we attempt to prove that, it is also a sufficient condition for both of the classes. Note that, Lemma 5.3.2.12 has made the task easier. The lemma implies that if the result is true for all functions of two variables of range of size two, it has to hold for all functions of any finite number of variables too. Thus, we just need to prove the result for the class $O^2$ .

### THEOREM 5.3.2.13

*Let $O_{12}$ is an objective function of two variables* $y_1$ and $y_2$, both of range $\{\alpha,\beta\}$. *If O is not regular, it cannot be an FNO – function.*

***Proof:*** We will prove that, if $O_{12}$ is an FNO – function, it must be an R-function. For that, we start with an FNO – function $O_{12}$.

$$\theta(O_{12}) = \gamma(\alpha,\alpha)O(\alpha,\alpha)+\gamma(\alpha,\beta)O(\alpha,\beta)+\gamma(\beta,\alpha)O(\beta,\alpha)+\gamma(\beta,\beta)O(\beta,\beta)$$

$$= O(\alpha,\alpha)-O(\alpha,\beta)-O(\beta,\alpha)+O(\beta,\beta).$$

To prove that, is an R-function, it is sufficient to prove that, $\theta(O_{12}) < 0$. If possible, assume that, $\theta(O_{12}) > 0$. We will prove the result by arriving at a contradiction that, $O_{12}$ is not an FNO – function.

Let us define functions $O_1', O_2'$ and $O'$ as follows:

$$O_1'(\alpha,\alpha)=0, O_1'(\alpha,\beta)=-O_{12}(\alpha,\beta), O_1'(\beta,\alpha)=0, O_1'(\beta,\beta)=-O_{12}(\alpha,\beta)$$

$$O_2'(\alpha,\alpha)=0, O_2'(\alpha,\beta)=0, O_2'(\beta,\alpha)=-O_{12}(\beta,\alpha), O_2'(\beta,\beta)=-O_{12}(\beta,\alpha)$$

$$O'(\alpha,\alpha)=0, O'(\alpha,\beta)=0, O'(\beta,\alpha)=0, O'(\beta,\beta)=\theta(O_{12}).$$

It can be verified that,

$$O' = O_{12}(\alpha,\alpha)+O_1'+O_2'+O_{12}$$
(5.15)

As all the terms on the R.H.S of (5.15) are FNO – functions, $O'$ must also be an FNO – function.

Thus, there must exist a network *G*(*V, E*) representing $O'$ where $\{v_1,v_2\}\subset V$. Thus, by definition, there exists a constant c such that, for every configuration $y_1', y_2'$ of $y_1, y_2$, we get,

$$O'' = O'(y_1', y_2')+c \tag{5.16}$$

Thus, $O''(\alpha,\alpha)=c, O''(\alpha,\beta)=c, O''(\beta,\alpha)=c, O''(\beta,\beta)=\theta(O_{12})+c$

As c is the minimum value among all the values assigned to various configurations by $O''$, a minimum cut C on the network G will satisfy $|C|=c$. This also means that, $c\geq 0$. By Max-flow min-cut theorem, this implies that, the maximum amount of flow that can be sent from source *s* to sink $t$ in the graph is $c$. Then, the residual network $\tilde{G}$ of G obtained after pushing the flow of amount c exactly represents the function $O'$.

For pushing the maximum flow through network G and getting residual network, consider the network $\tilde{G}(y_1 = y_1', y_2 = y_2')$ i.e. push the maximum possible flow without using the edges corresponding to $y_1 = y_1'$ and $y_2 = y_2'$. This will yield a residual network $\tilde{G}(y_1 = y_1', y_2 = y_2')$. Let the maximum amount of flow that can be pushed through the residual network $\tilde{G}(y_1 = y_1', y_2 = y_2')$ be $\tilde{O}(y_1', y_2')$.

Then,

$$O'' = \tilde{O}(y_1', y_2') + c \tag{5.17}$$

From (5.16) and (5.17), we get,

$$O'(y_1', y_2') = \tilde{O}(y_1', y_2')$$

i.e. the network G exactly represents the function $O'(y_1', y_2')$ and thus, c = 0.

