Department of Applied Mathematics

Applications of Non-Linear Differential Equations in the Study of Superdense Stars on Geometrically Significant Spacetimes

A Thesis Submitted to

The M. S. University of Baroda

For The Degree Of

Doctor of Philosophy

in

Applied Mathematics

by

Bharat S Ratanpal

under the guidance of

Prof. (Dr.) S. Ramamohan

April, 2013

Dedicated to

My Grandmother
(Amma)

Missing your love and care...

Declaration

I hereby declare that:

(i) the thesis comprises only my original work towards the PhD except where indicated,

(ii) due acknowledgement has been made in the text to all other materials used,

(iii) this work has not formed the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Certificate

This is to certify that Mr Bharat S Ratanpal has worked under my guidance to prepare the thesis entitled "Applications of Non-Linear Differential Equations in the Study of Superdense Stars on Geometrically Significant Spacetimes" which is being submitted herewith towards the requirement for the degree of Doctor of Philosophy in Applied Mathematics.

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This thesis entitled "Applications of Non-Linear Differential Equations in the Study of Superdense Stars on Geometrically Significant Spacetimes" submitted by Bharat S Ratanpal in Applied Mathematics is hereby approved for the degree of Doctor Of Philosophy.

EXAMINERS

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Acknowledgement

At this important stage of my life first of all I would like to say

गुरुब्रहमा गुरुविष्णुः गुरुदेवो महेश्वरः |
गुरुः साक्षात् परब्रहम तस्मै श्रीगुरवे नमः ||

for Swamiji (Shri Swami Shishu Satyavidhehanand Saraswati).

The following Sanskrit sloka best describes my supervisior Prof. S. Ramamohan, and my teachers Prof. R. K. George, Dr. V. O. Thomas:

चैतन्यः शाश्वत् शान्तहो |
व्योमातीतो निरन्जनः ||
बिन्दु नाद कलातीतः |
तस्मै श्री गुरवे नमः ||

The meaning of above mentioned Sanskrit sloka is: That Guru who is the representative of the unchangeable, ever present, peaceful spirit, who is one pointed and beyond the realm of space and time, whose vision is always enchanting, I salute such a Guru.

This work is a result of constant encouragement of my Grandmother, my parents and all my teachers. My parents taught me the value of education. All the members of my joint family supported me by sharing household and social responsibilities. I am thankful to all my family members for extending their support to me. My daughter Drishti's smile made me fresh whenever I was tired. This work took lot of efforts and my wife Sonia extended all possible support. I am thankful to her.

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Bharat S Ratanpal

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1.1 Introduction

Mathematics forms the basis for understanding various physical sciences. Tensor calculus plays an important role in Einstein's theory of general relativity through Einstein's field equations

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -\frac{8\pi G}{c^2}T_{ij}, \tag{1.1.1}$$

where $i,\ j$ take the values from 0 to 3 and g_{ij}, $\Re_{ij}$, $\Re$, G, c are metric tensor, Ricci tensor, Ricci scalar, Newton's gravitational constant and speed of light in vacuum respectively, T_{ij} is energy-momentum tensor, throughout the thesis we have used geometrized units ($c^2 = G = 1$), unless otherwise stated, and used Einstein's field equations in the form

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -8\pi T_{ij}, \tag{1.1.2}$$

Tensor calculus was originally presented by Ricci in 1892 (Résumé de quelque travaux sur les sytémes variables de fonctions associćes á une forme différentielle quadratique, *Bulletin des Sciences Mathématiques* **2** (16):167-189) and later by Tullio Levi-Civita in their classic text *Methods de calcul differentiel absolu et leurs applications* (Methods of absolute differential calculus and their application) in 1900. The Einstein's theory of general relativity is one of the applications of tensor calculus.

Einstein's field equations consist of a system of 16 highly nonlinear differential equations. For spherically symmetric spacetime metric the number of equations is reduced to 4. Getting the singularity free exact solution of these nonlinear differential field equations is highly difficult. That is why Tolman [97] said "It is difficult to obtain explicit solutions of Einstein's gravitational field equations, in terms of known analytic functions, on account of their complicated and nonlinear character".

Einstein's field equations connect the geometry of the spacetime with the matter content of the distribution. Therefore geometry plays an important role in general theory of relativity and hence spacetime metric having a definite geometry is of mathematical as well as physical importance in general relativity.

The spacetime metric having geometrical significance was first studied by Karl Schwarzschild [78] and obtained first exact solution of Einstein's field equations for empty spacetime. The study of interior of stellar objects began with Schwarzschild [79] interior solution, in which matter density was assumed to be constant, which is good model for stellar structures in which pressure is relatively low. Schwarzschild used spherical, spherically symmetric spacetime metric to describe interior of relativistic star. In the recent past Vaidya and Tikekar [99], Tikekar and Thomas [93] & Tikekar and Jotania [89] used spheroidal, pseudo spheroidal and paraboloidal spacetimes respectively and found that these spacetimes are useful in describing models of superdense stars. The spacetime metrics used by them are spherically symmetric spacetime metrics of the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{1.1.3}$$

with different ansatz for $e^{\lambda(r)}$. Vaidya and Tikekar [99] considered $e^{\lambda(r)} = \frac{1-K\frac{r^2}{R^2}}{1-\frac{r^2}{R^2}}$, which represents a 3-spheroid

$$\frac{w^2}{b^2} + \frac{x^2 + y^2 + z^2}{R^2} = 1, \tag{1.1.4}$$

immersed in a 4-dimensional flat space having metric

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{1.1.5}$$

The parametrization

$$\left.\begin{aligned} x &= R\sin\lambda\sin\theta\cos\phi \\ y &= R\sin\lambda\sin\theta\sin\phi \\ z &= R\sin\lambda\cos\theta \\ w &= b\cos\lambda \end{aligned}\right\}, \tag{1.1.6}$$

of 3-spheroid, leads to the spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - \left(\frac{1 - K\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}}\right) dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{1.1.7}$$

The exact solution of Einstein's field equations for a perfect fluid on spheroidal spacetime metric (1.1.7) was obtained by Vaidya and Tikekar [99] and Energy conditions were also examined by them. Knutsen [46] examined dynamical stability of model of superdense star on spheroidal spacetime, and found that the model is stable with respect to infinitesimal radial oscillations.

Tikekar and Thomas [93] have taken $e^{\lambda(r)}$ in the form $e^{\lambda(r)} = \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}$. With this choice of $e^{\lambda(r)}$, the $t = constant$ sections of spacetime metric (1.1.3) have geometry of 3-pseudo spheroid with cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1, \tag{1.1.8}$$

immersed in four-dimensional Euclidean space with metric (1.1.5). The space part of the metric is obtained by introducing parametric equations

$$\left.\begin{aligned} x &= R\sinh\lambda\sin\theta\cos\phi \\ y &= R\sinh\lambda\sin\theta\sin\phi \\ z &= R\sinh\lambda\cos\theta \\ w &= b\cosh\lambda \end{aligned}\right\}. \tag{1.1.9}$$

Tikekar and Thomas [93] also have found that the model of stars on pseudo spheroidal spacetime are stable under radial modes of pulsation.

Tikekar and Jotania [89] used $e^{\lambda(r)} = 1+\frac{r^2}{R^2}$. The 3-space of spacetime metric (1.1.3), obtained as $t = constant$, has geometry of 3-paraboloid immersed in 4-dimensional flat space having metric (1.1.5) with Cartesian equation $x^2 + y^2 + z^2 = 2wR$.

Ever since Schwarzschild [78] obtained solution for Einstein's field equations, a number of exact solutions of Einstein's field equations were obtained describing models of isotropic stars, anisotropic stars, collapsing stars accompanied by radiation, charged stars, charged anisotropic stars and core-envelope models of superdense stars.

A method was developed by Tolman [97] to find exact solution of Einstein's field equations in terms of known functions for static fluid spheres. Delgaty and Lake [13] analysed physical plausibility conditions for 127 solutions of Einstein's field equations and found that only 16 of them satisfies all the conditions and only for 9 solutions sound speed is decreasing with radius. Pant and Sah [69] generalized Tolman's I, IV and V solutions and the de Sitter solution, also obtained class of new static solutions assuming equation of state. Durgapal [21] obtained class of new exact solutions for spherically symmetric static fluid spheres with the ansatz $e^{\nu} \propto (1 + x)^n$, and found that for each integer value of n, one can have new exact solution. Tikekar [88] obtained new exact solution for a static fluid sphere on spheroidal spacetime. Chattopadhyay and Paul [10] obtained the solutions of static compact stars on higher dimensional spacetime. The space part of spacetime metric considered by them is $(D - 1)$ pseudo spheroid immersed in D-dimensional Euclidean space.

The locally anisotropic equation of state for relativistic spheres was considered by Bowers and Liang [7]. Pant and Sah [68] obtained analytic solution for charged fluid on spherically symmetric spacetime, in their analysis, if charge is absent, the solution is Tolman's solution VI with $B = 0$. Consenza *et. al.* [12] developed the procedure to obtain solution of Einstein's field equations for anisotropic matter from known solutions of isotropic matter. The charged analog of Vaidya-Tikekar [99] solution on spheroidal spacetime was obtained by Patel and Koppar [70]. Bayin [5] found the solution for anisotropic fluid sphere by generalizing equation of state $p = \alpha\rho$ and also studied radiating anisotropic fluid sphere. Tikekar and Thomas [94] found exact solution of Einstein's field equations for anisotropic fluid sphere on pseudo spheroidal spacetime. The key feature of their model is the high variation of density from centre to boundary of stellar configuration also radial and tangential pressure are equal at the centre and boundary of the star. Mak and Harko [61] obtained classes of exact anisotropic solutions of Einstein's field equations on spherically symmetric spacetime metric. Komathiraj and Maharaj [47] studied analytical models of quark stars where they found a class of solutions of Einstein-Maxwell system by considering linear equation of state. Karmakar *et. al.* [43] analysed the role of pressure anisotropy for Vaidya-Tikekar [99] model. The exact solutions for Einstein-Maxwell system were extensively studied by Komathiraj and Maharaj [48][57] & Thirukkanesh and Maharaj [85].

Non-adiabatic gravitational collapse of a radiating star on the background of spheroidal spacetime was studied by Tikekar and Patel [91]. Tikekar and Patel [92] also studied non-adiabatic gravitational collapse of a charged radiating stellar structure, where they formulated equations governing shear free non-adiabatic collapse of spherical charged anisotropic matter in the presence of heat flow in the radial direction. Maharaj and Govender [56] considered effect of shear in charged radiating gravitational collapse. Non-adiabatic charged gravitational collapse by considering effect of viscosity is studied by Prisco *et. al.* [73]. The gravitation collapse with heat flux and shear on spherically symmetric spacetime metric is studied by Rajah and Maharaj [75], they found that gravitational behavior is described by Riccati equation, also found two new closed form solution. Misthry *et. al.* [63] found several new classes of exact solutions for radiative collapse.

Koppar and Patel [49] obtained the models of stars with two density distributions. Paul and Tikekar [72] obtained core-envelope models of stars on spheroidal spacetime and core-envelope models of stars on pseudo spheroidal spacetime are obtained by Tikekar and Thomas [95].

These studies show that core-envelope models having core and envelope with different physical features, collapse of radiating stars, dynamical stability of models of superdense stars and anisotropic stars are important in general theory of relativity. In this thesis we have studied core-envelope models of superdense stars on pseudo spheroidal spacetime, core-envelope model of a collapsing radiating star, dynamical stability of the model of superdense stars on paraboloidal spacetime, core-envelope models of superdense stars on paraboloidal spacetime, anisotropic stars on paraboloidal spacetime and also generated quadratic equation of state for anisotropic models of superdense stars on paraboloidal spacetime.

Following preliminaries are provided to facilitate comprehension of the results presented in later chapters.

1.2 Preliminaries

Definition 1.2.1. *The expression $a_1x^1 + a_2x^2 + ... + a_kx^k$ is denoted by symbol $\sum\limits_{i=1}^{k} a_ix^i$. By **summation convention** we mean that if a suffix occurs twice in a term, once in the lower position and once in the upper position then that suffix implies summation over the range under consideration. Hence we drop the summation sign and write a_ix^i to denote above expression.*

Definition 1.2.2. *If a suffix occurs twice in a term, once in the lower position and once in the upper position, then that suffix is called **dummy suffix**, Hence $a_ix^i = a_jx^j$.*

Definition 1.2.3. *If A^i be the set of quantities defined in the coordinate system $(x^1, x^2, ..., x^n)$ and A'^α be the set of quantities defined in the coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities A^i is said to be **contravariant vector** or **contravariant tensor of rank 1** if it satisfies the transformation law*

$$A'^\alpha = A^i \frac{\partial x'^\alpha}{\partial x^i}.$$

Example 1.2.1. *dx^i is a contravariant vector. If we transform dx^i from coordinate system $(x^1, x^2, ..., x^n)$ to $(x'^1, x'^2, ..., x'^n)$, the transformation gives*

$$dx'^\alpha = dx^i \frac{\partial x'^\alpha}{\partial x^i},$$

which obays the law of transformation of coordinates.

Definition 1.2.4. *If A_j be the set of quantities defined in the coordinate system $(x^1, x^2, ..., x^n)$ and A'_β be the set of quantities defined in the coordinate system $(x'^1, x'^2, ..., x'^n)$ then the set of quantities A_j is said to be **covariant vector** or **covariant tensor of rank 1** if it satisfies the transformation law*

$$A'_\beta = A_j \frac{\partial x^j}{\partial x'^\beta}.$$

Example 1.2.2. $\frac{\partial\phi}{\partial x^j}$ *is a covariant vector. If we transform* $\frac{\partial\phi}{\partial x^j}$ *from coordinate system* $(x^1, x^2, ..., x^n)$ *to* $(x'^1, x'^2, ..., x'^n)$*, the transformation gives*

$$\frac{\partial\phi}{\partial x'^\beta} = \frac{\partial\phi}{\partial x^j}\frac{\partial x^j}{\partial x'^\beta},$$

which obays the law of transformation of coordinates.

Definition 1.2.5. *If* $A^{i_1,i_2,.....,i_p}$ *be the set of quantities defined in coordinate system* $(x^1, x^2, ..., x^n)$ *and* $A'^{\alpha_1,\alpha_2,......,\alpha_p}$ *be the set of quantities defined in coordinate system* $(x'^1, x'^2, ..., x'^n)$ *then the set of quantities* $A^{i_1,i_2,.....,i_p}$ *is said to be* ***contravariant tensor of rank p*** *if it satisfies the transformation law*

$$A'^{\alpha_1,\alpha_2,......,\alpha_p} = A^{i_1,i_2,.....,i_p}\frac{\partial x'^{\alpha_1}}{\partial x^{i_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{i_2}}.....\frac{\partial x'^{\alpha_p}}{\partial x^{i_p}}.$$

Definition 1.2.6. *If* $A_{j_1,j_2,.....,j_q}$ *be the set of quantities defined in coordinate system* $(x^1, x^2, ..., x^n)$ *and* $A'_{\beta_1,\beta_2,......,\beta_q}$ *be the set of quantities defined in coordinate system* $(x'^1, x'^2, ..., x'^n)$ *then the set of quantities* $A_{j_1,j_2,.....,j_q}$ *is said to be* ***covariant tensor of rank q*** *if it satisfies the transformation law*

$$A'_{\beta_1,\beta_2,......,\beta_q} = A_{j_1,j_2,.....,j_p}\frac{\partial x^{j_1}}{\partial x'^{\beta_1}}\frac{\partial x^{j_2}}{\partial x'^{\beta_2}}.....\frac{\partial x^{j_q}}{\partial x'^{\beta_q}}.$$

Definition 1.2.7. *If* $A^{i_1,i_2,.....,i_p}_{j_1,j_2,.....,j_q}$ *be the set of quantities defined in coordinate system* $(x^1, x^2, ..., x^n)$ *and* $A'^{\alpha_1,\alpha_2,......,\alpha_p}_{\beta_1,\beta_2,......,\beta_q}$ *be the set of quantities defined in coordinate system* $(x'^1, x'^2, ..., x'^n)$ *then the set of quantities* $A^{i_1,i_2,.....,i_p}_{j_1,j_2,.....,j_q}$ *is said to be* ***mixed tensor of rank p+q*** *if it satisfies the transformation law*

$$A'^{\alpha_1,\alpha_2,......,\alpha_p}_{\beta_1,\beta_2,......,\beta_q} = A^{i_1,i_2,.....,i_p}_{j_1,j_2,.....,j_q}\frac{\partial x'^{\alpha_1}}{\partial x^{i_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{i_2}}.....\frac{\partial x'^{\alpha_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x'^{\beta_1}}\frac{\partial x^{j_2}}{\partial x'^{\beta_2}}.....\frac{\partial x^{j_q}}{\partial x'^{\beta_q}}.$$

Definition 1.2.8. *A covariant tensor* A_{ij} *is said to be* ***symmetric*** *if*

$$A_{ij} = A_{ji},$$

similarly a contravariant tensor $A^{\alpha\beta}$ *is said to be* ***symmetric*** *if*

$$A^{\alpha\beta} = A^{\beta\alpha}.$$

Definition 1.2.9. *A covariant tensor A_{ij} is said to be **anti-symmetric** if*

$$A_{ij} = -A_{ji},$$

*similarly a contravariant tensor $A^{\alpha\beta}$ is said to be **anti-symmetric** if*

$$A^{\alpha\beta} = -A^{\beta\alpha}.$$

Definition 1.2.10. *Consider two tensors $A^{\alpha}_{\beta\gamma}$ and $B^{\alpha}_{\beta\gamma}$ of same rank and character then their **sum** is a tensor $C^{\alpha}_{\beta\gamma}$ of same rank and character defined as*

$$C^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma} + B^{\alpha}_{\beta\gamma}.$$

Definition 1.2.11. *The **outer product** of a tensor $A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n}$ having covariant rank n and contravariant rank m with a tensor $B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}$ having covaraint rank p and contravariant rank q, is a tensor having covariant rank $n+q$ and contravariant rank $m+p$ and is defined as*

$$C^{i_1,i_2,\ldots,i_m,\alpha_1,\alpha_2,\ldots,\alpha_p}_{j_1,j_2,\ldots,j_n\beta_1,\beta_2,\ldots,\beta_q} = A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}.$$

Definition 1.2.12. *The process of setting one covariant and one contravariant suffixes equal is called **contraction**. Contraction reduce the tensor rank by 2.*

Definition 1.2.13. *The **inner product** of a tensor $A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n}$ having covariant rank n and contravariant rank m with a tensor $B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_1,\beta_2,\ldots,\beta_q}$ having covaraint rank p and contravariant rank q, is a tensor having covariant rank $n+q-1$ and contravariant rank $m+p-1$ and is defined as*

$$C^{i_2,\ldots,i_m,\alpha_1,\alpha_2,\ldots,\alpha_p}_{j_1,j_2,\ldots,j_n,\beta_2,\ldots,\beta_q} = A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{i_1\beta_2,\ldots,\beta_q} = A^{i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_n} B^{\alpha_1,\alpha_2,\ldots,\alpha_p}_{\beta_2,\ldots,\beta_q}$$

*Hence **inner product** of two tensors is their outer product followed by contraction.*

Definition 1.2.14. ***Quotient law of tensors** says that, a set of quantities, whose inner product with an arbitrary tensor is a tensor, is tensor itself.*

Definition 1.2.15. ***Kronecker Delta** is denoted by δ^i_j and is defined as*

$$\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

which is mixed tensor of rank 2.

Definition 1.2.16. *The metric of the form*

$$ds^2 = g_{ij} dx^i dx^j,$$

where i and j varies from 1 *to n is called* ***Riemannian metric*** *in n dimensional* ***Riemannian space***, g_{ij} *is called* ***fundamental tensor***. *The* ***reciprocal*** *of* g_{ij} *is denoted by* g^{ij} *and is defined as*

$$g^{ij} = \frac{cofactor\ of\ g_{ij}\ in\ |g_{ij}|}{g},$$

where $g = |g_{ij}|$.

Definition 1.2.17. *The process of multiplying covariant tensor* A_i *with contravariant metric tensor* g^{ij} *is called* ***raising an index*** *and resultant is contravariant tensor*

$$A^j = g^{ij} A_i.$$

Definition 1.2.18. *The process of multiplying contravariant tensor* A^i *with covariant metric tensor* g_{ij} *is called* ***lowering an index*** *and resultant is covariant tensor*

$$A_j = g_{ij} A^i.$$

Definition 1.2.19. ***Christoffel symbol of first kind*** *is denoted by* $\Gamma_{\mu\nu\sigma}$ *and is defined as*

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) = \frac{1}{2}\left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}\right).$$

The lower suffix preceding by comma denote a derivative in this way. ***Christoffel symbol of second kind*** *is denoted by* $\Gamma^\sigma_{\mu\nu}$ *and is defined as*

$$\Gamma^\sigma_{\mu\nu} = g^{\sigma\beta}\Gamma_{\mu\nu\beta}.$$

Both the Christoffel symbol are not tensor quantities.