Note that, the maximum flow that can be pushed through the residual network $\tilde{G}$ is zero, which is also the cost of the minimum cut on $\tilde{G}$. This wipes out any possibility of path connecting the terminal vertices in $\tilde{G}$. If edges $e_{v_1}^t$ and $e_{v_2}^t$ are included in $\tilde{G}$ with, there must exist residual path in the graph as $O'(\beta, \beta) = \theta(O_{12}) > 0$. The augmenting path P must reach to terminal vertex t through at least one of the vertices $v_1$ and $v_2$. Without loss of generality, we assume that, the path contains at least $v_1$. Then we can construct a new augmenting path $P' \cup e_{v_1}^t$ by joining a segment $P'$ with $e_{v_1}^t$ in the graph $\tilde{G}(y_1 = \beta, y_2 = \alpha)$ obtained by inserting edges $e_{v_1}^t$ and $e_{v_2}^s$ with $\left|e_{v_1}^t\right| = \infty = \left|e_{v_2}^s\right|$. This implies that, some positive flow can be pushed through this augmenting path containing $e_{v_1}^t$. This implies that, $O'(\beta, \alpha) > 0$. This is contradiction with the fact that, $O'(\beta, \alpha) = 0$.

Thus, our assumption that, $\theta(O_{12}) > 0$ is false. Hence, $\theta(O_{12})$ is non-positive.

This proves that, the function $O_{12}$ is an R-function.

### THEOREM 5.3.2.14

*Optimization of objective functions of variables with range of size two is an NP – hard problem if the objective function is not an R – function.*

***Proof:*** We will prove that, optimization of objective function of class $O^2$ is NP – hard, if it is not an R-function, which will imply that the general result for the class $O^n\left(\forall n \in \mathbb{N}\right)$. Let O be an objective function of n binary variables $y_1, y_2, ....., y_n$ given by

$$O\left(y_1, y_2, ....., y_n\right) = \sum_{1<i<j<n} O_{ij}\left(y_i, y_j\right)$$

Without loss of generality, we assume that, there is no term of single variables. As O is not an R – function, there must exist a term $O_{ij}$ (with fixed *i* and *j*) which is not an R – function.

Consider the following function:

$$O'_{ij}\left(\alpha,\alpha\right)=0, O'_{ij}\left(\alpha,\beta\right)=0, O'_{ij}(\beta,\alpha)=0, O'_{ij}(\beta,\beta)=O_{ij}(\beta,\beta)+O_{ij}\left(\alpha,\alpha\right)-O_{ij}\left(\alpha,\beta\right)-O_{ij}(\beta,\alpha)$$

Note that, the function $O'_{ij}$ and $O_{ij}$ are equivalent functions of $O^2$ and has the same value under $\theta$. In other words,

$\theta\left(O_{ij}\right)=O_{ij}(\beta,\beta)+O_{ij}\left(\alpha,\alpha\right)-O_{ij}\left(\alpha,\beta\right)-O_{ij}(\beta,\alpha)=\theta\left(O'_{ij}\right)$. But, $O_{ij}$ is not an R-function. This implies that, $O'_{ij}$ can also not be an R-function. It also follows that, both $\theta\left(O_{ij}\right)$ and $\theta\left(O'_{ij}\right)$ are non- negative and are equal. Let $\rho=\theta\left(O_{ij}\right)=\theta\left(O'_{ij}\right)$

Let G(*V*,*E*) be a graph where, $V=\{v_1, v_2, ....., v_n\}$. The problem is to determine the maximum independent subset $V'$ of *V*. i.e. a largest possible subset $V'$ of *V* which does not contain any pair of vertices of V which are directly connected by means of an edge in G. We will show that, optimization of O is equivalent to maximum independent set problem. As functions of single variables are R – functions and hence FNO – functions and adding such functions do not change the class of O (non R - function), we may add such terms to it.

Let's define $O_i\left(y_i\right)=\dfrac{\rho}{2n}$ for $1 \le i \le n$. Thus, now define $\tilde{O}$ as,

$$\tilde{O}\left(y_1, y_2, ....., y_n\right) = \sum_{1<i<n} O_i\left(y_i\right) + \sum_{1<i<j<n} O_{ij}\left(y_i, y_j\right)$$

We will show that, minimization of $\tilde{O}$ is equivalent to determination of maximum independent set problem. As the problem of determination of maximum independent set is known to be NP – hard and both O and $\tilde{O}$ are equivalent, it will prove that, optimization of O is an NP – hard problem.

First of all, note that, there is a one to one correspondence between the set of all possible configurations $y'_1, y'_2, ....., y'_n$ of the n – variables $y_1, y_2, ....., y_n$ and set of all independent subsets of V. Given any configuration $y'_1, y'_2, ....., y'_n$ of $y_1, y_2, ....., y_n$, it naturally corresponds to an independent set $\left\{v_i \mid y'_i=\beta\right\}$ and conversely, every independent set

$V' = \left\{ v_i \in \mathrm{V} / \left( v_i, v_j \right) \notin V \right\}$ naturally corresponds to a configuration $y_1', y_2', \ldots.., y_n'$ of $y_1, y_2, \ldots.., y_n$ where, $y_i' = \beta$ iff $v_i \in V'$. Note that,

$$\left| \sum_{1<i<n} O_i \left( y_i \right) \right| = \sum_{1<i<n} \left| O_i \left( y_i \right) \right| = \frac{\rho}{2n} \eta_0 \text{, where } \eta_0 = \left| V' \right| \left( \because O_i \left( y_i \right) \geq 0 \right)$$

And $\left| \sum_{1<i<j<n} O_{ij} \left( y_i, y_j \right) \right| = \sum_{1<i<j<n} \left| O_{ij} \left( y_i, y_j \right) \right| = 0$ if $V' = V \quad \left( \because O_{ij} \left( y_i, y_j \right) \geq 0 \right)$

Note that, $\left| \sum_{1<i<j<n} O_{ij} \left( y_i, y_j \right) \right| \geq \rho$ if $V' \neq V$.

Thus, minimum value of $\left| \tilde{O} \right|$ leads to the independent set $V'$ of maximum size. This proves the theorem.

## THEOREM 5.3.2.15

*If $O_1$ and $O_2$ are FNO – functions of n – variables $y_1, y_2, \ldots.., y_n$, $O_1 + O_2$ is also an FNO – function of $y_1, y_2, \ldots.., y_n$.*

***Proof:*** Let $V = \{ v_1, v_2, \ldots.., v_n \}$.

As $O_1$ is an FNO – function, there exist network $G_1(\mathrm{V}, \mathrm{E}_1)$ such that, the difference between $O_1(y_1', y_2', \ldots., y_n')$ and the cost of a minimum cut $C_1$ (considered over all the cuts C on the flow network $G_1$ in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C) on G differs by a constant say $c_1$.

i.e. $O_1(y_1', y_2', \ldots., y_n') = \left| C_1 \right| + c_1$

Similarly, there exist network $G_2(\mathrm{V}, \mathrm{E}_2)$ such that, the difference between $O_1(y_1', y_2', \ldots., y_n')$ and the cost of a minimum cut $C_2$ (considered over all the cuts C on the flow network $G_1$ in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C) on $G_2$ differs by a constant say $c_2$.

i.e. $O_1(y_1', y_2', \ldots., y_n') = \left| C_2 \right| + c_2$

If we consider a new network flow G(V, $E$ ) where $E = E_1 \cup E_2$, then it is easy to check that, the cut $C = C_1 \cup C_2$ is minimum among all the cuts C on the flow network G in which, $y_i' = \alpha$, if $v_i$ is connected to terminal vertex $s$ in the induced graph G \ C; $y_i' = \beta$, if $v_i$ is connected to terminal vertex $t$ in the induced graph G \ C and the difference between $(O_1 + O_2)(y_1', y_2', ...., y_n')$ and the cost of C differs by a constant and that constant is $c_1 + c_2$.

i.e., $(O_1 + O_2)(y_1', y_2', ...., y_n') == |C| + (c_1 + c_2)$.

This proves that, $O_1 + O_2$ is also an FNO – function.

### 5.3.3 SUMMARY OF NETWORK CONSTRUCTION FOR OBJECTIVE FUNCTIONS

In this sub-section, the summary and example of how the networks corresponding to objective functions of one, two and three variables can be constructed is presented.