Definition 1.2.20. ***Geodesic*** *is a curve for which variation in the arc length is* *zero if end points are kept fixed that is*

$$\delta \int_P^Q ds = 0 \Rightarrow \int_P^Q ds \text{ is stationary.}$$

*The equation of **geodesic** for Riemannian metric* $ds^2 = g_{ij}dx^i dx^j$ *is*

$$\frac{d^2x^k}{ds^2} + \frac{dx^i}{ds}\frac{dx^j}{ds}\Gamma^k_{ij} = 0.$$

Definition 1.2.21. ***Covariant derivative*** *of covariant tensor* A_μ *of rank* 1 *is*

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\alpha_{\mu\nu}A_\alpha = A_{\mu,\nu} - \Gamma^\alpha_{\mu\nu}A_\alpha.$$

Here semi-colon denote covariant derivative and comma denote ordinary derivative.
Covariant derivative *of contravariant tensor* A^μ *of rank* 1 *is*

$$A^\mu_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu}A^\alpha = A^\mu_{,\nu} + \Gamma^\mu_{\alpha\nu}A^\alpha.$$

Covariant derivative *of covariant tensor* $A_{\mu\nu}$ *of rank* 2 *is*

$$A_{\mu\nu;\gamma} = \frac{\partial A_{\mu\nu}}{\partial x^\gamma} - A_{\alpha\nu}\Gamma^\alpha_{\mu\gamma} - A_{\mu\alpha}\Gamma^\alpha_{\nu\gamma} = A_{\mu\nu,\gamma} - A_{\alpha\nu}\Gamma^\alpha_{\mu\gamma} - A_{\mu\alpha}\Gamma^\alpha_{\nu\gamma}.$$

Covariant derivative *of contravariant tensor* $A^{\mu\nu}$ *of rank* 2 *is*

$$A^{\mu\nu}_{;\gamma} = \frac{\partial A^{\mu\nu}}{\partial x^\gamma} + A^{\alpha\nu}\Gamma^\mu_{\alpha\gamma} + A^{\mu\alpha}\Gamma^\nu_{\alpha\gamma} = A^{\mu\nu}_{,\gamma} + A^{\alpha\nu}\Gamma^\mu_{\alpha\gamma} + A^{\mu\alpha}\Gamma^\nu_{\alpha\gamma}.$$

*In general the **covariant derivative** of tensor having contravariant rank* l *and covariant rank* m *is*

$$\begin{aligned}
A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m;\beta} &= \frac{\partial A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}}{\partial x^\beta} + A^{\alpha\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_1}_{\alpha\beta} + ... + A^{\mu_1.....\mu_{l-1}\alpha}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_l}_{\alpha\beta} - \\
&\quad A^{\mu_1\mu_2.....\mu_l}_{\alpha\nu_2.....\nu_m}\Gamma^\alpha_{\nu_1\beta} - ..... - A^{\mu_1\mu_2.....\mu_l}_{\nu_1.....\nu_{m-1}\alpha}\Gamma^\alpha_{\nu_m\beta} \\
&= A^{\mu_1\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m,\beta} + A^{\alpha\mu_2.....\mu_l}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_1}_{\alpha\beta} + ... + A^{\mu_1.....\mu_{l-1}\alpha}_{\nu_1\nu_2.....\nu_m}\Gamma^{\mu_l}_{\alpha\beta} - \\
&\quad A^{\mu_1\mu_2.....\mu_l}_{\alpha\nu_2.....\nu_m}\Gamma^\alpha_{\nu_1\beta} - ..... - A^{\mu_1\mu_2.....\mu_l}_{\nu_1.....\nu_{m-1}\alpha}\Gamma^\alpha_{\nu_m\beta}.
\end{aligned}$$

Definition 1.2.22. *The **Riemann-Christoffel tensor** or **curvature tensor** is* *defined as*

$$\Re^{\beta}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\beta}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\beta}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\sigma} = \Gamma^{\beta}_{\mu\sigma,\nu} - \Gamma^{\beta}_{\mu\nu,\sigma} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\sigma}.$$

Definition 1.2.23. *The* ***Bianci relation*** *is given by*

$$\Re^{\beta}_{\mu\nu\sigma;\tau} + \Re^{\beta}_{\mu\sigma\tau;\nu} + \Re^{\beta}_{\mu\tau\nu;\sigma} = 0,$$

which states that Riemann-Christoffel tensor satisfies these differential equations and symmetry conditions.

Definition 1.2.24. *The Riemann-Christoffel tensor with the contraction is called* ***Ricci tensor****, that is*

$$\Re_{\mu\nu} = \Re^{\beta}_{\mu\nu\beta}.$$

The explicit form of ***Ricci tensor*** *is given by*

$$\Re_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha},$$

again contractin we get ***Ricci scalar***

$$\Re = \Re^{\nu}_{\nu} = g^{\mu\nu}\Re_{\mu\nu}.$$

Definition 1.2.25. *The* ***principle of covariance*** *says that the laws must be expressible in a form which is independent of the particular spacetime coordinate chosen that is laws of nature remains invariant with respect to any spacetime coordinate system.*

Definition 1.2.26. *The* ***principle of equivalance*** *says that at every spacetime point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated cartesian coordinate systems in the absence of gravitation.*

Definition 1.2.27. *The* ***energy-momentum tensor*** *for a perfect fluid is of the form*

$$T_{ij} = (\rho + p)\, u_i u_j - g_{ij} p,$$

where ρ and p denotes density and pressure of fluid respectively and $u_i = \frac{dx^i}{dt}$.

Definition 1.2.28. *The **Einstein's Tensor** is defined as*

$$G_{ij} = \Re_{ij} - \frac{1}{2}\Re g_{ij}.$$

Theorem 1.2.1. ***Schwarzschild exterior solution:*** *For empty spacetime the static spherically symmetric spacetime metric*

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(dr^2 + \sin^2\theta d\phi^2\right),$$

possesses a solution of the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

□

Proof. Consider the spherically symmetric spacetime metric of the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(dr^2 + \sin^2\theta d\phi^2\right), \tag{1.2.1}$$

Here the coordinates are

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \tag{1.2.2}$$

the components of fundamental tensor for spacetime metric (1.2.1) are

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad g_{\mu\nu} = 0 \ for \ \mu \neq \nu, \tag{1.2.3}$$

and

$$g = g_{00}g_{11}g_{22}g_{33} = -e^{\lambda+\nu}r^2\sin^2\theta. \tag{1.2.4}$$

The Christoffel's symbol of second kind is given by

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\beta}\left(g_{\nu\beta,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}\right). \tag{1.2.5}$$

The non-vanishing components of Christoffel's symbol of second kind are

$$\begin{array}{lll} \Gamma^1_{00} = \nu' e^{\nu-\lambda} & \Gamma^0_{10} = \frac{\nu'}{2} & \Gamma^1_{11} = \frac{\lambda'}{2} \\ \\ \Gamma^2_{12} = \frac{1}{r} & \Gamma^1_{22} = -re^{-\lambda} & \Gamma^3_{13} = \frac{1}{r} \\ \\ \Gamma^3_{23} = \cot\theta & \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda} & \Gamma^2_{33} = -\sin\theta\cos\theta \end{array}$$

The Ricci tensor is given by

$$\Re_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{1.2.6}$$

The non-vanishing components of Ricci tensor are

$$\Re_{00} = e^{\nu-\lambda}\left(-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r}\right), \tag{1.2.7}$$

$$\Re_{11} = \frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r}, \tag{1.2.8}$$

$$\Re_{22} = e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1, \tag{1.2.9}$$

$$\Re_{33} = \Re_{22}\sin^2\theta. \tag{1.2.10}$$

For empty spacetime Einstein's field equations are given by $\Re_{ij} = 0$, therefore

$$e^{\nu-\lambda}\left(-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r}\right) = 0, \tag{1.2.11}$$

$$\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} = 0, \tag{1.2.12}$$

$$e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1 = 0, \tag{1.2.13}$$

$$\left\{e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) - 1\right\}\sin^2\theta = 0. \tag{1.2.14}$$

Equations (1.2.13) and (1.2.14) are dependent and hence there three independent equations (1.2.11) - (1.2.14). Dividing (1.2.11) by $e^{\nu-\lambda}$ and adding in (1.2.12) gives

$$\lambda' + \nu' = 0, \tag{1.2.15}$$

whose solution is

$$\lambda + \nu = K, \tag{1.2.16}$$

where K is constant of integration. For a large value of r, space must be approximately flate that is as $r \to \infty$, the unknowns $\lambda, \nu \to 0$. Hence $K = 0$ and therefore

$$\lambda + \nu = 0 \Rightarrow \lambda = -\nu. \tag{1.2.17}$$

From equation(1.2.13)

$$e^{-\lambda}\left(1 - \frac{r\lambda'}{2} + \frac{r\nu'}{2}\right) = 1, \tag{1.2.18}$$

Using equation (1.2.17) in equation (1.2.18),

$$e^{-\lambda}\left(1 + r\nu'\right) = 1 \Rightarrow \frac{d}{dr}\left(re^{\nu}\right) = 1 \Rightarrow re^{\nu} = r - 2m, \tag{1.2.19}$$

where m is constant of integration, therefore

$$e^{\nu} = 1 - \frac{2m}{r} \; and \; e^{\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}, \tag{1.2.20}$$

therefore complete solution is of the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.2.21}$$

□

Remark 1.2.1. *Schwarzschild exterior solution gave rise to corrections in Newtonian theory for planetary motion. These corrections are notable in the case of mercury, the planet nearest to sun.*

Remark 1.2.2. *Schwarzschild exterior solution has two singularities at $r = 0$ and at $r = 2m$.*

1.3 Layout of the thesis

The thesis is organized as follows:

Chapter 1 deals with general introduction of the thesis.

Chapter 2 deals with core-envelope model of superdense stars with the feature - core consisting of isotropic fluid distribution and envelope consisting of anisotropic fluid distribution on pseudo spheroidal spacetime. In the case of superdense stars core-envelope models with isotropic pressure in the core and anisotropic pressure in the envelope, may not be unphysical. We have used existing solution of Tikekar and Thomas [95] for developing the model. The core radius is found to be $b = \sqrt{2}R$, R being geometric parameter and for positivity of tangential pressure $p_{\perp}$ it is required that $\frac{a^2}{R^2} > 2$, where a is the radius of the star. This requirement restrict the value of density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)} \leq 0.093$. Further it is observed that thickness of the envelope decreases with increasing value of λ, and radius of the star increases as λ increases.

Chapter 3 describes the non-adiabatic gravitational collapse of a spherical distribution of matter having radial heat flux on pseudo spheroidal spacetime. The star is divided into two regions: core consisting of anisotropic fluid distribution and an envelope consisting of isotropic fluid distribution, various aspects of the collapse have been studied. The variation of polytropic index γ with respect to time, at the centre and on the boundary is calculated for the model with density variation parameter $\lambda = 0.05$. The polytropic index at the centre is less than $\frac{4}{3}$ and at the boundary is much higher than $\frac{4}{3}$ during initial stage of collapse. This indicates that the central region is dynamically unstable. Assuming the evolution of heat flow is governed by Maxwell-Cattaneo heat transport equation and by making suitable assumptions equation governing temperature profile have been derived.

In Chapter 4, we investigate stability of superdense star on paraboloidal spacetime. The stability of models of stars on paraboloidal spacetime is investigated by integrating Chandrasekhar's pulsation equation and it is found that the models with $0.26 < \frac{m}{a} < 0.36$ will be stable under radial modes of pulsation.

In Chapter 5, we have reported two core-envelope models with the feature core consisting of isotropic fluid and envelope consisting of anisotropic fluid distribution on the background of paraboloidal spactime. For both the models thickness of envelope increases as $\frac{m}{a}$ increases. A noteworthy feature these models is, they admits thin envelope, hence is significant in the study of glitches and star quakes.

Chapter 6 describes an anisotropic model of superdense star on paraboloidal spacetime. A particular choice is made on radial pressure p_r to integrate Einstein's field equations. The central pressure in this model is $\frac{p_0}{R^2}$. The bounds on p_0 is obtained such that physical plausibility conditions are satisfied and Herrera's [35] overtuning method is applied to check the stability of the star. It is found that prescribed model is stable for $\frac{1}{3} < p_0 < 0.3944$. Further equation of state is generated and we found that model satisfies quadratic equation of state.

Chapter 6 is followed by summery of thesis, appendices, list of publications and references used during the course of research.

Chapter 2

Core-Envelope Models of Superdense Star with Anisotropic Envelope

Contents

In this chapter, a core-envelope model for superdense matter distribution with the feature - core consisting of isotropic fluid distribution and envelope with anisotropic fluid distribution - is studied on the background of pseudo spheroidal spacetime. In the case of superdense stars core-envelope models, with isotropic pressure in the core and anisotropic pressure in the envelope, may not be unphysical. Physical plausibility of the models have been examined analytically and using programming.

2.1 Introduction

Theoretical investigations of Ruderman [76] and Canuto [8] on compact stars having densities much higher than nuclear densities led to the conclusion that matter may be anisotropic at the central region of the distribution. Maharaj and Maartens [59] have obtained models of spherical anisotropic distribution with uniform density. Gokhroo and Mehra [28] have extended this model to include anisotropic distributions with variable density. Dev and Gleiser [14] have obtained exact solutions for various forms of equation of state connecting the radial and tangential pressure.

When matter density of spherical objects are much higher than nuclear density, it is difficult to have a definite description of matter in the form of an equation of state. The uncertainty about the equation of state of matter beyond nuclear regime led to the consideration of a complementary approach called core-envelope models. In this approach, a relativistic stellar configuration is made up of two regions - a core surrounded by an envelope - containing matter distribution with different physical features. The first core-envelope model was obtained by Bondi [6] in 1964. A detailed analysis of such models are discussed by Hartle [32], Iyer and Vishveshwara [42]. A common feature of the core-envelope models reported in literature was that their core and envelope regions contain distributions of perfect fluids in equilibrium with different density distributions. Negi, Pande and Durgapal [65] have developed core-envelope models where both pressure and density are continuous across the core boundary.

Core-envelope models with anisotropic pressure distribution in the core and isotropic pressure distribution in the envelope are available [95]. We shall investigate whether the prescription of isotropic pressure in the core and anisotropic pressure in the envelope leads to a solution consistent with the physical requirements. Such an assumption may not be unphysical because in the case of superdense stars with core consisting of degenerate fermi fluid, the core can be considered as isotropic while its outer envelope may consist of fluid with anisotropic pressure. Further, the study of glitches and starquakes are important in stars having thin envelopes. It is, therefore, pertinent to investigate the physical viability of spherical distributions of matter with isotropic pressure in the core and anisotropic pressure in the envelope.

2.2 The Field Equations

We begin with a static spherically symmetric spacetime described by the metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.2.1}$$

with an ansatz

$$e^{\lambda(r)} = \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}, \tag{2.2.2}$$

where K and R are geometric parameters. The $t = const.$ section of (2.2.1) has the geometry of a 3-pseudo spheroid.

We consider the energy-momentum tensor of the form

$$T_{ij} = (\rho + p)\,u_i u_j - p g_{ij} + \pi_{ij}, \qquad u_i u^j = 1 \tag{2.2.3}$$

where ρ, p, u_i respectively denote matter density, isotropic pressure, unit four velocity field of matter. The anisotropic stress tensor π_{ij} is given by

$$\pi_{ij} = \sqrt{3}S\left[C_i C_j - \frac{1}{3}\left(u_i u_j - g_{ij}\right)\right]. \tag{2.2.4}$$

For radially symmetric anisotropic fluid distribution of matter, $S = S(r)$ denotes the magnitude of the anisotropic stress tensor and $C^i = \left(0, -e^{-\lambda/2}, 0, 0\right)$, which is a radial vector. For equilibrium models,

$$u_i = \left(e^{\nu/2}, 0, 0, 0\right) \tag{2.2.5}$$

and the energy-momentum tensor (2.2.3) has non-vanishing components

$$T_0^0 = \rho \quad T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right) \quad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right). \tag{2.2.6}$$

The pressure along the radial direction

$$p_r = p + \frac{2S}{\sqrt{3}}, \tag{2.2.7}$$

is different from the pressure along the tangential direction

$$p_\perp = p - \frac{S}{\sqrt{3}}. \tag{2.2.8}$$

The magnitude of anisotropy is given by

$$S = \frac{p_r - p_\perp}{\sqrt{3}}. \tag{2.2.9}$$

The field equation (1.1.2) corresponding to the metric (2.2.1) using ansatz (2.2.2) is given by a set of three equations:

$$8\pi\rho = \frac{K-1}{R^2}\left(3 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2}, \tag{2.2.10}$$

$$8\pi p_r = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1}, \tag{2.2.11}$$

$$\begin{aligned} 8\pi\sqrt{3}S \quad = \quad & -\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right)\left(1 + \frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-1} + \\ & \frac{(K-1)}{R^2} r\left(\frac{\nu'}{2} + \frac{1}{r}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2} + \\ & \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right)^{-1}. \end{aligned} \tag{2.2.12}$$

Equation (2.2.10) provides the law of variation of density of matter from which it follows that the density gradient

$$\frac{d\rho}{dr} = -\frac{2K(K-1)}{8\pi R^4} r\left(5 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-3} \tag{2.2.13}$$

is negative.

We consider a star with isotropic core and anisotropic envelope with radial pressure p_r and tangential pressure $p_\perp$. The anisotropy starts developing from the core boundary having radius $r = b$. The radial pressure decreases in the enveloping region and it becomes zero at the surface (say $r = a$, where a is the radius of the star under consideration). We describe the core upto the radius $r = b$, throughout which $S(r) = 0$. The radius of the star is taken as a and we divide it into two parts:

(i) $0 \leq r \leq b$ as the core of the star described by a fluid distribution with isotropic pressure.

(ii) $b \leq r \leq a$ as the outer envelope of the core which can be described by a fluid distribution with anisotropic pressure.

2.3 The Core of the Star

The core of the distribution is characterized by the isotropic distribution of matter. So throughout the core region $0 \leq r \leq b$ the radial pressure p_r is equal to the tangential pressure $p_\perp$ and hence $S(r) = 0$. Then equation (2.2.12) reduces to

$$\begin{aligned} \left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right)\left(1 + \frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right) - \\ \frac{(K-1)}{R^2} r \left(\frac{\nu'}{2} + \frac{1}{r}\right) - \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right) &= 0. \end{aligned} \tag{2.3.1}$$

Which is a non-linear differential equation, if we choose new independent variable z and dependent variable F defined by:

$$z = \sqrt{1 + \frac{r^2}{R^2}}, \tag{2.3.2}$$

$$F = e^{\nu/2}, \tag{2.3.3}$$

equation (2.3.1) takes the linear form

$$\left(1 - K + Kz^2\right)\frac{d^2F}{dz^2} - Kz\frac{dF}{dz} + K\left(K-1\right)F = 0. \tag{2.3.4}$$

We again make the transformation

$$x = \sqrt{\frac{K}{K-1}}z, \tag{2.3.5}$$

which reduces the equation (2.3.4) in the form

$$\left(1-x^2\right)\frac{d^2F}{dx^2}+x\frac{dF}{dx}-(K-1)\,F=0. \tag{2.3.6}$$

We observe that $x=\pm 1$ are two regular singular points. We assume the solution in the form $F=\sum\limits_{n=0}^{\infty}C_n x^n$. Substituting this in the equation (2.3.6), gives the recurrence relation

$$(n+2)+(n+1)\,C_{n+2}-\left(n^2-2n+K-1\right)C_n=0. \tag{2.3.7}$$

If $n^2-2n+K-1=0$, then the integral values of n forms a solution of one of two sets of coefficients $(C_0, C_2, C_4.....)\,,\ (C_1, C_3, C_5.....)$ containing finite number of terms for equation (2.3.6), n is a positive integer only when $K=2$ and hence $F_1=C_1x=Ax$ (without loss of generality replacing C_1 by A) is a finite polynomial solution of equation (2.3.6). For finding second linearly independent solution to the second order linear differential equation (2.3.6), we use the method of variation of parameters and assume $F_2=y(x)x$ as the other linearly independent solution and substitute F_2 in equation (2.3.6) which results into

$$x\left(1-x^2\right)\frac{d^2y}{dx^2}+\left(2-x^2\right)\frac{dy}{dx}=0, \tag{2.3.8}$$

which admits the solutions

$$y=\left[ln\left(x+\sqrt{x^2-1}\right)-\frac{\sqrt{x^2-1}}{x}\right]. \tag{2.3.9}$$

Hence the general solution of (2.3.6) is

$$F=F_1+F_2=Ax+B\left[x\;ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}\right]. \tag{2.3.10}$$

The back substitution of variable x gives closed form solution of equation (2.3.4) as

$$F=e^{\nu/2}=A\sqrt{1+\frac{r^2}{R^2}}+B\left[\sqrt{1+\frac{r^2}{R^2}}\mathbb{L}\left(r\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right], \tag{2.3.11}$$

for $K = 2$, where A and B are constants of integration and

$$\mathbb{L}(r) = ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}} + \sqrt{1+2\frac{r^2}{R^2}}\right). \tag{2.3.12}$$

Thus the spacetime metric of the core region $0 \leq r \leq b$ is described by:

$$\begin{aligned} ds^2 &= \left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2 dt^2 - \\ &\left(\frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\right) dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2. \end{aligned} \tag{2.3.13}$$

The density and pressure of the distribution are given by:

$$8\pi\rho = \frac{1}{R^2}\left(3+2\frac{r^2}{R^2}\right)\left(1+2\frac{r^2}{R^2}\right)^{-2}, \tag{2.3.14}$$

$$8\pi p = \frac{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\mathbb{L}(r) + \frac{1}{\sqrt{2}}\sqrt{1+\frac{r^2}{R^2}}\right]}{R^2\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}}. \tag{2.3.15}$$

The constants A and B are to be determined by requiring that the pressure and metric coefficients must be continuous across the core boundary $r = b$; and this is done in section 2.5.