#### 5.3.3.1 FUNCTION OF SINGLE VARIABLE OF RANGE OF SIZE TWO

Note that, all objective functions $O$ of single variables of range of size two are R – functions and hence are FNO – functions. They can be represented by a network with vertex set $V = \{s, t, v\}$ and edge set $E = \{e\}$ where, $e = \begin{cases} e_v^s, \text{if } O(\beta) \geq O(\alpha) \\ e_v^t, \text{if } O(\beta) < O(\alpha) \end{cases}$

The weights of the edge $e$ is defined as $|e_v^s| = O(\beta) - O(\alpha)$ and $|e_v^s| = O(\alpha) - O(\beta)$.

#### 5.3.3.2 FUNCTION OF TWO VARIABLES OF RANGE OF SIZE TWO

For an R - function $O_{12}$ of two variables $y_1$ and $y_2$, the network flow with vertex set $\{v_1, v_2, s, t\}$ and edge set $E = \{e_1, e_2, e_3\}$ is constructed as follows.

$$e_1 = e_{v_1}^s \text{ with } |e_{v_1}^s| = O_{12}(\beta, \alpha) - O_{12}(\alpha, \alpha) \text{ if } O_{12}(\beta, \alpha) \geq O_{12}(\alpha, \alpha)$$

$$e_1 = e_{v_1}^t \text{ with } |e_{v_1}^t| = O_{12}(\alpha, \alpha) - O_{12}(\beta, \alpha) \text{ if } O_{12}(\beta, \alpha) < O_{12}(\alpha, \alpha)$$

$$e_2 = e_{v_2}^s \text{ with } |e_{v_2}^s| = O_{12}(\beta, \beta) - O_{12}(\beta, \alpha) \text{ if } O_{12}(\beta, \beta) \geq O_{12}(\beta, \alpha)$$

$$e_2 = e_{v_2}^t \text{ with } |e_{v_2}^t| = O_{12}(\beta, \alpha) - O_{12}(\beta, \beta) \text{ if } O_{12}(\beta, \beta) < O_{12}(\beta, \alpha)$$

$$e_3 = e_{v_1v_2}^n \text{ with } |e_{v_1v_2}^n| = -\theta(O_{12}) \text{ if } O_{12}(\beta, \beta) \geq O_{12}(\beta, \alpha).$$

Note that, $|e_{v_1v_2}^n| = -\theta(O_{12}) > 0$ as $O_{12}$ is an R-function.

### 5.3.3.3 FUNCTION OF THREE VARIABLES OF RANGE OF SIZE TWO

For an R-function $O_{123}$ of three variables $y_1$, $y_2$ and $y_3$, the network flow with vertex set $V=\{s,t,v_1,v_2,...,v_{10}\}$ and edge set $E=\{e_i\}_{i=1}^{10}$ is constructed as follows:

***Case (i):*** $\theta\left(O_{123}\right)$ ***is non-negative***

| Edge | Constraint | Weight of the edge |
|---|---|---|
| $e_1=e_{v_1}^s$ | $O_{123}(\beta,\alpha,\beta)\geq O_{123}(\alpha,\alpha,\beta)$ | $O_{123}(\beta,\alpha,\beta)-O_{123}(\alpha,\alpha,\beta)$ |
| $e_1=e_{v_1}^t$ | $O_{123}(\beta,\alpha,\beta)<O_{123}(\alpha,\alpha,\beta)$ | $O_{123}(\alpha,\alpha,\beta)-O_{123}(\beta,\alpha,\beta)$ |
| $e_2=e_{v_2}^s$ | $O_{123}(\beta,\beta,\alpha)\geq O_{123}(\beta,\alpha,\alpha)$ | $O_{123}(\beta,\beta,\alpha)-O_{123}(\beta,\alpha,\alpha)$ |
| $e_2=e_{v_2}^t$ | $O_{123}(\beta,\beta,\alpha)<O_{123}(\beta,\alpha,\alpha)$ | $O_{123}(\beta,\alpha,\alpha)-O_{123}(\beta,\beta,\alpha)$ |
| $e_3=e_{v_3}^s$ | $O_{123}(\alpha,\beta,\beta)\geq O_{123}(\alpha,\beta,\alpha)$ | $O_{123}(\alpha,\beta,\beta)-O_{123}(\alpha,\beta,\alpha)$ |
| $e_3=e_{v_3}^t$ | $O_{123}(\alpha,\beta,\beta)<O_{123}(\alpha,\beta,\alpha)$ | $O_{123}(\alpha,\beta,\alpha)-O_{123}(\alpha,\beta,\beta)$ |
| $e_4=e_{v_2v_3}^n$ | No constraint | $\theta(proj[O_{123}(y_1=\alpha)])$ |
| $e_5=e_{v_3v_1}^n$ | No constraint | $\theta(proj[O_{123}(y_2=\alpha)])$ |
| $e_6=e_{v_1v_2}^n$ | No constraint | $\theta(proj[O_{123}(y_3=\alpha)])$ |
| $e_7=e_{v_1a}^n$ | No constraint | $\theta(O_{123})$ |
| $e_8=e_{v_2a}^n$ | No constraint | $\theta(O_{123})$ |
| $e_9=e_{v_3a}^n$ | No constraint | $\theta(O_{123})$ |
| $e_{10}=e_a^t$ | No constraint | $\theta(O_{123})$ |