2.4 The Envelope of the Star

The envelope of the star is characterized by the anisotropic distribution of matter. So throughout the enveloping region $b \leq r \leq a$ the radial pressure p_r is different from the tangential pressure $p_\perp$, and hence $S(r) \neq 0$. To obtain the solution of equation (2.2.12), in this case, we introduce new variables z and ψ defined by:

$$z = \sqrt{1+\frac{r^2}{R^2}} \qquad \psi = \frac{e^{\nu/2}}{(1-K+Kz^2)^{1/4}} \tag{2.4.1}$$

in terms of which equation (2.2.12) assumes the form:

$$\frac{d^2\psi}{dz^2}+\left[\frac{2K\left(2K-1\right)\left(1-K+Kz^2\right)-5K^2z^2}{4\left(1-K+Kz^2\right)^2}+\frac{8\sqrt{3}\pi R^2S\left(1-K+Kz^2\right)}{z^2-1}\right]\psi=0 \tag{2.4.2}$$

On prescribing

$$8\pi\sqrt{3}S=-\frac{\left(z^2-1\right)\left[2K\left(2K-1\right)\left(1-K+Kz^2\right)-5K^2z^2\right]}{4R^2\left(1-K+Kz^2\right)^3} \tag{2.4.3}$$

the second term of (2.4.2) vanishes and the resulting equation is

$$\frac{d^2\psi}{dz^2}=0, \tag{2.4.4}$$

which has solution of the form

$$\psi=Cz+D \tag{2.4.5}$$

where C and D are constants of integration. From equation (2.4.1) we get

$$e^{\nu/2}=\left(1+K\frac{r^2}{R^2}\right)^{\frac{1}{4}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right). \tag{2.4.6}$$

Thus the spacetime metric of the enveloping region $b\leq r\leq a$ is described by:

$$\begin{aligned} ds^2 \;=\; & \sqrt{1+K\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2dt^2-\left(\frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\right)dr^2- \\ & r^2\left(d\theta^2+sin^2\theta d\phi^2\right). \end{aligned} \tag{2.4.7}$$

The radial pressure p_r and anisotropy parameter $S(r)$ having explicit expressions:

$$8\pi p_r=\frac{C\sqrt{1+\frac{r^2}{R^2}}\left[3+2K\frac{r^2}{R^2}+K\left(2-K\right)\frac{r^2}{R^2}\right]+D\left[1+K\left(2-K\right)\frac{r^2}{R^2}\right]}{R^2\left(1+K\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)} \tag{2.4.8}$$

$$8\pi\sqrt{3}S=-\frac{\frac{r^2}{R^2}\left[2K\left(2K-1\right)\left(1+K\frac{r^2}{R^2}\right)-5K^2\left(1+\frac{r^2}{R^2}\right)\right]}{4R^2\left(1+K\frac{r^2}{R^2}\right)^3} \tag{2.4.9}$$

In order to have the same 3-space geometry throughout the distribution, we shall set $K=2$.

The matter density, fluid pressure and anisotropy parameter take the simple forms:

$$8\pi\rho = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}, \tag{2.4.10}$$

$$8\pi p_r = \frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)}, \tag{2.4.11}$$

$$8\pi p_\perp = \frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}, \tag{2.4.12}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}. \tag{2.4.13}$$

The constants C and D are to be determined by matching the solution with the Schwarzschild exterior spacetime metric:

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2+sin^2\theta d\phi^2\right) \tag{2.4.14}$$

across the boundary $r=a$ of the star, where $p_r(a)=0$.
The continuity of metric coefficients and pressure along radial direction across $r=a$ imply the following relations:

$$e^{\nu(a)} = \frac{1+\frac{a^2}{R^2}}{1+2\frac{a^2}{R^2}} = 1-\frac{2m}{a}, \tag{2.4.15}$$

$$C\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)+D = 0. \tag{2.4.16}$$

Equations (2.4.15) and (2.4.16) determine constants m, C and D as:

$$m = \frac{a^3}{2R^2\left(1+2\frac{a^2}{R^2}\right)} \tag{2.4.17}$$

$$C = -\frac{1}{2}\left(1+2\frac{a^2}{R^2}\right)^{-\frac{7}{4}} \tag{2.4.18}$$

$$D = \frac{1}{2}\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)\left(1+2\frac{a^2}{R^2}\right)^{-\frac{7}{4}} \tag{2.4.19}$$

substituting for C and D in the expression (2.4.11) and (2.4.12), we get

$$8\pi p_r = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right) - \sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right) - \sqrt{1+\frac{r^2}{R^2}}\right]} \tag{2.4.20}$$

$$\begin{aligned} 8\pi p_\perp \;=\; & \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right) - \sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right) - \sqrt{1+\frac{r^2}{R^2}}\right]} - \\ & \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3} \end{aligned} \tag{2.4.21}$$

2.5 Physical Plausibility

This approach does not assume any equation of state for matter. Hence it is pertinent to examine the physical plausibility of the solution. A physically plausible solution for the core-envelope model is expected to fulfil the following requirements in its region of validity.

(i) The spacetime metric (2.3.13) in the core should continuously match with the spacetime metric (2.4.7) in the envelope across the core boundary $r = b$.

(ii) $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \le r \le a$.

(iii) $p > 0$, $\frac{dp}{dr} < 0$, $\frac{dp}{d\rho} < 1$, $\rho - 3p > 0$ for $0 \le r \le b$.

(iv) $p_r \ge 0$, $p_\perp > 0$, $\frac{dp_r}{dr} < 0$ for $b \le r \le a$.

(v) $\frac{dp_r}{d\rho} < 1$, $\frac{dp_\perp}{d\rho} < 1$, $\rho - p_r - 2p_\perp \ge 0$ for $b \le r \le a$.

At the core boundary $r = b$, the anisotropy parameter vanishes. And consequently from equation (2.4.13) we get $\frac{b^2}{R^2} = 2$. Further the positivity of the tangential pressure demands that $\frac{a^2}{R^2} > 2$. The continuity of metric coefficients and the continuity of pressure across $r = b$ of the distribution lead to:

$$\sqrt{3}A + B\left[\sqrt{3}L\left(b\right) + \sqrt{2.5}\right] = 5^{-\frac{3}{4}}\left[11\sqrt{3}C + D\right], \tag{2.5.1}$$

$$\sqrt{3}A + B\left[\sqrt{3}L\left(b\right) - \sqrt{2.5}\right] = 5^{\frac{1}{4}}\left[\sqrt{3}C + D\right], \tag{2.5.2}$$

where

$$L\left(b\right) = ln\left(\sqrt{5} + \sqrt{6}\right). \tag{2.5.3}$$

Equations (2.5.1) and (2.5.2) determine A and B in terms of C and D as:

$$A = \frac{\sqrt{2}}{5^{\frac{5}{4}}}\left[3\sqrt{3}C - 2D\right] \tag{2.5.4}$$

$$B = \frac{\left[5\sqrt{5} - 3\sqrt{6}\left(\sqrt{3}L\left(b\right) - \sqrt{2.5}\right)\right]C + \left[5\sqrt{5} + 2\sqrt{2}\left(\sqrt{3}L\left(b\right) - \sqrt{2.5}\right)\right]D}{5^{\frac{5}{4}}} \tag{2.5.5}$$

substituting these values of A and B in (2.3.15), we get pressure in the core of the star. The requirement (ii) is satisfied in the light of equation (2.3.14) and (2.2.13). Verification of the requirements (iii) analytically is highly difficult due to the complicated expression for pressure given by (2.3.15). Hence we adopt programming approach.

It is evident from (2.4.20) that p_r is positive throughout the envelope. The tangential pressure $p_\perp\left(= p_r - \sqrt{3}S\right)$ is positive if $a \geq \sqrt{2}R$. After a lengthy but straight forward computation one finds that $\rho - p_r - 2p_\perp > 0$ throughout the envelope. Owing to the complexity of expressions, programming is used to verify $\frac{dp_r}{dr} < 0$, $\frac{dp_r}{d\rho} < 1$ and $\frac{dp_\perp}{d\rho} < 1$ in the envelope.

2.6 Discussion

The scheme given by Tikekar [88], for estimating the mass and size of the fluid spheres on the background of spheroidal spacetime can be used to determine the mass and size of the fluid distribution consisting of core and envelope.

Following this scheme we adopt $\rho\left(a\right) = 2 \times 10^{14} gm/cm^3$, and introduce a density variation parameter λ given by:

$$\lambda = \frac{\rho(a)}{\rho(0)} = \frac{1 + \frac{2}{3}\frac{a^2}{R^2}}{\left(1 + 2\frac{a^2}{R^2}\right)^2} \tag{2.6.1}$$

Since ρ is a decreasing function of r, $\lambda < 1$. The condition $p_\perp \geq 0$ in the envelope

$$i.e.\ \ \frac{a^2}{R^2} = \frac{1 - 6\lambda + \sqrt{24\lambda + 1}}{12\lambda} \geq 2 \tag{2.6.2}$$

then restricts λ to comply with the requirement $\lambda \leq 0.093$. Thus the introduction of anisotropy in the envelope results in a high degree of density variation as one moves from the centre to the boundary.

Equation (2.3.14) implies that the matter density at the centre is explicitly related with the curvature parameter R as

$$8\pi\rho(0) = \frac{3}{R^2}. \tag{2.6.3}$$

Equation (2.6.3) determines R in terms of $\rho(a)$ and λ. The size of the configuration can be obtained from (2.6.2) in terms of surface density $\rho(a)$ and density variation parameter λ.

We take the matter density on the boundary $r = a$ of the star as $\rho(a) = 2 \times 10^{14} gm/cm^3$. Choosing different values of $\lambda \leq 0.093$, we determine the boundary radius (in kilometers) using (2.6.2) and the total mass of the star (in kilometers) using (2.4.17).

The mass of the star in grams is obtained using $M = \frac{mc^2}{G}$. Results of these calculations, the thickness of the envelope (in kilometers) together with some relevant quantities are given in table 2.1. It has been noticed that the thickness of the envelope decreases as λ increases in the range $0 < \lambda \leq 0.093$ of table 2.1. The plots showing radial variations of pressure, density and sound speed for the model with $\lambda = 0.05$ are shown in Figure 2.1, Figure 2.2 and Figure 2.3 respectively.

Table 2.1: Masses and equilibrium radii of superdense star models corresponding to $K = 2$, $\rho(a) = 2 \times 10^{14} gm/cm^3$.

λ	R	a	$\frac{a^2}{R^2}$	m	C	D	Thickness of the envelope
0.010000	2.838109	11.740629	17.112941	2.851833	-0.000982	0.298517	7.726937
0.020000	4.013692	11.862912	8.735635	2.805169	-0.003038	0.359674	6.186694
0.030000	4.915749	11.961340	5.920799	2.757472	-0.008891	0.402905	5.009422
0.040000	5.676218	12.041076	4.500000	2.709242	-0.008891	0.437896	4.013692
0.050000	6.346204	12.105678	3.638733	2.660798	-0.012378	0.468003	3.130790
0.060000	6.951919	12.157716	3.058403	2.612351	-0.016124	0.494825	2.326219
0.070000	7.508930	12.199111	2.639370	2.564047	-0.020077	0.519260	1.579880
0.080000	8.027384	12.231340	2.321667	2.515986	-0.024198	0.541869	0.878905
0.090000	8.514327	12.255567	2.071888	2.468242	-0.028459	0.563027	0.214491
0.093000	8.655069	12.261420	2.006968	2.453988	-0.029762	0.569132	0.021304

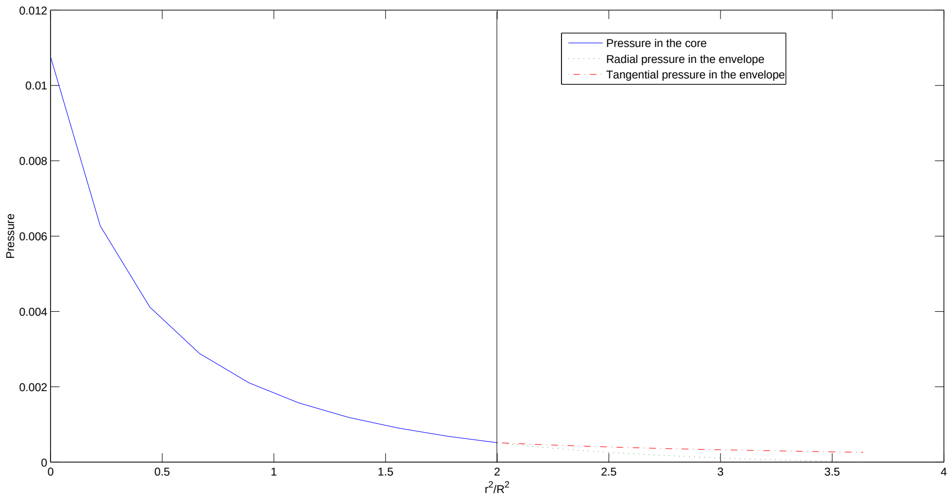

Figure 2.1: Variation of p against $\frac{r^2}{R^2}$ in the core and variation of p_r, $p_\perp$ against $\frac{r^2}{R^2}$ in the envelope.

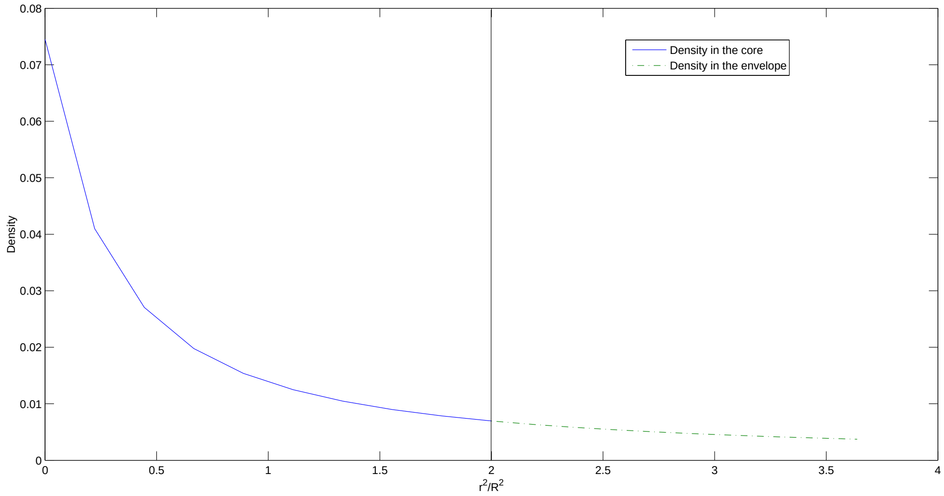

Figure 2.2: Variation of ρ against $\frac{r^2}{R^2}$ throughout the distribution.

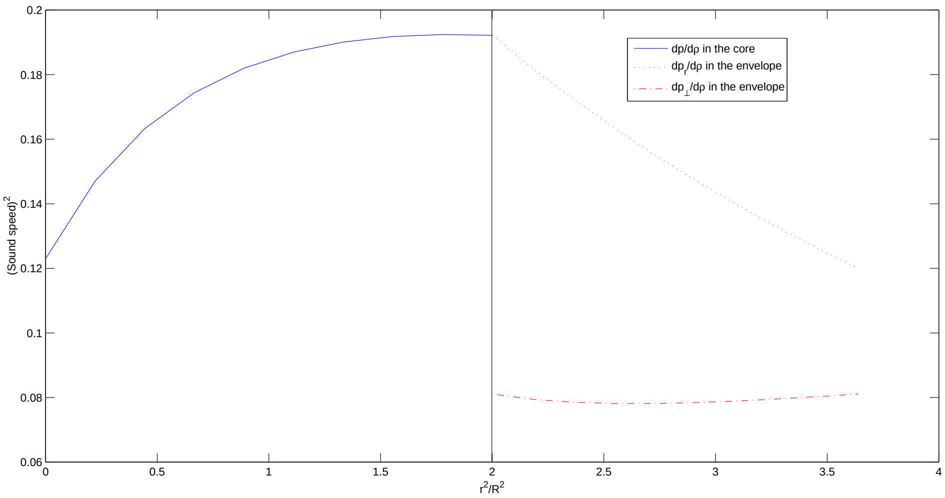

Figure 2.3: Variation of $\frac{dp}{d\rho}$ against $\frac{r^2}{R^2}$ in the core and variation of $\frac{dp_r}{d\rho}$, $\frac{dp_\perp}{d\rho}$ against $\frac{r^2}{R^2}$ in the envelope.

All the requirements stipulated in section 2.5 for the physical plausibility of the distribution are satisfied in the core as well as in the envelope.

The core-envelope models described on the background of a pseudo spheroidal space-time have the following salient features:

1. The core region contains a distribution of isotropic fluid and is surrounded by an envelope of anisotropic fluid at rest.
2. The density profile is continuous throughout, even at the core boundary.
3. The pressure is continuous throughout and continuously join across the core boundary to the anisotropic pressure of the fluid in the envelope.
4. The models admit a high degree of density variation from centre to boundary.

Chapter 3

Non-Adiabatic Gravitational Collapse with Anisotropic Core

Contents

In this chapter, we study the non-adiabatic gravitational collapse of a spherical distribution of matter accompanied by radial heat flux on the background of a pseudo spheroidal spacetime. The spherical distribution is divided into two regions: a core consisting of anisotropic pressure distribution and envelope consisting of isotropic pressure distribution. Various aspects of the collapse have been studied using both analytic and programming approach.

3.1 Introduction

When a thermonuclear sources of energy in a star are exhausted it begins to collapse due to absence of outward force to balance the inward gravitational force. The final stage of such massive stars is either a white dwarf, a neutron star or a black hole, depending upon the mass of the configuration.

The gravitational collapse problems of spherical distribution of matter, like stars, are important problems in relativistic astrophysics. If the collapse is accompanied by heat flux which radiates out through the surface of the star, then the problem is realistic in nature. The first attempt in this direction was made by Oppenheimer and Snyder [67], when they studied an idealized problem of the gravitational collapse of a spherical dust distribution for adiabatic flow. Thereafter several authors have worked on problems of gravitational collapse considering different idealized situations.

The junction conditions for a more relativistic gravitational collapse with non - adiabatic heat flow has been first studied by Santos [77]. An important consequence of this study was that, at the boundary, the pressure is proportional to the magnitude of the heat flow vector. Based on this approach Oliveira, Santos and Kolassis [66] proposed mathematical model for a collapsing radiating star with unpolarized outgoing radiation and studied the physical conditions and thermodynamic relations. Gravitational collapse solutions with shear and radial heat flow were first obtained by Glass [27]. Dynamical equations governing the gravitational non-adiabatic collapse of a shear-free spherical distribution of anisotropic matter in the presence of charge has been studied by Tikekar and Patel [92].

When we consider the gravitational collapse of spherical distributions consisting of superdense matter distribution, the pressure may not be isotropic throughout the distribution. For such stars the core region may be anisotropic in pressure (Ruderman [76] and Canuto [8]). Therefore study of models by with isotropic pressure throughout the distribution may not be physically realistic. Keeping this in view we have studied the gravitational collapse of spherical distribution of perfect fluid accompanied by heat flux in the radial direction. The whole region is divided into two parts - a core surrounded by an envelope. The core consisting of matter with anisotropic pressure distribution and the envelope with isotropic pressure distribution.