***Case (ii):*** $\theta\left(O_{123}\right)$ ***is negative***

| Edge | Constraint | Weight of the edge |
|---|---|---|
| $e_1=e_{v_1}^s$ | $O_{123}(\beta,\beta,\alpha)\geq O_{123}(\alpha,\beta,\alpha)$ | $O_{123}(\beta,\beta,\alpha)-O_{123}(\alpha,\beta,\alpha)$ |

| | | |
|---|---|---|
| $e_1 = e^t_{v_1}$ | $O_{123}(\beta,\beta,\alpha) < O_{123}(\alpha,\beta,\alpha)$ | $O_{123}(\alpha,\beta,\alpha) - O_{123}(\beta,\beta,\alpha)$ |
| $e_2 = e^s_{v_2}$ | $O_{123}(\alpha,\beta,\beta) \geq O_{123}(\alpha,\alpha,\beta)$ | $O_{123}(\alpha,\beta,\beta) - O_{123}(\alpha,\alpha,\beta)$ |
| $e_2 = e^t_{v_2}$ | $O_{123}(\alpha,\beta,\beta) < O_{123}(\alpha,\alpha,\beta)$ | $O_{123}(\alpha,\alpha,\beta) - O_{123}(\alpha,\beta,\beta)$ |
| $e_3 = e^s_{v_3}$ | $O_{123}(\beta,\alpha,\beta) \geq O_{123}(\beta,\alpha,\alpha)$ | $O_{123}(\beta,\alpha,\beta) - O_{123}(\beta,\alpha,\alpha)$ |
| $e_3 = e^t_{v_3}$ | $O_{123}(\beta,\alpha,\beta) < O_{123}(\beta,\alpha,\alpha)$ | $O_{123}(\beta,\alpha,\alpha) - O_{123}(\beta,\alpha,\beta)$ |
| $e_4 = e^n_{v_3 v_2}$ | No constraint | $-\theta(proj[O_{123}(y_1 = \beta)])$ |
| $e_5 = e^n_{v_1 v_3}$ | No constraint | $-\theta(proj[O_{123}(y_2 = \beta)])$ |
| $e_6 = e^n_{v_2 v_1}$ | No constraint | $-\theta(proj[O_{123}(y_3 = \beta)])$ |
| $e_7 = e^n_{a v_1}$ | No constraint | $-\theta(O_{123})$ |
| $e_8 = e^n_{a v_2}$ | No constraint | $-\theta(O_{123})$ |
| $e_9 = e^n_{a v_3}$ | No constraint | $-\theta(O_{123})$ |
| $e_{10} = e^s_a$ | No constraint | $-\theta(O_{123})$ |

### 5.3.4 EXAMPLE OF OPTIMIZATION OF R - FUNCTION OF CLASS $O^3$ USING THE TERMINOLOGY

Let's consider an example for clarity of the network construction process. Consider the function $O(y_1, y_2, y_3) = y_1 - 2y_2 + 3(1 - y_3) + 4y_1 y_2 + 5|y_2 - y_3|$. All the variables $y_1, y_2$ and $y_3$ have range $\{1, 0\}$. First, Let's try to analyze the function. First, we need to decide whether this objective function is FNO – function or not. But, we know that, the function is FNO only if it is an R – function. O is a function of three variables and thus, it is an R – function, only if, all the projections of O of two variables are R – functions. As every variable has two possible values and there are three such variables, there will be total six projections of two variables. We need to check that all of these are R – functions. For that, it is sufficient to check that, their values under $\theta$ is non-positive. Details shown in the following table prove that, all these six projections are R – functions.