3.2 The Interior Spacetime

The spacetime in the interior of non-adiabatically collapsing fluid sphere with outgoing radial heat-flow is denoted by $\mathbb{M}_{(i)}$. The spacetime metric of $\mathbb{M}_{(i)}$ is taken as spherically symmetric metric in the form:

$$ds^2_{(i)} = e^{\nu(r,t)}dt^2 - e^{\mu(t)}\left(e^{\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{3.2.1}$$

with an ansatz

$$e^{\lambda(r)} = \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}} \tag{3.2.2}$$

where K and R are geometric parameters.

When $\mu = 0$, the spacetime reduces to the static pseudo spheroidal spacetime used for describing static core-envelope models of superdense fluid spheres described by Thomas, Ratanpal and Vinodkumar [87].

The energy-momentum tensor for the interior spacetime $\mathbb{M}_{(i)}$ is taken as

$$T_{ij} = (\rho + p)\, u_iu_j - pg_{ij} + \pi_{ij} + q_iu_j + q_ju_i \tag{3.2.3}$$

where ρ, p, u^i and q^i denote the matter density, isotropic fluid pressure, components of unit time-like flow vector field of matter and components of space-like radial heat flux vector orthogonal to u^i, respectively. The anisotropic stress tensor π_{ij} is given by:

$$\pi_{ij} = \sqrt{3}S\left[C_iC_j - \frac{1}{3}\left(u_iu_j - g_{ij}\right)\right], \tag{3.2.4}$$

where $S = S(r)$ denotes the magnitude of anisotropy and

$$C^i = \left(0, -e^{-\lambda/2}, 0, 0\right), \tag{3.2.5}$$

which is a radial vector. For a comoving observer

$$u^i = \left(e^{-\nu/2}, 0, 0, 0\right) \tag{3.2.6}$$

and

$$q^i = (0, q, 0, 0). \tag{3.2.7}$$

The heat flux vector q^i is orthogonal to u^i with magnitude $q = q\,(r,t)$.

The energy-momentum tensor (3.2.3) has the following non-vanishing components

$$T^0_0 = \rho \quad T^1_1 = -\left(p + \frac{2S}{\sqrt{3}}\right) \quad T^2_2 = T^3_3 = -\left(p - \frac{S}{\sqrt{3}}\right) \quad T^1_0 = qe^{\nu/2}. \tag{3.2.8}$$

The pressure along the radial direction

$$p_r = p + \frac{2S}{\sqrt{3}}, \tag{3.2.9}$$

is different from the pressure along the tangential direction

$$p_\perp = p - \frac{S}{\sqrt{3}}. \tag{3.2.10}$$

The magnitude of anisotropy is given by:

$$S = \frac{p_r - p_\perp}{\sqrt{3}}. \tag{3.2.11}$$

Equations (1.1.2) corresponding to metric (3.2.1) with the energy-momentum tensor (3.2.3) is equivalent to the following set of four equations:

$$8\pi\rho = \frac{K-1}{R^2}\left(3 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2} e^{-\mu} + \frac{3}{4}e^{-\nu}\dot{\mu}^2, \tag{3.2.12}$$

$$8\pi p_r = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1} e^{-\mu} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right), \tag{3.2.13}$$

$$\begin{aligned} 8\pi\sqrt{3}S &= -\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right)\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1}+ \\ & \frac{(K-1)}{R^2}r\left(\frac{\nu'}{2}+\frac{1}{r}\right)\left(1+K\frac{r^2}{R^2}\right)^{-2}- \\ & \frac{K-1}{R^2}\left(1+K\frac{r^2}{R^2}\right)^{-1}e^{-\mu}, \end{aligned} \tag{3.2.14}$$

$$8\pi q = -\frac{1}{2}\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1}e^{-\nu/2}e^{-\mu}\nu'\dot{\mu}. \tag{3.2.15}$$

Here a dot and a prime denote differentiation with respect to t and r, respectively.

We shall divide the interior $\mathbb{M}_{(i)}$ of the star into two parts core and envelope of the interior.

1. The region $0 \leq r \leq b$ as the core of the star characterized by anisotropic fluid distribution.
2. The region $b \leq r \leq a$ as the envelope of the interior of the star characterized by isotropic fluid distribution.

Further discussion of the core and envelope of the interior spacetime is done in sections 3.6 and 3.7 respectively.

3.3 The Exterior Spacetime

The spacetime in the exterior of non-adiabatically collapsing fluid sphere with outgoing radial heat-flow is denoted by $\mathbb{M}_{(e)}$. We consider Vaidya [98] metric in the exterior $\mathbb{M}_{(e)}$ of the star

$$ds^2_{(e)} = \left(1-\frac{2m(v)}{y}\right)dv^2 - 2dvdy - y^2d\theta^2 - y^2\sin^2\theta d\phi^2, \tag{3.3.1}$$

where $m = m(v)$ denotes the total mass enclosed within the spherical region.

The energy-momentum tensor in $\mathbb{M}_{(e)}$ is given by

$$T_i^j = \varepsilon\zeta^j\zeta_i, \tag{3.3.2}$$

where $\zeta_i = (1, 0, 0, 0)$. The non-vanishing components of T_i^j is

$$T_0^1 = \varepsilon, \tag{3.3.3}$$

where ε is the radiation density.

A time-like 3-hypersurface $\Sigma_{(b)}$ separates the interior from the exterior and the dividing hypersurface Σ distinguishes the two spacetime manifolds $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$, both of which contain $\Sigma_{(b)}$ as a part of their boundaries. The intrinsic metric on $\Sigma_{(b)}$ is:

$$ds_\Sigma^2 = d\tau^2 - \mathbb{R}(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.3.4}$$

3.4 Boundary Conditions

We follow the method of Israel [38] in matching the interior with the exterior on the boundary $\Sigma_{(b)}$. In order to have a unique intrinsic geometry at the boundary hypersurface $\Sigma_{(b)}$, we must have

$$ds_{(i)}^2 = ds_{(e)}^2 = ds_{(b)}^2. \tag{3.4.1}$$

This conditions guarantee the continuity of metric coefficients across the boundary surface $\Sigma_{(b)}$.

The second boundary condition imposed on $\Sigma_{(b)}$ is

$$\kappa_{ij(e)} - \kappa_{ij(i)} = 0, \tag{3.4.2}$$

where κ_{ij} are components of the extrinsic curvature. This guarantees the continuity of the first derivatives of the metric coefficients in $\mathbb{M}_{(e)}$ and $\mathbb{M}_{(i)}$ across $\Sigma_{(b)}$. The components of the extrinsic curvature κ_{ij} are given by (Eisenhart [22]):

$$\kappa_{ij} = -\eta_\alpha \frac{\partial^2 \chi^\alpha}{\partial \xi^i \partial \xi^j} - \eta_\alpha \Gamma_{ab}^\alpha \frac{\partial \chi^a}{\partial \xi^i} \frac{\partial \chi^b}{\partial \xi^j}, \tag{3.4.3}$$

where ξ^i are coordinates θ, ϕ, τ on $\Sigma_{(b)}$, χ^α are coordinates appropriate to $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$ and η_α are unit normals to $\Sigma_{(b)}$ in $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$.

The equation of the boundary surface $\Sigma_{(b)}$ in terms of the interior coordinates is given by

$$f\left(r,t\right) = r - r_{(b)}, \tag{3.4.4}$$

where r_b is a constant. The equation of the boundary surface in terms of the exterior coordinates is given by

$$f\left(y,\upsilon\right) = y - r_{(b)}\left(\upsilon\right). \tag{3.4.5}$$

The unit space-like normals $\eta_{\alpha(i)}$ and $\eta_{\alpha(e)}$ to $\Sigma_{(b)}$ in $\mathbb{M}_{(i)}$ and $\mathbb{M}_{(e)}$, respectively, are given by

$$\eta_{\alpha(i)} = \left(0, e^{\frac{\mu+\lambda}{2}}, 0, 0\right), \tag{3.4.6}$$

$$\eta_{\alpha(e)} = \left(-\frac{\partial y}{\partial \tau}, \frac{\partial \upsilon}{\partial \tau}, 0, 0\right). \tag{3.4.7}$$

The boundary conditions (3.4.1) give the following relations:

$$\left(e^{\nu/2}\right)_{(b)} \frac{dt}{d\tau} = 1, \tag{3.4.8}$$

$$\left(re^{\mu/2}\right)_{(b)} = y_{(b)}\left(\upsilon\right) = \mathbb{R}\left(\tau\right), \tag{3.4.9}$$

$$\left[\frac{d\upsilon}{d\tau}\right]^{-2}_{(b)} = \left[2\frac{dy}{d\upsilon} + 1 - \frac{2m}{y}\right]_{(b)}. \tag{3.4.10}$$

Equations (3.4.2) and (3.4.3) yield the following equations

$$\left[-e^{-\left(\frac{\mu+\lambda}{2}\right)}\frac{\nu'}{2}\right]_{(b)} = \left[\frac{\frac{d^2\upsilon}{d\tau^2}}{\frac{d\upsilon}{d\tau}} - \frac{m}{y^2}\frac{d\upsilon}{d\tau}\right]_{(b)}, \tag{3.4.11}$$

$$\left[re^{\frac{\mu-\lambda}{2}}\right]_{(b)} = \left[y\frac{dy}{d\tau} + y\left(1 - \frac{2m}{y}\right)\frac{d\upsilon}{d\tau}\right]_{(b)}. \tag{3.4.12}$$

Using (3.4.10) and (3.4.12), we get the total mass $m(\upsilon)$ of the collapsing fluid sphere as

$$m(\upsilon) = \left[\frac{re^{\mu/2}}{2}\left(1 - e^{-\lambda} + \frac{r^2 e^{\mu-\nu}}{4}\dot{\mu}\right)\right]_{(b)}. \tag{3.4.13}$$

Substituting equations (3.4.10) and (3.4.11) in equations (3.2.13) and (3.2.15), we can obtain

$$p_{(b)} = \left[qe^{\frac{\mu+\lambda}{2}}\right]_{(b)}. \tag{3.4.14}$$

This relation shows that the radial pressure at the boundary is directly related to

the heat flux q at the boundary. The pressure at the boundary becomes zero only when there is no heat flux along the radial direction across the boundary.
The energy density of radiation measured by an observer on $\Sigma_{(b)}$ with four-velocity u^{α} is given by (Lindquist, Schwartz and Misner [54])

$$\varepsilon = u^{\alpha}u^{\beta}T_{\alpha\beta}, \tag{3.4.15}$$

where

$$u^{\alpha} = \left(\frac{dv}{d\tau}, \frac{dy}{d\tau}, 0, 0\right). \tag{3.4.16}$$

The Einstein's field equations (1.1.2) for the spacetime metric (3.3.1) and energy-momentum tensor (3.3.2) yield

$$8\pi T_{00} = -\frac{2}{y^2}\frac{dm}{dv} \tag{3.4.17}$$

as the only surviving component of Einstein tensor. Now equation (3.4.15) becomes

$$8\pi\varepsilon = \left[-\frac{2}{y^2}\frac{dm}{dv}\left(\frac{dv}{d\tau}\right)^2\right]_{(b)}. \tag{3.4.18}$$

The total luminosity for an observer at rest at infinity (Lindquist, Schwartz and Misner [54]) is:

$$L_{\infty} = \lim_{y\to\infty, \frac{dy}{d\tau}\to 0} 4\pi y^2\varepsilon = -\frac{dm}{dv}. \tag{3.4.19}$$

The luminosity observed on $\Sigma_{(b)}$ is:

$$L = 4\pi y^2\varepsilon. \tag{3.4.20}$$

The boundary red shift $Z_{(b)}$ is given by

$$\frac{dv}{d\tau} = 1 + Z_{(b)}. \tag{3.4.21}$$

From (3.4.18) and (3.4.19), we can write

$$L_{\infty} = 4\pi y^2\varepsilon = \frac{1}{\dot{v}^2}. \tag{3.4.22}$$

From (3.4.12) and (3.4.9), we get

$$\dot{\upsilon} = \left[\frac{re^{\frac{\mu-\lambda}{2}} - r^2 e^{\mu}\frac{\dot{\mu}}{2}}{re^{\mu/2} - 2m}\right]_{(b)}. \tag{3.4.23}$$

It is observed from equation (3.4.22) that $L_\infty \to 0$ as $\dot{\upsilon} \to \infty$. That is when $re^{\mu/2} \to 2m$. Thus when the collapsing star becomes a black hole i.e. when $re^{\mu/2} = 2m$, the boundary redshift becomes infinity.

3.5 Solution of Field Equations

If $\mu(t) = 0$ and $\nu(r,t) = \nu(r)$ in the spacetime metric (3.2.1), we get the usual spherically symmetric static metric in Schwarzschild coordinates. If the matter content of the spacetime is in the form of perfect fluid, then the field equations are:

$$8\pi\rho_0 = \frac{K-1}{R^2}\left(3 + K\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2}, \tag{3.5.1}$$

$$8\pi p_0 = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{K-1}{R^2}\right]\left(1 + K\frac{r^2}{R^2}\right)^{-1}, \tag{3.5.2}$$

where ρ_0 and p_0 are the proper density and radial pressure of the fluid.

Equations (3.2.12), (3.2.13) will become

$$8\pi\rho = 8\pi\rho_0 e^{-\mu} + \frac{3}{4}e^{-\nu}\dot{\mu}^2, \tag{3.5.3}$$

$$8\pi p = 8\pi p_0 e^{-\mu} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right). \tag{3.5.4}$$

On using the equation (3.4.14) in (3.5.4), we get

$$\left[\frac{1}{2}e^{\frac{\nu-\lambda}{2}}\dot{\mu}\nu'\right]_{(b)} = \left[e^{\mu/2}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{1}{2}\dot{\mu}\dot{\nu}\right)\right]_{(b)}, \tag{3.5.5}$$

if we choose $f(t) = e^{\mu/2}$ equation (3.5.5) reduces to

$$2f\ddot{f} + \dot{f}^2 - 2\alpha\dot{f} = 0, \tag{3.5.6}$$

where

$$\alpha = \left(\frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}} \right)_{(b)} . \tag{3.5.7}$$

Equation (3.5.6) possesses a first integral

$$\dot{f} = -\frac{2\alpha}{b} \frac{1 - b\sqrt{f}}{\sqrt{f}}, \tag{3.5.8}$$

and admits the solution

$$t - t_0 = \frac{f}{2\alpha} + \frac{\sqrt{f}}{\alpha b} + \frac{1}{\alpha b} ln \mid b\sqrt{f} - 1 \mid, \tag{3.5.9}$$

where b and t_0 are arbitrary constants of integration.

We choose $b = 1$ and re-parametrize t, to get

$$\dot{f} = -\frac{2\alpha}{\sqrt{f}} \left(1 - \sqrt{f} \right), \tag{3.5.10}$$

$$t = \frac{f}{2\alpha} + \frac{\sqrt{f}}{\alpha} + \frac{1}{\alpha} ln \left(1 - \sqrt{f} \right). \tag{3.5.11}$$

Here $t = -\infty$ corresponds to $f = 1$. When t gradually increases from $-\infty$ (i.e $f = 1$) the fluid gradually starts collapsing. Further we note that the spacetime metric (3.2.1) corresponds to a static metric when $t = -\infty$.

Equation (3.4.13) gives the total mass of the collapsing fluid sphere as

$$m(v) = \left[2\alpha^2 r^3 e^{-\nu} \left(1 - \sqrt{f} \right)^2 + m_0 f \right]_{(b)}, \tag{3.5.12}$$

where

$$m_0 = \left[\frac{r}{2} \left(1 - e^{-\lambda} \right) \right]_{(b)}, \tag{3.5.13}$$

is the mass inside $\Sigma_{(b)}$ when $t = -\infty$ (i.e. $f = 1$). The expression for luminosity (3.4.22) in this case reads

$$L_\infty = \left\{ \frac{\left[\alpha r^2 e^{-\frac{\lambda+\nu}{2}} \nu' \left(1 - \sqrt{f} \right) \right] \left[r e^{\mu/2} - 2m \right]}{r f \left(e^{-\frac{\lambda}{2}} - r\dot{f} \right)} \right\}_{(b)} . \tag{3.5.14}$$

It follows from equation (3.5.14) that when the collapsing body becomes a black hole, that is, $re^{\mu/2} = 2m$, $L_\infty = 0$.

3.6 The Core of the Star

The core of the spherically symmetric fluid distribution is considered to be anisotropic. Hence in the core region $0 \leq r \leq b$ the radial pressure p_r is different from the tangential pressure $p_\perp$, and hence $S(r) \neq 0$. Defining new variables z and ψ as

$$z = \sqrt{1 + \frac{r^2}{R^2}} \qquad \psi = \frac{e^{\nu/2}}{(1 - K + Kz^2)^{1/4}}, \tag{3.6.1}$$

equation (3.2.14) assumes the closed form

$$\frac{d^2\psi}{dz^2} + \left[\frac{2K(2K-1)(1-K+Kz^2) - 5K^2z^2}{4(1-K+Kz^2)^2} + \frac{8\sqrt{3}\pi R^2 S(1-K+Kz^2)}{z^2-1}\right]\psi = 0, \tag{3.6.2}$$

choosing

$$8\pi\sqrt{3}S = -\frac{(z^2-1)\left[2K(2K-1)(1-K+Kz^2) - 5K^2z^2\right]}{4R^2(1-K+Kz^2)^3}, \tag{3.6.3}$$

the second term of equation (3.6.2) vanishes and solution of the resulting equation takes the simple form

$$\psi = Cz + D, \tag{3.6.4}$$

where C and D are constants of integration. From equation (3.6.1) for a particular choice of curvature parameter $K = 2$ we get

$$e^{\nu/2} = \left(1 + 2\frac{r^2}{R^2}\right)^{\frac{1}{4}}\left(C\sqrt{1 + \frac{r^2}{R^2}} + D\right). \tag{3.6.5}$$

Hence the spacetime metric in the core $(0 \leq r \leq b)$ is described by

$$\begin{aligned} ds^2_{(i)_{(c)}} = & \sqrt{1 + 2\frac{r^2}{R^2}}\left(C\sqrt{1 + \frac{r^2}{R^2}} + D\right)^2 dt^2 - \left(\frac{1 + 2\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}\right) f^2 dr^2 - \\ & r^2 f^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \end{aligned} \tag{3.6.6}$$

Utilizing equation (3.6.5), the matter density, fluid pressure, anisotropy parameter and heat flux takes the form

$$8\pi\rho = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}\frac{1}{f^2} + \frac{12\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)^2}{f^3}, \tag{3.6.7}$$

$$\begin{aligned} 8\pi p_r &= \frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)}\frac{1}{f^2} + \\ &\quad \frac{4\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.6.8}$$

$$\begin{aligned} 8\pi p_\perp &= \left[\frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}\right]\frac{1}{f^2} + \\ &\quad \frac{4\alpha^2}{\sqrt{1+2\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.6.9}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}\frac{1}{f^2}, \tag{3.6.10}$$

$$8\pi q = 4\alpha\frac{r}{R^2}\frac{\sqrt{1+\frac{r^2}{R^2}}}{\left(1+2\frac{r^2}{R^2}\right)^{9/4}}\frac{\left[C\left(2+3\frac{r^2}{R^2}\right)+D\sqrt{1+\frac{r^2}{R^2}}\right]}{\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2}\frac{\left(1-\sqrt{f}\right)}{f^{7/2}}. \tag{3.6.11}$$

Continuity of pressure and metric coefficients across the core boundary $r = b$ will give the expression of the constants C and D, and this is done in section 3.8.

The polytropic index $\gamma = \frac{d(ln\ p)}{d(ln\ \rho)}$ at the centre is given by

$$\gamma_0 = \frac{\left[k_1 k_2 f^{3/2} + \alpha^2 R^2 \left(5 - 4\sqrt{f}\right)\right] \left[3k_2^2 f + 12\alpha^2 R^2 \left(1 - \sqrt{f}\right)^2\right]}{\sqrt{f}\left[k_1 k_2 \sqrt{f} + 4\alpha^2 R^2 \left(1 - \sqrt{f}\right)\right] \left[6k_2^2 f + 12\alpha^2 R^2 \left(1 - \sqrt{f}\right)\left(3 - 2\sqrt{f}\right)\right]}, \tag{3.6.12}$$

where,

$$k_1 = A + B\left(ln[\sqrt{2} + 1] + \frac{1}{\sqrt{2}}\right),$$
$$k_2 = A + B\left(ln[\sqrt{2} + 1] - \frac{1}{\sqrt{2}}\right).$$

3.7 The Envelope of the Star

The envelope of the star is characterized by the isotropic distribution of matter. So throughout the enveloping region $b \leq r \leq a$ the radial pressure p_r is equal to the tangential pressure $p_\perp$, so that $S(r) = 0$. Then equation (3.2.14) reduces to

$$\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right)\left(1 + \frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right) - \frac{(K-1)}{R^2} r \left(\frac{\nu'}{2} + \frac{1}{r}\right) + \frac{K-1}{R^2}\left(1 + K\frac{r^2}{R^2}\right) = 0. \tag{3.7.1}$$

Choosing new independent variable z and dependent variable F defined by

$$z = \sqrt{1 + \frac{r^2}{R^2}}, \tag{3.7.2}$$

$$F = e^{\nu/2}, \tag{3.7.3}$$

equation (3.7.1) takes the form

$$\left(1 - K + Kz^2\right)\frac{d^2F}{dz^2} - Kz\frac{dF}{dz} + K(K-1)F = 0. \tag{3.7.4}$$

Equation (3.7.4) admits a closed form solution

$$F = e^{\nu/2} = A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\mathbb{L}(r) - \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right], \tag{3.7.5}$$

for $K=2$, where A and B are constants of integration and

$$\mathbb{L}(r)=\sqrt{1+\frac{r^2}{R^2}}ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right). \tag{3.7.6}$$

Hence the spacetime metric in the envelope region $b\leq r\leq a$ is described by

$$\begin{aligned} ds^2_{(i)_{(e)}} &= \left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2 dt^2- \\ &\left(\frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\right)f^2dr^2-r^2f^2\left(d\theta^2+sin^2\theta d\phi^2\right). \end{aligned} \tag{3.7.7}$$

The density, pressure and heat flux are given by the expressions

$$\begin{aligned} 8\pi\rho &= \frac{1}{R^2}\left[3+2\frac{r^2}{R^2}\right]\left[1+2\frac{r^2}{R^2}\right]^{-2}\frac{1}{f^2}+ \\ &\frac{12\alpha^2}{\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)^2}{f^3}, \end{aligned} \tag{3.7.8}$$

$$\begin{aligned} 8\pi p &= \frac{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)+\frac{1}{\sqrt{2}}\sqrt{1+\frac{r^2}{R^2}}\right]}{R^2\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}}\frac{1}{f^2}+ \\ &\frac{4\alpha^2}{\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)}{f^{5/2}}, \end{aligned} \tag{3.7.9}$$

$$8\pi q=4\alpha\frac{r}{R^2}\frac{\sqrt{1+\frac{r^2}{R^2}}}{1+2\frac{r^2}{R^2}}\frac{\left\{A+Bln\left[\sqrt{2}\sqrt{1+2\frac{r^2}{R^2}}+\sqrt{1+\frac{r^2}{R^2}}\right]\right\}}{\left\{A\sqrt{1+\frac{r^2}{R^2}}+B\left[\mathbb{L}(r)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2}\frac{\left(1-\sqrt{f}\right)}{f^{7/2}}. \tag{3.7.10}$$

The constants A and B are to be determined by matching the static solution with Schwarzschild exterior spacetime metric

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2\left(d\theta^2+sin^2\theta d\phi^2\right), \tag{3.7.11}$$

at the boundary surface $r = a$ of the distribution when static term of pressure is zero i.e $(p_0)_\Sigma = 0$. The continuity of the metric coefficients give

$$\sqrt{\frac{1+\frac{a^2}{R^2}}{1+2\frac{a^2}{R^2}}} = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\mathbb{L}(a) - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right), \tag{3.7.12}$$

where

$$L(a) = \sqrt{1+\frac{a^2}{R^2}} ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right). \tag{3.7.13}$$

The continuity of pressure across $r = a$ requires, p_0 to vanish on the boundary implying that

$$A\sqrt{1+\frac{a^2}{R^2}} = -B\left(\mathbb{L}(a) + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right). \tag{3.7.14}$$

The constants A and B are determined from equations (3.7.12) and (3.7.14) as:

$$A = \frac{\mathbb{L}(a) + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}, \tag{3.7.15}$$

$$B = -\frac{\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}. \tag{3.7.16}$$

The polytropic index $\gamma = \frac{d(ln\ p)}{d(ln\ \rho)}$ at the surface is given by

$$\gamma_s = \frac{\left[u_a v_a f^{3/2} + \alpha^2 R^2 w_a^2\left(5-4\sqrt{f}\right)\right]\left[\left(2+w_a^2\right)v_a^2 f + 12\alpha^2 R^2 w_a^4\left(1-\sqrt{f}\right)^2\right]}{\sqrt{f}\left[u_a v_a\sqrt{f} + 4\alpha^2 R^2 w_a^2\left(1-\sqrt{f}\right)\right]\left[2\left(2+w_a^2\right)v_a^2 f + 12\alpha^2 R^2 w_a^4\left(1-\sqrt{f}\right)\left(3-2\sqrt{f}\right)\right]}, \tag{3.7.17}$$

where,

$$u_a = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\sqrt{1+\frac{a^2}{R^2}}\ln\left[\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right] + \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right),$$

$$v_a = A\sqrt{1+\frac{a^2}{R^2}} + B\left(\sqrt{1+\frac{a^2}{R^2}}\ln\left[\sqrt{2}\sqrt{1+\frac{a^2}{R^2}} + \sqrt{1+2\frac{a^2}{R^2}}\right] - \frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}\right),$$

and

$$w_a = \sqrt{1+2\frac{a^2}{R^2}}.$$

3.8 Physical Plausibility

Since our approach does not assume any equation of state for matter, it is necessary to examine the physical plausibility of the solution in the light of energy conditions. A physically plausible solution for the core-envelope model is expected to fulfil the following requirements:

(i) The spacetime metric (3.6.6) in the core should continuously match with the spacetime metric (3.7.7) in the envelope across the core boundary $r = b$.

(ii) $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$,

(iii) $p_r \geq 0$, $p_\perp > 0$, $\frac{dp_r}{dr} < 0$, $\rho - p_r - 2p_\perp \geq 0$ for $0 \leq r \leq b$,

(iv) $\frac{dp_r}{d\rho} < 1$, $\frac{dp_\perp}{d\rho} < 1$ for $0 \leq r \leq b$,

(v) $p > 0$, $\frac{dp}{dr} < 0$, $\frac{dp}{d\rho} < 1$, $\rho - 3p > 0$ for $b \leq r \leq a$.

At the core boundary $r = b$, the anisotropy parameter vanishes hence from equation (3.6.10) we get $\frac{b^2}{R^2} = 2$. The continuity of matric coefficients and the continuity of static pressure across $r = b$ of the distribution leads to

$$11\sqrt{3}C + D = 5^{\frac{3}{4}} \left[\sqrt{3}A + B\left(\mathbb{L}(b) + \sqrt{2.5}\right)\right], \tag{3.8.1}$$

$$\sqrt{3}C + D = 5^{-\frac{1}{4}} \left[\sqrt{3}A + B\left(\mathbb{L}(b) - \sqrt{2.5}\right)\right], \tag{3.8.2}$$

where

$$\mathbb{L}(b) = \sqrt{3}ln\left(\sqrt{5} + \sqrt{6}\right). \tag{3.8.3}$$

Equations (3.8.1) and (3.8.2) determines C and D in terms of A and B

$$C = \frac{2A + B\left(\frac{2}{\sqrt{3}}\mathbb{L}(b) + \sqrt{7.5}\right)}{5^{\frac{5}{4}}}, \tag{3.8.4}$$

$$D = \frac{3\sqrt{3}A + B\left(3\mathbb{L}(b) - 8\sqrt{2.5}\right)}{5^{\frac{5}{4}}}, \tag{3.8.5}$$

substituting these values of C and D in (3.6.8) and (3.6.9) we get radial and tangential pressure in the core of the star. Owing to the complexity of expressions, programming is used to verify requirements (ii) to (v).

3.9 Discussion

We have discussed certain aspects of a non-adiabatically collapsing spherical distribution of matter associated with radial heat flow. We found that the density $\rho(r,t)$, radial pressure $p_r(r,t)$, tangential pressure $p_\perp(r,t)$ are positive throughout the distribution during its collapse from equilibrium to blackhole. Density and radial pressure decreases along the radially outward direction.

The plots showing variations of pressure, density and sound speed for the model with $\lambda = 0.05$ and $f = 0.7$ are shown in Figures 3.1 - 3.3 respectively.

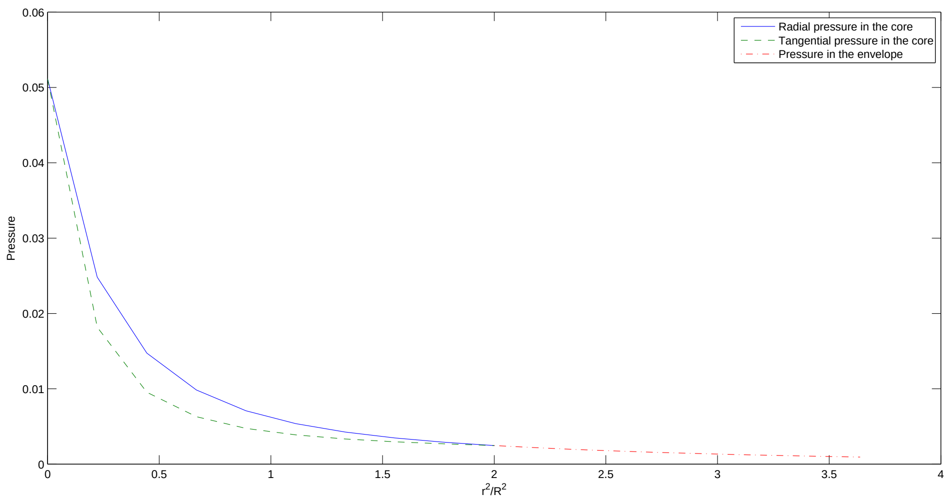

Figure 3.1: Variation of p_r, $p_\perp$ against $\frac{r^2}{R^2}$ in the core and variation of p against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

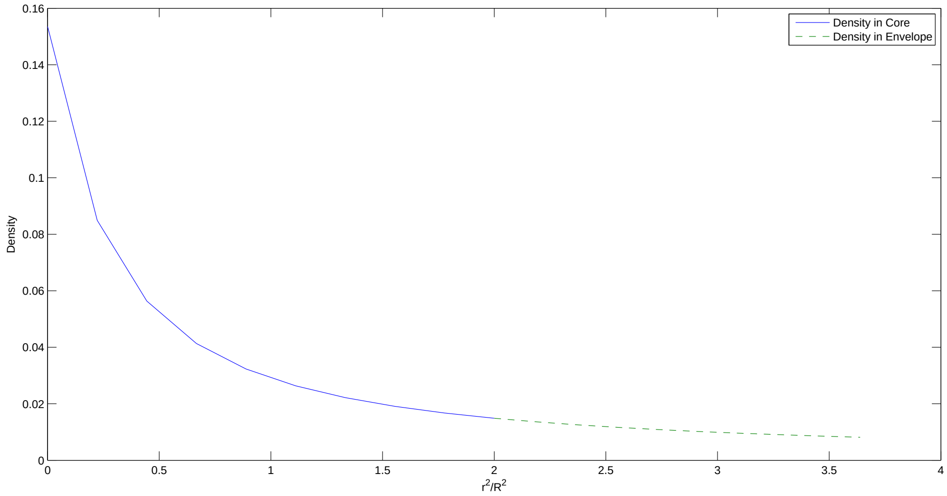

Figure 3.2: Variation of ρ against $\frac{r^2}{R^2}$ throughout the distribution for $f = 0.7$.

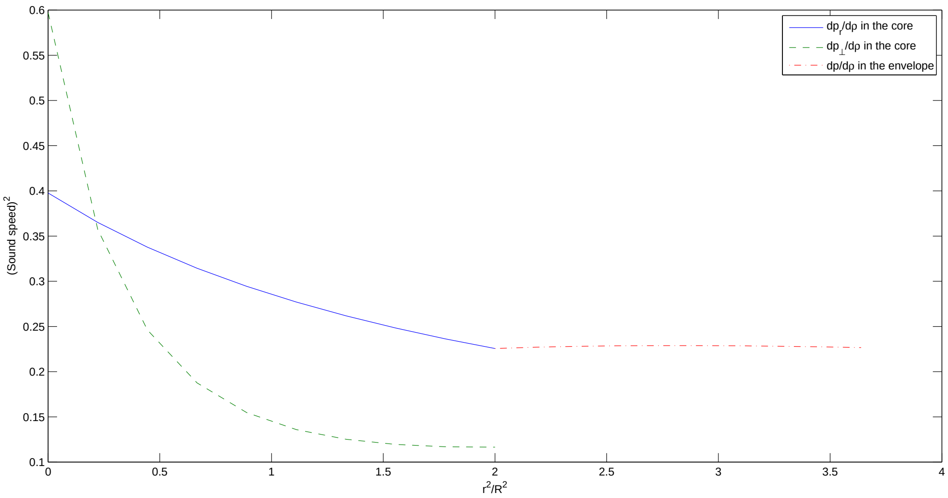

Figure 3.3: Variation of $\frac{dp_r}{d\rho}$, $\frac{dp_\perp}{d\rho}$ against $\frac{r^2}{R^2}$ in the core and variation of $\frac{dp}{d\rho}$ against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

Figure 3.4 indicates that strong energy condition is satisfied throughout the distribution. The variation of the polytropic index γ with respect to time function $f(t)$ is calculated numerically for the model with $\lambda = 0.05$ at centre and on the boundary and these variations are shown in figure 3.5. The polytropic index at the centre is less than $\frac{4}{3}$ and at the boundary is much larger than $\frac{4}{3}$ during the initial stage of collapse. This indicates that the central region is dynamically unstable. The collapsing star becomes a black hole when f takes the value 0.5356.

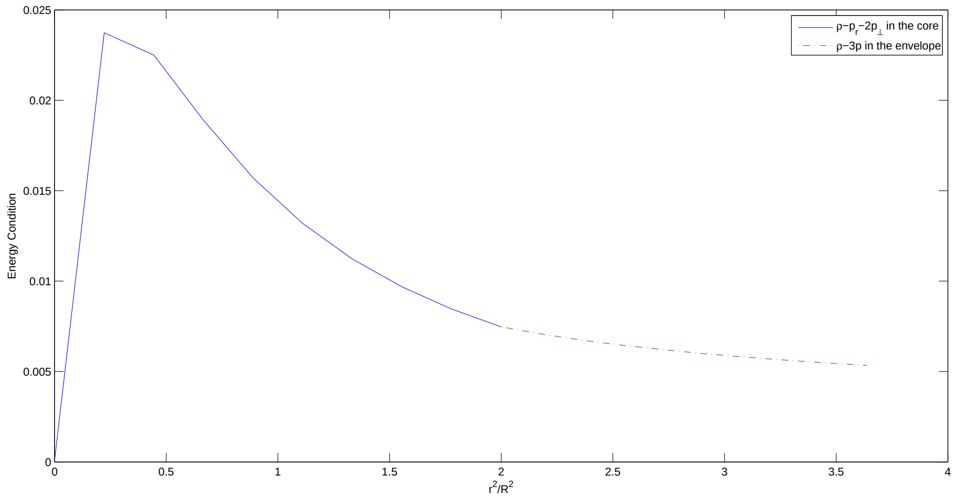

Figure 3.4: Variation of $\rho - p_r - 2p_\perp$ against $\frac{r^2}{R^2}$ in the core and variation of $\rho - 3p$ against $\frac{r^2}{R^2}$ in the envelope for $f = 0.7$.

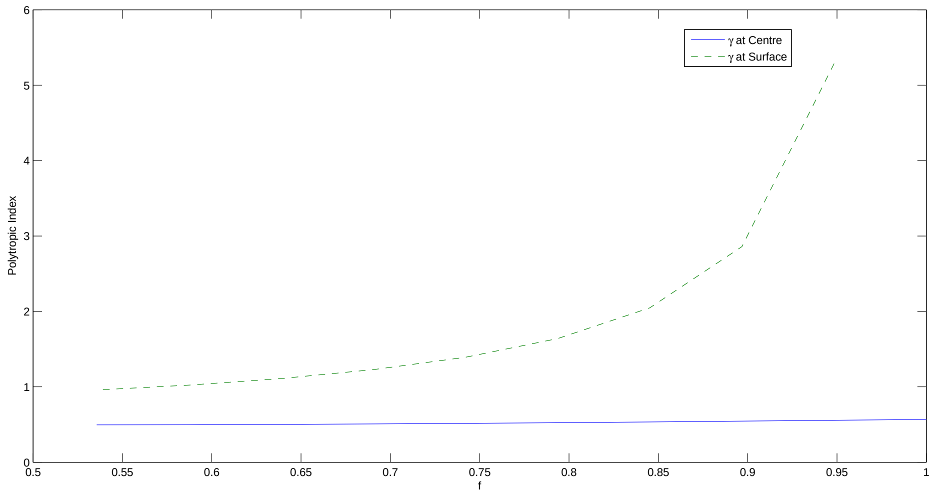

Figure 3.5: Variation of polytropic index γ at the centre and at the surface against f.

For temperature, we assume the evolution of heat flow governed by the Maxwell-Cattaneo transport equation

$$\tau_0 h^{\alpha\beta} u^c q_{\beta;c} + q^\alpha = K h^{\alpha\beta} \left(T_{,\beta} - T\dot{u}_\beta\right), \tag{3.9.1}$$

where $h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$ is the projection tensor, K is the thermal conductivity coefficient, τ_0 is the relaxation time and $\dot{u}_\beta = u_{\beta;c}u^c$. For the spacetime metric (3.2.1), transport equation (3.9.1) takes the form,

$$\tau_0 \dot{q} + e^{\nu/2} q = -K e^{-(\mu+\lambda)/2} \frac{d}{dr}\left(T e^{\nu/2}\right). \tag{3.9.2}$$

If the neutrinos are generated by thermal emission the τ_0 depends on temperature as

$$\tau_0 \propto T^{-3/2}. \tag{3.9.3}$$

Following J. Martínez [62], thermal conductivity $K = \frac{4}{3} b T^3 \tau_0$ and $b_0 = \frac{7a_0}{8}$, where $a_0 = 6.252 \times 10^{-64} cm^{-2} k^{-4}$ is a radiation constant. τ_0 can be expressed in dimensionless form as

$$\tau_0 \approx \mathbb{A} \frac{M_0}{\bar{\rho}\sqrt{Y_e T^3}}, \tag{3.9.4}$$

where $\mathbb{A} = 10^9 K^{3/2} m^{-1}$, M_0 is the initial mass of star in meters, T is the kelvin temperature, $\bar{\rho}$ is dimensionless energy density and $0.2 \leq Y_e \leq 0.3$.
Now equation (3.9.2) takes the form:

$$\frac{\mathbb{A} M_0}{\bar{\rho}\sqrt{Y_e}} T^{-3/2}\dot{q} + e^{\nu/2} q = -\left(\frac{7a}{6}\right)\left(\frac{\mathbb{A} M_0}{\bar{\rho}\sqrt{Y_e}}\right) T^{3/2} e^{-(\mu+\lambda)/2} \frac{d}{dr}\left(T e^{\nu/2}\right). \qquad (3.9.5)$$

The temperature profile can be obtained by solving equation (3.9.5) using appropriate initial conditions.

Chapter 4

Stability of Superdense Star on Paraboloidal Spacetime

Contents

In this chapter, we have discussed dynamical stability of superdense stars on paraboloidal spacetime under radial modes of pulsation. The paraboloidal spacetime metric is a particular case of Duorah and Ray [20] spacetime metric. Duorah and Ray spacetime metric was discussed in detail by Finch and Skea [26]. Our analysis indicates models with $0.26 \leq \frac{m}{a} \leq 0.36$ are stable for radial modes of pulsation. Here mass m and radius a are in kilometers as per geometrization convention. The paraboloidal geometry for its spatial sections $t = constant$ thus admits the possibilities of describing spacetime of superdense stars in equilibrium. The field equation and its solution is discussed in section 4.2. The dynamical stability of superdense star is discussed in section 4.3.

4.1 Introduction

The solution of Einstein's field equations for a perfect fluid sphere in thermodynamic equilibrium is not sufficient as the equilibrium may be stable equilibrium or unstable equilibrium. Buchdahl's theorem for stable star says that

$$a \geq \frac{8}{9} R_s,$$

where a is radius of star and R_s is the Schwarzschild radius. Hence the radius of a stable star exceeds the Schwarzschild radius. Buchdahl's theorem is independent of any equation of state $p = p(\rho)$.

Chandrasekhar considered the perturbation the solution of the star in stellar equilibrium resulting in non-zero off-diagonal elements in energy-momentum tensor by considering non-zero radial velocity for the fluid. Chandrasekhar assumed amplitude of oscillations ξ in the time dependent form $e^{i\sigma t}$ and applying Rayleigh-Ritz method of variational approach, Chandrasekhar then obtained pulsation equation. In that Pulsation equation if the frequency σ^2 is negative then the amplitude ξ does not have an upper bound. Hence for stable stars frequency must be positive.