| Projection *p* | *p* (1,1) | *p* (0,0) | *p* (0,1) | *p* (1,0) | $\theta(p)$ |
|---|---|---|---|---|---|
| $p = proj[O(y_1 = 1)]$ | -1 | 4 | 6 | 7 | -10 |
| $proj[O(y_1 = 0)]$ | -2 | 3 | 5 | 6 | -10 |
| $proj[O(y_2 = 1)]$ | -1 | 6 | -2 | 7 | 0 |
| $proj[O(y_2 = 0)]$ | 6 | 3 | 5 | 4 | 0 |
| $proj[O(y_3 = 1)]$ | -1 | 5 | -2 | 6 | 0 |
| $proj[O(y_3 = 0)]$ | 7 | 3 | 6 | 4 | 0 |

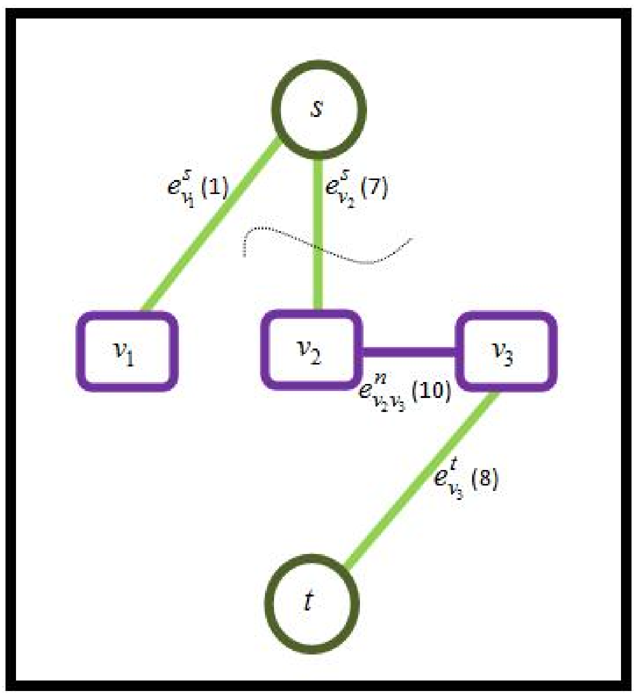


*Figure 5.4: Network and cut for the objective function O*

As $\theta(O) = 0$, it is non-negative. Thus, we refer to the procedure of construction of network presented in case (i) of 5.3.3.3.

We construct three edges $e^s_{v_1}, e^s_{v_2}$ and $e^t_{v_3}$ with weights $O(1,0,1) - O(0,0,1) = 1$, $O(1,1,0) - O(1,0,0) = 7$ and $O(0,1,0) - O(0,1,1) = 8$ respectively. The first two terminal edges correspond to terminal *s* because $O(1,0,1) > O(0,0,1)$ and $O(1,1,0) > O(1,0,0)$.

Then, we have to add edges $e^n_{v_2v_3}, e^n_{v_3v_1}$ and $e^n_{v_1v_2}$ with weights $-\theta\left(proj[O(y_1=0)]\right)=10$, $-\theta\left(proj[O(y_2=0)]\right)=0$ and $-\theta\left(proj[O(y_3=0)]\right)=0$. As the weights of last two edges are zero, we will not insert them in the network.

Lastly, we have to add a non-terminal reserve vertex a and edges $e^n_{v_1a}, e^n_{v_2a}, e^n_{v_3a}$ and $e^t_a$ of weights $\theta\left(O\right)=0$. As the weight of all these edges are zero, we will not insert them in the network. The network, so produced, is shown in Figure 5.4. Note that, among all cuts, the minimum cut is $\left\{e^s_{v_2}\right\}$, which leaves vertices $v_1$ and $v_2$ connected with terminal s and the vertex $v_3$ connected to terminal t in the induced graph. Thus, the configuration $y_1=0, y_2=0, y_3=1$ of the variables minimizes the objective function O.