We investigate the stability of models of superdense stars on paraboloidal spacetime under radial modes of pulsation. Tikekar and Jotania [89] have shown that the paraboloidal spacetime metric

$$ds^2 = e^{\nu(r)} dt^2 - \left(1 + \frac{r^2}{R^2}\right) dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.1.1}$$

is suitable for describing the interior of strange stars (star consisting of strange matter). The spacetime metric (4.1.1) is a particular case of spacetime metric used by Duorah and Ray [20] which has the form

$$ds^2 = A^2 y^2(x) dt^2 - Z^{-1}(x) dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.1.2}$$

with

$$x \equiv Cr^2.$$

In spacetime metric (4.1.2) if we set $A^2 y^2(x) = e^{\nu(r)}$, $C = \frac{1}{R^2}$ and $Z^{-1}(x) = 1 + x$, we get spacetime metric (4.1.1).

Finch and Skea [26] showed that stellar models of Duorah and Ray [20] do not satify Einstein's field equations and they obtained solution satisfying Einstein's field equation.

4.2 Solution of Field Equations

We take the energy-momentum tensor of the form

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij}, \qquad u^i = \left(e^{-\nu/2}, 0, 0, 0\right), \tag{4.2.1}$$

where ρ and p respectively denote the matter density and fluid pressure. The Einstein's field equations for spacetime metric (4.1.1) in view of (4.2.1) takes the following form:

$$8\pi\rho = \frac{1}{R^2}\left(1 + \frac{r^2}{R^2}\right)^{-2}\left(3 + \frac{r^2}{R^2}\right), \tag{4.2.2}$$

$$8\pi p = \left(1 + \frac{r^2}{R^2}\right)^{-1}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \tag{4.2.3}$$

$$\left(1 + \frac{r^2}{R^2}\right)\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'}{2r}\right] - \frac{r\nu'}{2R^2} + \frac{r^2}{R^4} = 0. \tag{4.2.4}$$

Following Finch and Skea [26] and Tikekar and Jotania [89], the solution of field equations (4.2.2) - (4.2.4) is given by,

$$e^{\nu/2} = (B - Az)\cos z + (A + Bz)\sin z, \tag{4.2.5}$$

where $z = \sqrt{1 + \frac{r^2}{R^2}}$ and A and B are constants of integration. Hence the matter density and fluid pressure take the following explicit forms,

$$8\pi\rho = \left(\frac{1}{R^2}\right)\left(\frac{z^2 + 2}{z^4}\right), \tag{4.2.6}$$

$$8\pi p = \frac{1}{z^2 R^2}\left[\frac{(A - Bz)\, sinz + (Az + B)\, cosz}{(A + Bz)\, sinz - (Az - B)\, cosz}\right], \tag{4.2.7}$$

and the spacetime metric (4.1.1) takes the form

$$ds^2 = \left[(B - Az)\, cosz + (A + Bz)\, sinz\right]^2 dt^2 - z^2 dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.2.8}$$

at the boundary of the star $r = a$, the interior spacetime metric (4.2.8) should continously match with Schwarzschild exterior spacetime metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{4.2.9}$$

and pressure must vanish at the boundary of the star $r = a$. These two conditions determines the constants of integration A and B as

$$A = \frac{\sqrt{1 - \frac{2m}{a}} \left(z_a sinz_a - cosz_a\right)}{\left(sinz_a + z_a cosz_a\right)\left(cosz_a + z_a sinz_a\right) - \left(sinz_a - z_a cosz_a\right)\left(cosz_a - z_a sinz_a\right)}, \tag{4.2.10}$$

$$B = \left[\frac{sinz_a + z_a cosz_a}{z_a sinz_a - cosz_a}\right] A, \tag{4.2.11}$$

where $z_a = \sqrt{1 + \frac{a^2}{R^2}}$. Substituting the values of A and B in (4.2.3), we get the expression for the pressure profile of the distribution. The expresssion for $\frac{dp}{d\rho}$ is given by

$$\frac{dp}{d\rho} = \frac{z^2 \left(Bcosz + Asinz\right) T\left(z\right)}{\left(z^2 + 4\right)\left(Asinz + Bzsinz + Bcosz - Azcosz\right)^2}, \tag{4.2.12}$$

where $T(z) = \left(z^2 \sin z + \sin z - z \cos z\right) A + \left(z^2 \cos z + \cos z + z \sin z\right) B$ and A, B are given by (4.2.10) and (4.2.11).

The scheme given by Tikekar [88] is used to compute mass and size of the star. The density at the centre of the star is given by

$$8\pi\rho(0) = \frac{3}{R^2}, \tag{4.2.13}$$

and at the surface of the star is

$$8\pi\rho(a) = \frac{1}{R^2}\left(3 + \frac{a^2}{R^2}\right)\left(1 + \frac{a^2}{R^2}\right)^{-2}. \tag{4.2.14}$$

The density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)}$ is then given by

$$\lambda = \frac{\rho(a)}{\rho(0)} = \left(1 + \frac{a^2}{3R^2}\right)\left(1 + \frac{a^2}{R^2}\right)^{-2}. \tag{4.2.15}$$

The continuity of interior spacetime metric (4.2.8) with Schwarzschild exterior spacetime metric (4.2.9) determines the mass radius relation,

$$\frac{m}{a} = \frac{a^2}{2R^2}\left(1+\frac{a^2}{R^2}\right)^{-1}. \tag{4.2.16}$$

Equations (4.2.15) and (4.2.16) give $\frac{a^2}{R^2}$ and $\frac{m}{a}$ in terms of density variation parameter λ as

$$\frac{a^2}{R^2} = \frac{1-6\lambda+\sqrt{1+24\lambda}}{6\lambda}, \tag{4.2.17}$$

and

$$\frac{m}{a} = \frac{1-6\lambda+\sqrt{1+24\lambda}}{2\left(1+\sqrt{1+24\lambda}\right)}. \tag{4.2.18}$$

Further we can express λ in terms of $\frac{m}{a}$ in the form

$$\lambda = \left(1-\frac{2m}{a}\right)\left(1-\frac{4m}{3a}\right). \tag{4.2.19}$$

4.3 Dynamic Stability

A sufficient condition for the dynamic stability of a spherically symmetric distribution of matter under small radial adiabatic perturbations has been developed by Chandrasekhar [9]. A normal mode of radial oscillations for an equilibrium configuration

$$\delta r = \xi\left(r\right) e^{i\sigma t}, \tag{4.3.1}$$

is stable if its frequency σ is real and is unstable if σ is imaginary. Chandrasekhar's pulsation equation for the spacetime metric (4.1.1) is given by

$$\begin{aligned}\sigma^2\int_0^a e^{(3\lambda+\nu)/2}\left(p+\rho\right)\frac{u^2}{r^2}dr^2 \;=\; & \int_0^a e^{(\lambda+3\nu)/2}\frac{4}{r^3}\left(\frac{dp}{dr}\right)u^2dr - \\ & \int_0^a e^{(\lambda+3\nu)/2}\frac{1}{r^2}\frac{1}{p+\rho}\left(\frac{dp}{dr}\right)^2u^2dr + \\ & \int_0^a e^{(\lambda+3\nu)/2}\left(\frac{p+\rho}{r^2}\right)8\pi e^{\lambda}pu^2dr + \\ & \int_0^a e^{(\lambda+3\nu)/2}\left(\frac{p+\rho}{r^2}\right)\frac{dp}{d\rho}\left(\frac{du}{dr}\right)^2dr, \end{aligned} \tag{4.3.2}$$

where $u = \xi r^2 e^{-\nu/2}$ and $e^{\lambda} = 1 + \frac{r^2}{R^2}$. The boundary condition to be satisfied at $r = a$ is that Lagrangian change in pressure should vanish at $r = a$ that is,

$$\Delta p = -e^{\nu/2}\left(\frac{\gamma p}{r^2}\right)\left(\frac{du}{dr}\right) = 0 \;\; at \;\; r = a,$$

where γ is the adiabatic index. Therefore we must have,

$$\frac{du}{dr} = 0 \;\; at \;\; r = a. \tag{4.3.3}$$

Following the method of Bardeen *et al.* [4], we choose

$$u = R^3 x^{3/2}\left(1 + a_1 x + b_1 x^2 + ...\right),$$

as a trial function, where $x = \frac{r^2}{R^2}$. The boundary condition $\frac{du}{dr} = 0$ at $r = a$ yields

$$3 + 5a_1 b + 7b_1 b^2 + ... = 0, \tag{4.3.4}$$

where $b = \frac{a^2}{R^2}$ and $a_1,\ b_1....$ are parameters. We consider here a three term approximation of (4.3.4). The pulsation equation (4.3.2) for the metric (4.1.1) now takes the form,

$$\begin{aligned}\sigma^2\int_0^a e^{(3\lambda+\nu)/2}(p+\rho)\frac{u^2}{R^2}dr &= \int_0^a \{T_1T_2(T_3+T_4)\}\{[(T_5T_9T_{10})-(T_{11}T_{12}T_{13})+T_{14}]T_{15}\}dr + \\ &\int_0^a \{T_1T_2(T_3+T_4)\}\{(T_{16}T_{17})\}dr, \end{aligned}\tag{4.3.5}$$

where,

$T_1 = \frac{1}{2R^2z^2}$,

$T_2 = z^2 + 3(B - Az)\,cosz + 3(Bz + A)\,sinz$,

$T_3 = \left[\frac{(A-Bz)sinz+(Az+B)cosz}{(A+Bz)sinz-(Az-B)cosz}\right]$,

$T_4 = \frac{z^2+2}{z^4}$,

$T_5 = 2T_1$,

$T_6 = Bcosz + Asinz$,

$T_7 = Az^2sinz + Asinz + Bzsinz - Azcosz + Bz^2cosz + Bcosz$,

$T_8 = (Asinz + Bzsiz + Bcosz - Azcosz)^2$,

$T_9 = \frac{T_6T_7}{T_8}$,

$T_{10} = \frac{1}{T_3+T_4}$,

$T_{11} = \frac{4r^2}{R^4z^4}$,

$T_{12} = T_{10}^2,$
$T_{13} = T_9^2,$
$T_{14} = \frac{8\pi}{R^4} T_3,$
$T_{15} = r^4 \left(1 + a_1 \frac{r^2}{R^2} + b_1 \frac{r^4}{R^4}\right)^2,$
$T_{16} = \left(\frac{z^2}{z^2+4}\right) T_9,$
$T_{17} = r^2 \left(3 + 5a_1 \frac{r^2}{R^2} + 7b_1 \frac{r^4}{R^4}\right)^2.$

4.4 Discussion

We have evaluated the integral on the right side of equation (4.3.5) numerically for different choices small, large, positive and negative values of the constants a_1, b_1. It is found that the integral admits positive value for the strange star models, $0.26 \leq \frac{m}{a} \leq 0.36$. Table 4.1 presents these numerical computations for certain specific choices of a_1 and b_1 for the model with $\frac{m}{a} = 0.27$. This analysis indicates that these models with $0.26 \leq \frac{m}{a} \leq 0.36$ will be stable for radial modes of pulsation. The static paraboloidal spacetime metric (4.2.8) for its spatial sections $t = constant$ admits the possibility of describing spacetime of superdense stars in equilibrium.

Table 4.1: The Values of the integral on the right side of the pulsation equation (4.3.5) for some specific choices of the constants a_1, b_1 with $\frac{m}{a} = 0.27$.

a_1	b_1	*Integral*
0.000	-0.717	10.4332
-0.776	0.000	4.8092
-0.858	1.000	34.8663
1.000	-1.641	22.3521
5.000×10^2	-4.627×10^2	6.6213×10^5
-5.410×10^2	5.000×10^2	7.7009×10^5
1.000×10^5	-9.240×10^4	2.6303×10^{10}
-1.000×10^5	9.240×10^4	2.6303×10^{10}

Chapter 5

Core-Envelope Models of Superdense Stars on Paraboloidal Spacetime

Contents

In this chapter, we study two core-envelope models of superdense stars on based on paraboloidal spacetime metric. Both the models satisfy all the physically plausible conditions. Core-envelope models with thin envelope are useful in the study of glitches and star quakes. The comparative study of both the models is done. A noteworthy feature of these models is that they admit thin envelope.

5.1 Introduction

The non-linear nature of Einstein's field equations is a consequence of the self interaction of the gravitational field. This makes it difficult to obtain relativistic models of spherical stars based on exact solutions of Einstein's field equations. The standard method for studying cold compact stars consists of integrating the Tolman, Oppenheimer and Volkoff (TOV) equation assuming an equation of state $p = p(\rho)$, where p is the proper pressure and ρ is the proper density, for the matter distribution. The integration continues till pressure drops down to zero for some value $r = a$ which is taken as the radius of the spherical distribution.

When the density exceeds twice nuclear density, the equation of state becomes uncertain. A widely accepted alternative approach to deal with such situations is the one suggested by Vaidya and Tikekar [99]. In this approach one assigns a geometry to the physical three space in place of the equation of state.

Tikekar and Jotania [89] have shown that the paraboloidal spacetime metric is suitable for describing relativistic models of strange stars and hybrid neutron stars. In this chapter we present two core-envelope models on paraboloidal spacetime.

The assumption of taking isotropic pressure distribution in the core and anisotropic pressure distribution in the envelope may not be unphysical in the case of core consisting of degenerate fermi fluid while its outer envelope may consist of fluid having anisotropic pressure. Further the study of glitches and quakes is important in stars having thin envelope. We investigate whether paraboloidal spacetime is usefull in describing spherical distribution of matter with isotropic pressure in the core and anisotropic pressure in the thin envelope.

The field equations for anisotroic models are described in section 5.2. Two different core-envelope models are discussed in sections 5.3 and 5.4. The physical plausibility conditions are checked in section 5.5 which also includes comparative study of the thickness of core and envelope of both the models.

5.2 The Core and Envelope of the Star

We consider the static spherically symmetric paraboloidal spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - \left(1 + \frac{r^2}{R^2}\right)dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{5.2.1}$$

Following Maharaj and Maartens [59], we write the energy momentum tensor for anisotropic fluid distribution in the form:

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij} + \pi_{ij}, \qquad u_i u^j = 1, \tag{5.2.2}$$

$$\pi_{ij} = \sqrt{3}S\left[C_i C_j - \frac{1}{3}\left(u_i u_j - g_{ij}\right)\right]. \tag{5.2.3}$$

The magnitude of anisotropic stress tensor is $S = S(r)$ and $C^i = \left(0, -\frac{1}{\sqrt{1+\frac{r^2}{R^2}}}, 0, 0\right)$ is a radial vector. For equilibrium models $u_i = \left(e^{\nu/2}, 0, 0, 0\right)$ and the non-vanishing components fo energy momentum tensor are

$$T_0^0 = \rho, \qquad T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right), \qquad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right). \tag{5.2.4}$$

The pressure along the radial and tangential direction respectively are given by

$$p_r = p + \frac{2S}{\sqrt{3}}, \tag{5.2.5}$$

and

$$p_\perp = p - \frac{S}{\sqrt{3}}. \tag{5.2.6}$$

Hence the magnitude of anisotropy is given by

$$8\pi\sqrt{3}S = p_r - p_\perp. \tag{5.2.7}$$

The field equations for the spacetime metric (5.2.1) and energy-momentum tensor (5.2.2) are equivalent to the following three equations

$$8\pi\rho = \frac{3 + \frac{r^2}{R^2}}{R^2\left[1 + \frac{r^2}{R^2}\right]^2}, \tag{5.2.8}$$

$$8\pi p_r = \left(1+\frac{r^2}{R^2}\right)^{-1}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}, \tag{5.2.9}$$

and

$$\begin{aligned} 8\pi\sqrt{3}S \;=\; & \left(1+\frac{r^2}{R^2}\right)^{-1}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}+\frac{1}{R^2}\left(1+\frac{r^2}{R^2}\right)^{-2}\left[1+\frac{\nu' r}{2}\right]- \\ & \left(1+\frac{r^2}{R^2}\right)^{-1}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'}{2r}\right]. \end{aligned} \tag{5.2.10}$$

By applying transformation

$$z = \sqrt{1+\frac{r^2}{R^2}}, \tag{5.2.11}$$

and

$$F_1 = e^{\nu/2}, \tag{5.2.12}$$

The nonlinear equation (5.2.10) takes the form

$$\frac{d^2F_1}{dz^2}-\frac{2}{z}\frac{dF_1}{dz}+\left(\frac{8\pi\sqrt{3}SR^2z^4+z^2-1}{z^2-1}\right)F_1 = 0. \tag{5.2.13}$$

The core of the star extends up to the radius $r = b(< a)$, where $S(r) = 0$ and the radius of the star is taken as $r = a$.

The isotropic pressure distribution is considered in the core region $0 \le r \le b$, hence magnitude of anisotropy parameter $S = 0$ in the core and the solution of field equations lead to the spacetime metric (4.2.8) and the expressions of density and pressure respectively are given by (4.2.2), (4.2.3).

We choose anisotropic pressure distribution in the envelope, hence the anisotropic parameter $S(r) \neq 0$ for $b \le r \le a$, where a is the boundary of the star.

5.3 Core-Envelope Model - 1

On prescribing

$$8\pi\sqrt{3}S = \frac{(z^2-1)(9-4z^2)}{4z^6R^2}, \tag{5.3.1}$$

the equation (5.2.13) takes the form

$$4z^2\frac{d^2F_1}{dz^2} - 8z\frac{dF_1}{dz} + 9F_1 = 0, \tag{5.3.2}$$

which admits the closed form solution as

$$e^{\nu/2} = Cz^{3/2}logz + Dz^{3/2}, \tag{5.3.3}$$

where C and D are constants of integration. Therefore the spacetime metric in the envelope region $b \leq r \leq a$ is described by:

$$ds^2 = \left(Cz^{3/2}logz + Dz^{3/2}\right)^2 dt^2 - z^2dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{5.3.4}$$

The matter density, radial pressure and tangential pressure take the following forms:

$$8\pi\rho = \frac{2+z^2}{R^2z^4}, \tag{5.3.5}$$

$$8\pi p_r = \frac{\left(3logz + 2 - z^2logz\right)C + \left(3 - z^2\right)D}{R^2z^4\left(Clogz + D\right)}, \tag{5.3.6}$$

$$8\pi p_\perp = \frac{\left(3logz + 2 - z^2logz\right)C + \left(3 - z^2\right)D}{R^2z^4\left(Clogz + D\right)} - \frac{9-4z^2}{4R^2z^6}. \tag{5.3.7}$$

At the boundary of the star $r = a$, the spacetime metric in the envelope (5.3.4) should continously match with Schwarzschild exterior spacetime metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{5.3.8}$$

Also radial pressure (5.3.6) must vanish at the boundary of the star $r = a$. The conditions yields the following relationships:

$$\frac{m}{a} = \frac{z_a^2 - 1}{2z_a^2}, \tag{5.3.9}$$

$$C = \frac{z_a^2 - 3}{2z_a^{5/2}}, \tag{5.3.10}$$

and

$$D = \frac{3logz_a + 2 - z_a^2logz_a}{2z_a^{5/2}}. \tag{5.3.11}$$

where $z_a = \sqrt{1+\frac{a^2}{R^2}}$. Substituting the value of C and D in (5.3.6) and (5.3.7) we get radial and tangential pressure in the envelope of the star.

At the core-envelope boundary $r = b$, due to assumption (5.3.1) gives core radius as $b = \frac{\sqrt{5}}{2}R$. Also at the core-envelope boundary, coefficients of the spacetime metric (4.2.8) must continuously match with spacetime metric (5.3.4) and $p(b) = p_r(b) = p_\perp(b)$. These conditions lead to the following values for A and B in terms of C and D as

$$A = 1.3367C + 1.1928D, \tag{5.3.12}$$

$$B = -0.2850C + 0.4938D. \tag{5.3.13}$$

Substituting the values of A and B in equation (4.2.3), we get pressure in the core of the star.

5.4 Core-Envelope Model - 2

By choosing

$$8\pi\sqrt{3}S = \frac{(z^2-1)(2-z^2)}{z^6R^2}, \tag{5.4.1}$$

where z is given by (5.2.11), Tikekar and Jotania [90] have obtained solution of (5.2.13) in the form

$$e^{\nu/2} = Ez^2 - 2Fz, \tag{5.4.2}$$

where E and F are constants of integration. The spacetime metric in the envelope region is described by the metric

$$ds^2 = \left(Ez^2-2Fz\right)^2 dt^2 - z^2dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right). \tag{5.4.3}$$

The expressions for density, radial pressure and tangential pressure in the envelope are respectively given by

$$8\pi\rho = \frac{2+z^2}{R^2z^4}, \tag{5.4.4}$$

$$8\pi p_r = \frac{Ez(4-z^2) - 2F(2-z^2)}{R^2z^2\left[Ez^3-2Fz^2\right]}, \tag{5.4.5}$$

$$8\pi p_\perp = \frac{Ez(4-z^2) - 2F(2-z^2)}{R^2z^2\left[Ez^3-2Fz^2\right]} - \frac{2-z^2}{R^2z^6}. \tag{5.4.6}$$

Equation (5.4.1) determines the core boundary as $b = R$. The constants E and F are to be determined by matching the spacetime metric (5.4.3) with Schwarzschild exterior spacetime metric (5.3.8), across the boundary $r = a$, where $p_r(a) = 0$, which gives

$$\frac{m}{a} = \frac{z_a^2 - 1}{2z_a^2}, \tag{5.4.7}$$

$$E = \frac{z_a^2 - 2}{2z_a^3}, \tag{5.4.8}$$

and

$$F = \frac{z_a^2 - 4}{4z_a^2}, \tag{5.4.9}$$

where $z_a = \sqrt{1 + \frac{a^2}{R^2}}$.

Substituting the values of E and F in (5.4.5) and (5.4.6), we get expressions of radial and tangential pressure in the envelope. At the core-envelope boundary $r = b$, the spacetime metric (4.2.8) must continuously match with spacetime metric (5.4.3) and $p(b) = p_r(b) = p_\perp(b)$. This gives

$$\left(sin\sqrt{2} - \sqrt{2}cos\sqrt{2}\right) A + \left(cos\sqrt{2} + \sqrt{2}sin\sqrt{2}\right) B = 2E - 2\sqrt{2}F, \tag{5.4.10}$$

and

$$\left(sin\sqrt{2} + \sqrt{2}cos\sqrt{2}\right) A + \left(cos\sqrt{2} - \sqrt{2}sin\sqrt{2}\right) B = 2E. \tag{5.4.11}$$

Solving (5.4.10) and (5.4.11) for A and B we get,

$$A = 1.9755E - 1.2410F, \tag{5.4.12}$$

$$B = 0.31185E - 1.2083F. \tag{5.4.13}$$

Substituting the values of A and B in equation (4.2.3), we can obtain the expression of pressure in the core of the star.

5.5 Discussion

Since we have not assumed any equation of state, the matter distribution in the core and envelope should satisfy the following conditions:

(i) $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$,

(ii) $p > 0$, $\frac{dp}{dr} < 0$, $\frac{dp}{d\rho} < 1$, $\rho - p > 0$ for $0 \leq r \leq b$,

(iii) $p_r \geq 0$, $p_\perp > 0$, $\frac{dp_r}{dr} < 0$ for $b \leq r \leq a$,

(iv) $\frac{dp_r}{d\rho} < 1$, $\frac{dp_\perp}{d\rho} < 1$, $\rho - p_r \geq 0$, $\rho - p_\perp \geq 0$ for $b \leq r \leq a$.

The scheme given by Tikekar [88], which is described in section 4.2 is used to determine the mass and size of the superdense star. It follows from the expression (5.3.5) that $\rho > 0$, $\frac{d\rho}{dr} < 0$ for $0 \leq r \leq a$ for both core-envelope models. The expressions of $\frac{a^2}{R^2}$, $\frac{m}{a}$ and density variation parameters λ are described by equations (4.2.17) - (4.2.19) respectively.

In Finch and Skea [26] approach, the ratio $\frac{A}{B}$ is restricted by the limits $0.217958 \leq \frac{A}{B} \leq 6.406980$. This restriction, in view of the arguments described in section 4.2 leads to the constraint $\frac{m}{a} < 0.3614955$. If $\frac{m}{a} > 0.3614955$, then $\frac{dp}{d\rho} > 1$ in the core, and therefore the models with $\frac{m}{a} > 0.3614955$, physical plausibility condition (ii) is not satisfied in the core. Further for core-envelope model - 1 having $\frac{m}{a} \geq 0.36$ it is observed that $\frac{dp_r}{d\rho} > 1$ in the envelope violating the condition (iv). Following Sharma *et al.* [81], we choose central density as $\rho(0) = 4.68 \times 10^{15} gmcm^{-3}$. From (5.3.5) it is observed that density is decreasing throughout the distribution. It is observed that conditions (ii) - (iv) are satisfied for the stars for which $0.28 \leq \frac{m}{a} \leq 0.35$ for the first model and $0.26 \leq \frac{m}{a} \leq 0.36$ for the second model, using programming and graphical methods.

Numerical estimates of the radius of the star (in kilometers), the core-radius (in kilometers), the mass of the star (in kilometers) and the thickness of the envelope (in kilometers) for the first model are given in table 5.1 and for the second model are given in table 5.2. The mass of the star in grams is obtained as $M = \frac{mc^2}{G}$. These models admit thin envelopes and the thickness of the envelope increases as $\frac{m}{a}$ increases.

Table 5.1: Masses and equilibrium radii of core-envelope model - 1 of superdense stars corresponding to $\rho(0) = 4.68 \times 10^{15} gm/cm^3$.

$\frac{m}{a}$	a	m	$b(= \frac{\sqrt{5}}{2}R)$	Thickness of the envelope
0.28	6.617267	1.852835	6.557918	0.059349
0.29	6.892874	1.998934	6.557918	0.334956
0.30	7.183840	2.155152	6.557918	0.625921
0.31	7.492298	2.322612	6.557918	0.934380
0.32	7.820774	2.502648	6.557918	1.262856
0.33	8.172285	2.696854	6.557918	1.614367
0.34	8.550480	2.907163	6.557918	1.992561
0.35	8.959822	3.135938	6.557918	2.401904

Table 5.2: Masses and equilibrium radii of core-envelope model - 2 of superdense stars corresponding to $\rho(0) = 4.68 \times 10^{15} gm/cm^3$.

$\frac{m}{a}$	a	m	$b(= R)$	Thickness of the envelope
0.26	6.105090	1.587323	5.865581	0.239509
0.27	6.355196	1.715903	5.865581	0.489616
0.28	6.617267	1.852835	5.865581	0.751687
0.29	6.892874	1.998934	5.865581	1.027294
0.30	7.183840	2.155152	5.865581	1.318259
0.31	7.492298	2.322612	5.865581	1.626718
0.32	7.820774	2.502648	5.865581	1.955194
0.33	8.172285	2.696854	5.865581	2.306705
0.34	8.550480	2.907163	5.865581	2.684899
0.35	8.959822	3.135938	5.865581	3.094242
0.36	9.405854	3.386107	5.865581	3.540274

The plots showing variation of ρ throughout the distribution, p in the core, p_r and $p_\perp$ in the envelope, $\frac{dp}{d\rho}$ in the core, $\frac{dp_r}{d\rho}$ and $\frac{dp_\perp}{d\rho}$ in the envelope against z for core-envelope model - 1 and model - 2 with $\frac{m}{a} = 0.29$ are depicted in Figures 5.1 - 5.5. Figure 5.2 shows that the pressure in the core for core-envelope model - 1 is always greater than that of model - 2. Figure 5.3 shows that tangential pressure is always greater than radial pressure in the envelope for both the models. It is also observed from figures 5.4 and 5.5 that speed of sound is less than speed of light.

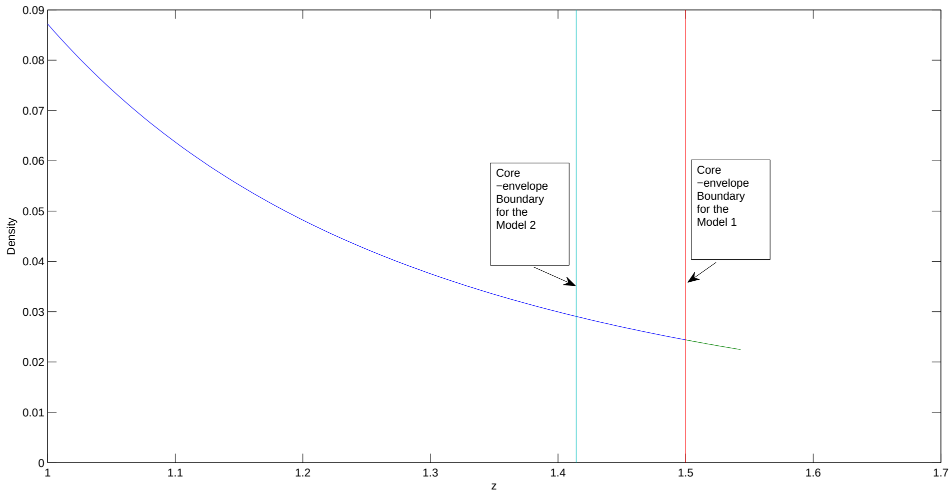

Figure 5.1: Variation of ρ against z throughout the distribution

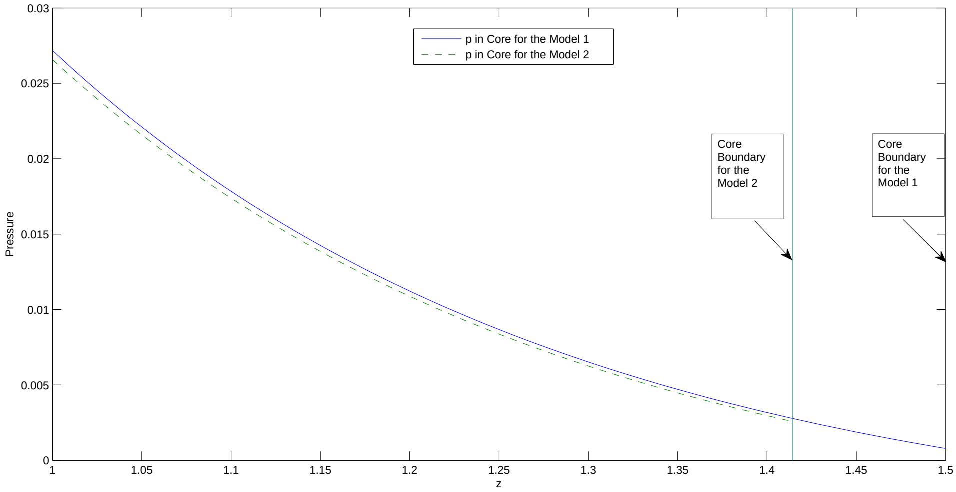

Figure 5.2: Variation of p against z in the core

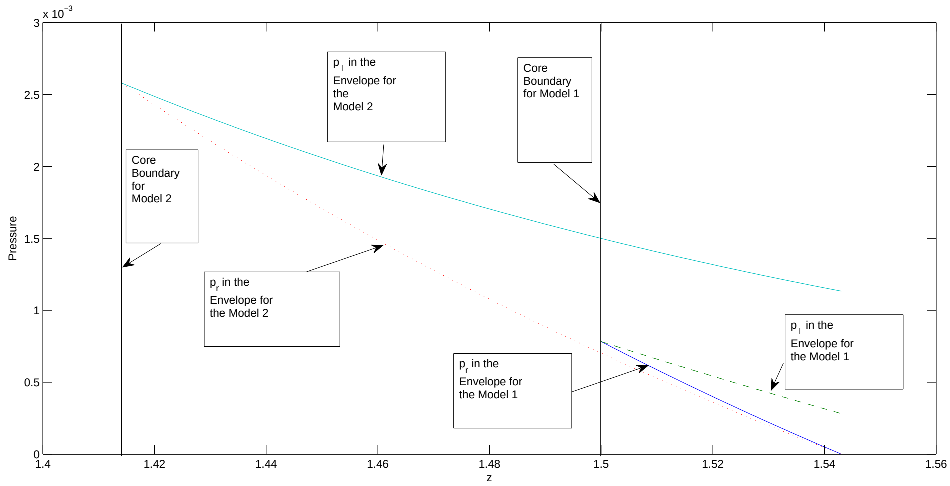

Figure 5.3: Variation of p_r and $p_\perp$ in the envelope

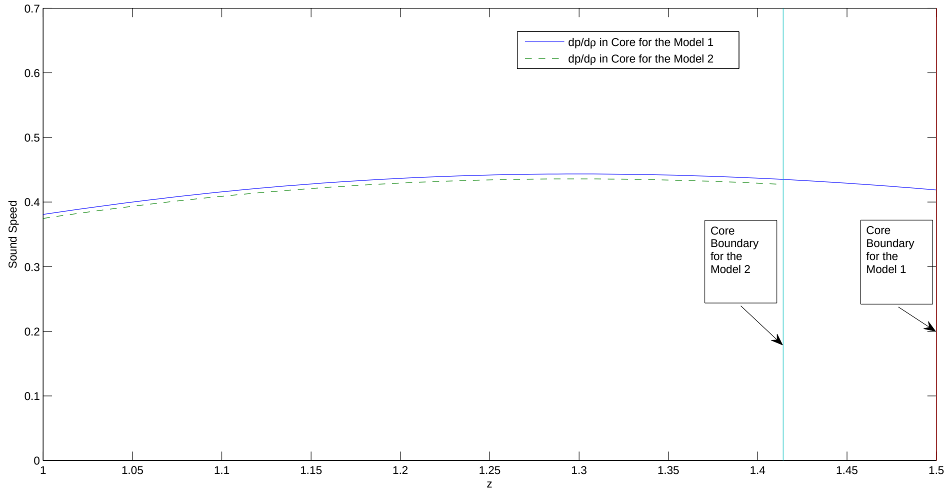

Figure 5.4: Variation of $\frac{dp}{d\rho}$ against z in the core

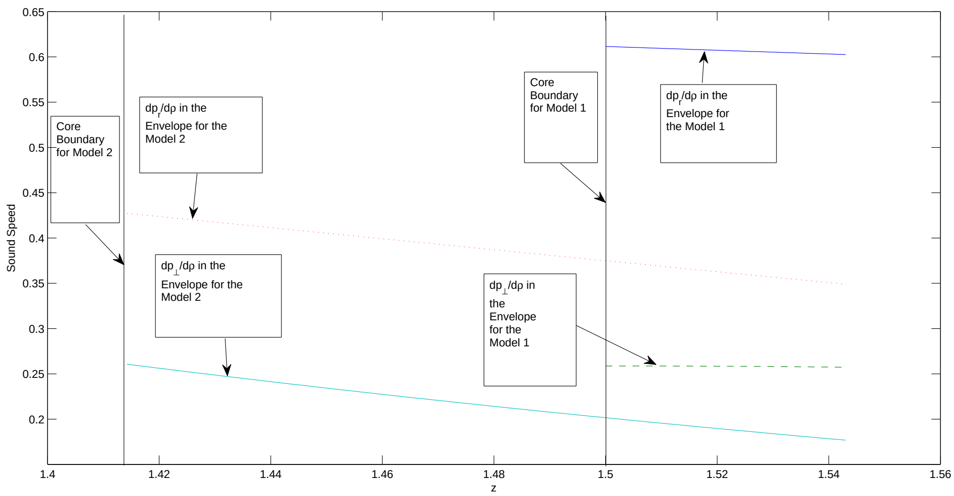

Figure 5.5: Variation of $\frac{dp_r}{d\rho}$ and $\frac{dp_\perp}{d\rho}$ against z in the envelope

These models are falling under Type I and Type II strange stars (Tikekar and Jotania [89]). From table 5.1 and table 5.2, it is obsereved that the first core-envelope model has very thin envelope. Thus we have presented core-envelope models with isotropic pressure in the core and anisotropic pressure in the envelope. A noteworthy feature of these models is that they admit thin envelope. Hence these models are significant in the study of glitches and star quakes.

Chapter 6

Relativistic Stellar Models Admitting a Quadratic Equation of State

Contents

In this chapter a class of solutions describing the interior of a static spherically symmetric compact anisotropic star based on paraboloidal spacetime is reported. Based on physical grounds appropriate bounds on the model parameters have been obtained and it has been shown that the model admits an equation of state which is quadratic in nature.

6.1 Introduction

A neutron star is assumed to be the final stage of a collapsing star whose gravitational attraction is counter balanced by its constituent degenerate neutron gas. However, observational studies in the recent past strongly point towards the existence of exotic class of compact stars which are more compact than ordinary neutron stars (see for example, [18], [19], [51], [52], [53], [104], [105]). To have a proper understanding of such ultra-compact objects, it is imperative to know the exact composition and nature of particle interactions at extremely high density regime. From general relativistic perspective, if the equation of state of the material composition of a compact star is known, one can easily integrate the Tolman-Oppenheimer-Volkoff (TOV) equations to analyze the physical features of the star. The problem is that we still lack reliable information about physics of particle interactions beyond nuclear density. In the class of stars having a density regime exceeding nuclear matter density, many exotic phases may exist in the interior, including a possible transition from hadronic to quark degrees of freedom ([1], [2], [11], [24], [101], [103]). However, even in the extreme case of a quantum cromo dynamics inspired model based equation of state, by and large, remains phenomenological till date.

The objective of this chapter is to construct models of equilibrium configurations of relativistic ultra-compact objects when no reliable information about the composition and nature of particle interactions are available. This can be achieved by generating exact solutions of Einsteins field equations describing the interior of a static spherically symmetric relativistic star. However, finding exact solutions of Einsteins field equations is extremely difficult due to highly non-linear nature of the governing field equations. Consequently, many simplifying assumptions are often made to tackle the problem. Since general relativity provides a mutual correspondence between the material composition of a relativistic star and its associated spacetime, we will adopt a geometric approach to deal with such a situation. In this approach, a suitable ansatz with a clear geometric characterization for one of the metric potentials of the associated spacetime metric will be prescribed to determine the other. Such a method was initially proposed by Vaidya and Tikekar [99]; subsequently the method was utilized by many to generate and analyze physically viable models of compact astrophysical objects (see for example, [43], [45], [58], [64], [81], [82], [83], [88] and references therein). In present work we consider paraboloidal spacetime metric described by Tikekar and Jotania [89].

We have incorporated a general anisotropic term in the stress-energy tensor representing the material composition of the star. Impact of anisotropy on stellar configurations may be found in the pioneering works of Bowers and Liang [7] and Herrera and Santos [37]. Local anisotropy at the interior of an extremely dense object may occur due various factors such as the existence of type 3A superfluid ([7], [76], [44]), phase transition ([84]), presence of electromagnetic field ([41]), etc. Mathematically, anisotropy provides an extra degree of freedom in our system of equations. Therefore, on top of paraboloidal spacetime metric, we shall utilize this freedom to assume a particular pressure profile to solve the system. In the past, a large class of exact solutions corresponding to spherically symmetric anisotropic matter distributions have been found and analyzed (see for example, [5], [28], [36], [50], [59], [61], [80]). Maharaj and Chaisi [55] have prescribed an algorithm to generate anisotropic models from known isotropic solutions. Dev and Gleiser ([14],[15],[17]) have studied the effects of anisotropy on the properties of spherically symmetric gravitationally bound objects and also investigated stability of such configurations. It has been shown that if the tangential pressure $p_{\perp}$ is greater than the radial pressure p_r of a stellar configuration, the system becomes more stable. Impact of anisotropy has also been investigated by Ivanov [40]. In an anisotropic stellar model for strange stars developed by Paul *et. al.* [71], it has been shown that the value of the bag constant depends on the anisotropic parameter. For a charged anisotropic stellar model governed by the Massachusetts Institute of Technology (MIT) bag model equation of state, Rahaman *et. al.* [74] have shown that the bag constant depends on the compactness of the star. A core-envelope type model describing a gravitationally bound object with an anisotropic fluid distribution has been obtained in [86], [87], [95].

In this chapter, we have constructed a non-singular anisotropic stellar model on paraboloidal spacetime, satisfying all the necessary conditions of a realistic compact star. Based on physical grounds, bounds on the model parameters are prescribed and the relevant equation of state for the system is worked out. An interesting feature of this model is that the solution admits a quadratic equation of state. Due to complexity, it is often very difficult to generate an equation of state ($p = p\,(\rho)$) from known solutions of Einsteins field equations. In fact, in most of the models involving an equation of state, the equation of state is prescribed a priori to generate the solutions. For example, Sharma and Maharaj [80] have obtained an analytic solution for compact anisotropic stars where a linear equation of state was assumed. Thirukkanesh and Maharaj [85] have assumed a linear equation of state to obtain solutions of an anisotropic fluid distribution. Feroze and Siddiqui [25] and Maharaj and Takisa [60] have separately utilized a quadratic equation of state to generate solutions for static anisotropic spherically symmetric charged distributions. A general approach to deal with anisotropic charged fluid systems admitting a linear or non-linear equation of state have been discussed by Varela *et. al.* [100]. In present model, we do not prescribe the equation of state; rather the solution imposes a constraint on the equation of state corresponding to the material composition of the highly dense system.

In section 6.2, the relevant field equations describing a gravitationally bound spherically symmetric anisotropic stellar configuration in equilibrium have been laid down. Solution to the system of equations is obtained in section 6.3 and analyzed bounds on the model parameters based on physical grounds are analyzed. Physical features of the model have been discussed in section 6.4. We have also generated an approximated equation of state in this section which has been found to be quadratic in nature. In section 6.5, we have concluded by pointing out some interesting features of our model.

6.2 Field Equations

We write the interior spacetime of a static spherically symmetric stellar configuration in the standard form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{6.2.1}$$

where $\nu(r)$ and $\lambda(r)$ are yet to be determined. We assume that the material composition of the configuration is anisotropic in nature and accordingly we write the energy-momentum tensor in the form

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij} + \pi_{ij}, \tag{6.2.2}$$

where ρ and p represent energy-density and isotropic pressure of the system and u^i is the 4-velocity of fluid. The anisotropic stress-tensor π_{ij} is assumed to be of the form

$$\pi_{ij} = \sqrt{3}S\left[C_iC_j - \frac{1}{3}\left(u_iu_j - g_{ij}\right)\right], \tag{6.2.3}$$

where $S = S(r)$ denotes the magnitude of anisotropy and $C^i = \left(0, -e^{-\lambda/2}, 0, 0\right)$ is a radially directed vector. We Calculate non-vanishing components of the energy-momentum tensor as

$$T_0^0 = \rho, \quad T_1^1 = -\left(p + \frac{2S}{\sqrt{3}}\right), \quad T_2^2 = T_3^3 = -\left(p - \frac{S}{\sqrt{3}}\right). \tag{6.2.4}$$

This implies that the radial presure and the tangential pressure can be obtained in the following forms

$$p_r = p + \frac{2S}{\sqrt{3}}, \tag{6.2.5}$$

$$p_\perp = p - \frac{S}{\sqrt{3}}, \tag{6.2.6}$$

respectively. Therefore, magnitude of the anisotroy is obtained as

$$p_r - p_\perp = \sqrt{3}S. \tag{6.2.7}$$

The Einstein's field equations corresponding to the spacetime metric (6.2.1) and the energy-momentum tensor (6.2.3) are obtained as

$$8\pi\rho = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{6.2.8}$$

$$8\pi p_r = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2}, \tag{6.2.9}$$

$$8\pi p_\perp = \frac{e^{-\lambda}}{4}\left[2\nu'' + (\nu' - \lambda')\left(\nu' + \frac{2}{r}\right)\right]. \tag{6.2.10}$$

Defining the mass within a radius r as

$$m(r) = \frac{1}{2}\int_0^r \tilde{r}^2\rho(\tilde{r})d\tilde{r}. \tag{6.2.11}$$

We rewrite the field equations (6.2.8) - (6.2.10) in the form

$$e^{-\lambda} = 1 - \frac{2m}{r}, \tag{6.2.12}$$

$$r\left(r - 2m\right)\nu' = 8\pi p_r r^3 + 2m, \tag{6.2.13}$$

$$\left(8\pi\rho + 8\pi p_r\right)\nu' + 2(8\pi p_r') = -\frac{4}{r}\left(8\pi\sqrt{3}S\right). \tag{6.2.14}$$

6.3 Interior Solution

To solve the system of equations (6.2.12) - (6.2.14), we make use of the ansatz for metric potential $e^{\lambda(r)}$ as

$$e^{\lambda(r)} = 1 + \frac{r^2}{R^2}, \tag{6.3.1}$$

where R is the curvature parameter. The $t = constant$ sections of (6.2.1) for ansatz (6.3.1) represent paraboloidal spacetimes immersed in 4-Euclidean spacetime.

The energy density and mass function are then obtained as

$$8\pi\rho = \frac{3 + \frac{r^2}{R^2}}{R^2\left(1 + \frac{r^2}{R^2}\right)^2}, \tag{6.3.2}$$

$$m(r) = \frac{r^3}{2R^2\left(1 + \frac{r^2}{R^2}\right)}. \tag{6.3.3}$$

Combining equations (6.2.13) and (6.3.3), we get

$$\nu' = \left(8\pi p_r\right) r\left(1 + \frac{r^2}{R^2}\right) + \frac{r}{R^2}. \tag{6.3.4}$$

To integrate equation (6.3.4), we assume $8\pi p_r$ in the form

$$8\pi p_r = \frac{p_0\left(1 - \frac{r^2}{R^2}\right)}{R^2\left(1 + \frac{r^2}{R^2}\right)^2}, \tag{6.3.5}$$

where $p_0 > 0$ is a parameter such that $\frac{p_0}{R^2}$ denotes the central pressure. The particular form of the radial pressure profile assumed here is reasonable due to the following facts:

1. Differentiation of equation (6.3.5) yields

$$8\pi \frac{dp_r}{dr} = \frac{-2rp_0\left(3 - \frac{r^2}{R^2}\right)}{R^4\left(1 + \frac{r^2}{R^2}\right)^3}. \qquad (6.3.6)$$

   For $p_0 > 0$, equation (6.3.6) implies that $\frac{dp_r}{dr} < 0$, i.e., the radial pressure is a decreasing function of radial parameter r. At a finite radial distance $r = R$ the radial pressure vanishes which is an essential criterion for the construction of a realistic compact star. The curvature parameter R is then identified as the radius of the star.

2. The particular choice (6.3.5) makes equation (6.3.4) integrable.

Substituting equation (6.3.5) in equation (6.3.4), we obtain

$$\nu' = \frac{2p_0 r}{R^2\left(1 + \frac{r^2}{R^2}\right)} + (1 - p_0)\frac{r}{R^2}, \qquad (6.3.7)$$

which is integrable and yields

$$e^{\nu} = C\left(1 + \frac{r^2}{R^2}\right)^{p_0} e^{(1-p_0)r^2/2R^2}, \qquad (6.3.8)$$

where C is a constant of integration. Thus the interior spacetime of the configuration takes the form

$$\begin{aligned} ds^2 &= C\left(1 + \frac{r^2}{R^2}\right)^{p_0} e^{(1-p_0)r^2/2R^2} dt^2 - \left(1 + \frac{r^2}{R^2}\right) dr^2 \\ &\quad -r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \end{aligned} \qquad (6.3.9)$$

which is non-singular at $r = 0$.

Making use of equations (6.2.14), (6.3.2), (6.3.5) and (6.3.7), the anisotropy can be determined as follows

$$\begin{aligned} 8\pi\sqrt{3}S = &-\frac{\frac{r^2}{R^2}}{4R^2\left(1 + \frac{r^2}{R^2}\right)^3}\left[\left((3 + p_0) + (1 - p_0)\frac{r^2}{R^2}\right)\right. \\ &\left.\times\left(2p_0 + (1 - p_0)\left(1 + \frac{r^2}{R^2}\right)\right) + 4p_0\left(\frac{r^2}{R^2} - 3\right)\right]. \end{aligned} \qquad (6.3.10)$$

Note that the anisotropy vanishes at the centre ($r = 0$) as expected. The tangential pressure takes the form

$$8\pi p_{\perp} = 8\pi p_r - 8\pi\sqrt{3}S = \frac{4p_0\left(1 - \frac{r^4}{R^4}\right) + \frac{r^2}{R^2} f\left(r, p_0, R\right)}{4R^2\left(1 + \frac{r^2}{R^2}\right)^3}, \tag{6.3.11}$$

where,

$$f\left(r, p_0, R\right) = \left[\left(3 + p_0 + (1 - p_0)\frac{r^2}{R^2}\right)\left(2p_0 + (1 - p_0)\left(1 + \frac{r^2}{R^2}\right)\right) + 4p_0\left(\frac{r^2}{R^2} - 3\right)\right].$$

This model has three independent parameters, namely, p_0, C and R. The requirement that the interior metric (6.3.9) should be matched to the Schwarzschild exterior spacetime metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{6.3.12}$$

across the boundary $r = R$ of the star together with the condition that the radial pressure should vanish at the surface ($p_r\left(r = R\right) = 0$) help us to determine these constants. Note that the form of the radial pressure profile is such that the condition ($p_r\left(r = R\right) = 0$) itself becomes the definition of the radius R of the star in this construction. Matching the relevant metric coefficients across the boundary R then yields

$$R = 4m, \tag{6.3.13}$$

$$C = \frac{e^{-(1-p_0)/2}}{2^{p_0+1}}, \tag{6.3.14}$$

where m is the total mass enclosed within the radius R from the centre of the star. If the radius of the star R is known, equation (6.3.13) can be utilized to determine the total mass m of the star and vice-versa. For a given value of p_0, equation (6.3.14) determines C. In this model $\frac{p_0}{R^2}$ corresponds to the central pressure. Therefore, for a given mass (m) or radius (R), if the central pressure is specified, the system is completely determined.

Following Delgaty and Lake [13], we impose the following conditions on our system so that it becomes a realistic and physical acceptable model.

(i) $\rho(r),\ p_r(r),\ p_\perp(r) \geq 0$, for $0 \leq r \leq R$.

(ii) $\rho - p_r - 2p_\perp \geq 0$, for $0 \leq r \leq R$.

(iii) $\frac{d\rho}{dr},\ \frac{dp_r}{dr},\ \frac{dp_\perp}{dr} < 0$, for $0 \leq r \leq R$.

(iv) $0 \leq \frac{dp_r}{d\rho} \leq 1$, $0 \leq \frac{dp_\perp}{d\rho} \leq 1$, for $0 \leq r \leq R$.

The requirements (i) and (ii) imply that the weak and dominant energy conditions are satisfied. Condition (iii) ensures regular behaviour of the energy density and the two pressures $(p_r,\ p_\perp)$. The condition (iv) is invoked to ensure that the sound speed does not exceed speed of light. In addition, for regularity, we demand that the anisotropy should vanish at the centre, i.e., $p_r = p_\perp$ at $r = 0$. From equation (6.3.10), we observe that the anisotropy vanishes at $r = 0$ and $S(r) > 0$ for $0 < r < R$. Interestingly, for the particular choice $p_0 = 1$, the anisotropy also vanishes at the boundary $r = R$ in this construction. From equation (6.3.2), it is obvious that $\rho > 0$, and

$$8\pi\frac{d\rho}{dr} = \frac{-2r\left(5 + \frac{r^2}{R^2}\right)}{R^4\left(1 + \frac{r^2}{R^2}\right)^3}, \tag{6.3.15}$$

which estabilishes that ρ decreases in the radially outward direction. We have already stated that $\frac{p_0}{R^2}$ corresponds to the central pressure which implies that $p_0 > 0$. It can be shown that for $p_\perp > 0$, we must have $p_0 < 1$. Thus the bounds on p_0 can be obtained as

$$0 < p_0 \leq 1. \tag{6.3.16}$$

To obtain a more stringent bound on p_0, we evaluate

$$8\pi\frac{dp_\perp}{dr} = \frac{r\left[(3 - 20p_0 + p_0^2) + (2 + 12p_0 - 6p_0^2)\frac{r^2}{R^2} + (-1 - 4p_0 + 5p_0^2)\frac{r^4}{R^4}\right]}{2R^4\left(1 + \frac{r^2}{R^2}\right)^4}, \tag{6.3.17}$$

at two different points. At the centre of the star $(r = 0)$ it takes the following form

$$\left(8\pi\frac{dp_\perp}{dr}\right)_{(r=0)} = 0, \tag{6.3.18}$$

and at the boundary of the star $(r = R)$, it takes the form

$$\left(8\pi\frac{dp_\perp}{dr}\right)_{(r=R)} = \frac{1 - 3p_0}{8R^3}, \tag{6.3.19}$$

which will be negative if $p_0 > \frac{1}{3}$. Therefore, a more stringent bound on the parameter p_0 is obtained as

$$\frac{1}{3} < p_0 \leq 1. \tag{6.3.20}$$

To verify whether the bound on p_0 satisfies the causality condition $0 < \frac{dp_r}{d\rho} < 1$, we combine equations (6.3.6) and (6.3.15), to yield

$$\frac{dp_r}{d\rho} = \frac{p_0\left(3 - \frac{r^2}{R^2}\right)}{5 + \frac{r^2}{R^2}}. \tag{6.3.21}$$

Now, at the centre of the star ($r = 0$), $\frac{dp_r}{d\rho} < 1$ if the condition $p_0 < 1.6667$ is satisfied and at the boundary of the star ($r = R$), $\frac{dp_r}{d\rho} < 1$ if the condition $p_0 < 3$ is satisfied. Both these restrictions are consistent with the requirement given in (6.3.20).

Similarly, we can obtain as

$$\frac{dp_\perp}{d\rho} = \frac{(-3 + 20p_0 - p_0^2) + (-2 - 12p_0 + 6p_0^2)\frac{r^2}{R^2} + (1 + 4p_0 - 5p_0^2)\frac{r^4}{R^4}}{4\left(1 + \frac{r^2}{R^2}\right)\left(5 + \frac{r^2}{R^2}\right)}, \tag{6.3.22}$$

At the centre ($r = 0$), the requirement $\frac{dp_\perp}{d\rho} < 1$ puts a constraint on p_0 such that $p_0 < 1.2250$. At the boundary of the star the corresponding requirement is given by $p_0 < 4.3333$. Both these requirements are also consistent with the bound $\frac{1}{3} < p_0 \leq 1$.

We now investigate the bound on the model parameters based on stability. To check stability of our model, we shall use Herrera's [35] overtuning technique which states that the region for which radial speed of sound is greater than the tangential speed of sound, is a potentially stable region. The radial and tangential sound speeds in our model are obtained as

$$v_{sr}^2 = \frac{dp_r}{d\rho} = \frac{p_0\left(3 - \frac{r^2}{R^2}\right)}{5 + \frac{r^2}{R^2}}, \tag{6.3.23}$$

$$v_{st}^2 = \frac{dp_\perp}{d\rho} = \frac{(-3 + 20p_0 - p_0^2) + (-2 - 12p_0 + 6p_0^2)\frac{r^2}{R^2} + (1 + 4p_0 - 5p_0^2)\frac{r^4}{R^4}}{4\left(1 + \frac{r^2}{R^2}\right)\left(5 + \frac{r^2}{R^2}\right)} \tag{6.3.24}$$

Herrera's [35] prescription demands that we must have $v_{st}^2 - v_{sr}^2 < 0$ throughout the star. Now, at the centre of the star

$$\left(v_{st}^2 - v_{sr}^2\right)_{(r=0)} = \frac{-3 + 8p_0 - p_0^2}{20}, \tag{6.3.25}$$

for $(v_{st}^2 - v_{sr}^2)_{(r=0)} < 0$, it is required that $-3 + 8p_0 - p_0^2 < 0$ i.e. $p_0 < 0.3944$. At the boundary of the star, we have

$$\left(v_{st}^2 - v_{sr}^2\right)_{(r=R)} = -\frac{(1 + p_0)}{12}, \tag{6.3.26}$$

which is negative for $\frac{1}{3} < p_0 < 0.3944$. Therefore, our model is physically reasonable and stable if the following bound is imposed: $\frac{1}{3} < p_0 < 0.3944$.

6.4 Physical Analysis

To check whether our model can accommodate realistic ultra-compact stars, let us first analyze the gross behaviour of the physical parameters such as energy density and pressure. For a particular choice $p_0 = 0.36$ (consistent with the bound), plugging in c and G at appropriate places, we have calculated the mass m (in terms of $M_\odot$), central density ρ_c (in MeV fm^{-3}), surface density ρ_R in (MeV fm^{-3}) of a star of radius R (in kilometers). This have been shown in Table 6.1.

Table 6.1: Values of physical parameters for different radii with $p_0 = 0.36$.

Case	R	M	ρ_c	ρ_R
I	6.55	1.11	2108.46	702.82
II	6.7	1.14	2015.11	671.70
III	7.07	1.20	1809.71	603.24
IV	8	1.36	1413.41	471.14
V	9	1.53	1116.77	372.26
VI	10	1.69	904.58	301.53
VII	11	1.86	747.59	249.20
VIII	12	2.03	628.18	209.39

We note that the central density in each case (except VIII, where we have assumed a comparatively larger radius which in turn has generated a bigger mass) lies above the deconfinement density [34] ~ 700 MeV fm^{-3} which implies that quark phases may exist at the interiors of such configurations. Variations of the physical parameters in (MeV fm^{-3}) for a particular case VI have been shown in Figures 6.1 - 6.5.

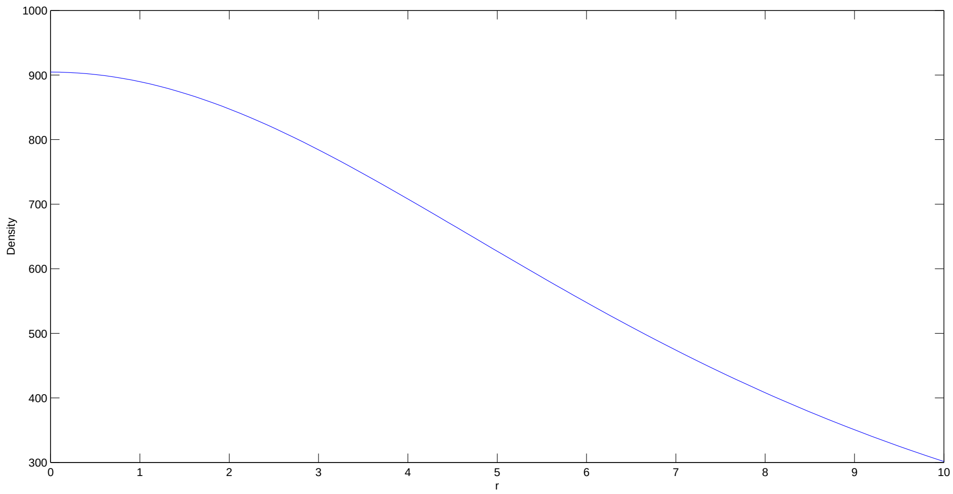

Figure 6.1: Variation of density (ρ) against the radial parameter r.

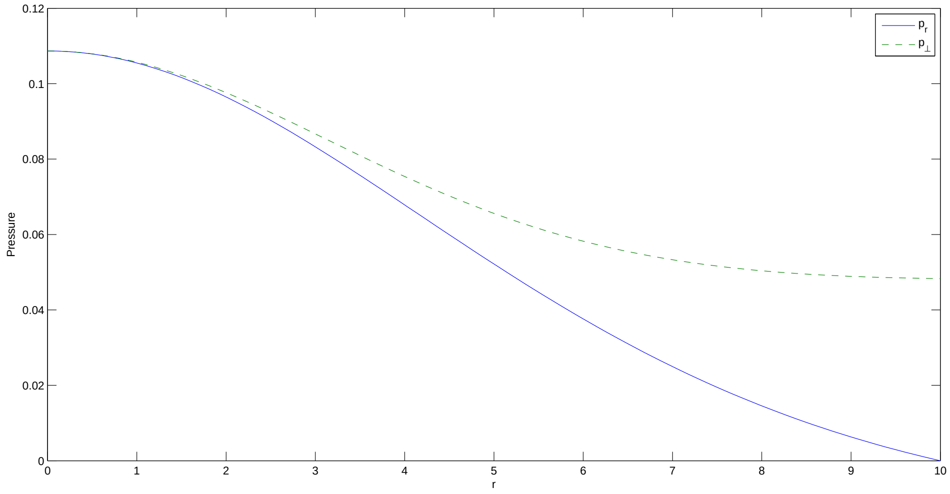

Figure 6.2: Variation of pressure (p_r and $p_\perp$) against the radial parameter r.

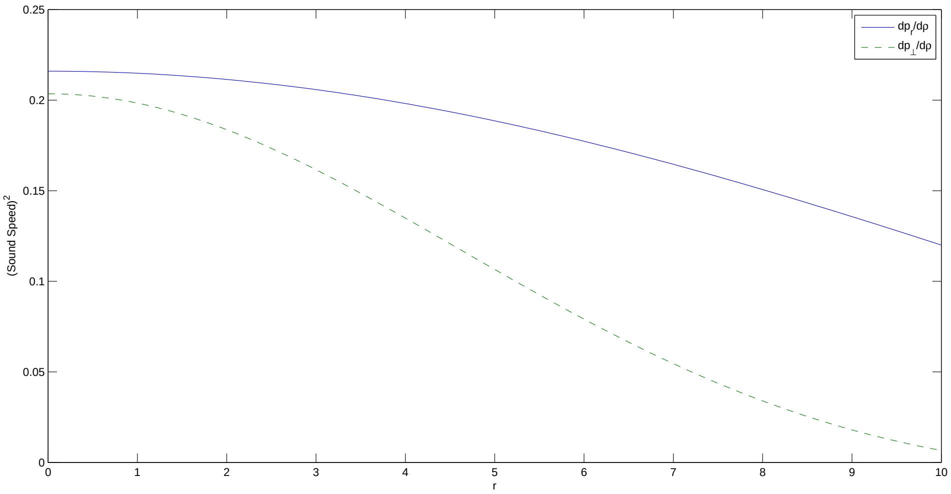

Figure 6.3: Variation of $\frac{dp_r}{d\rho}$ against the radial parameter r.

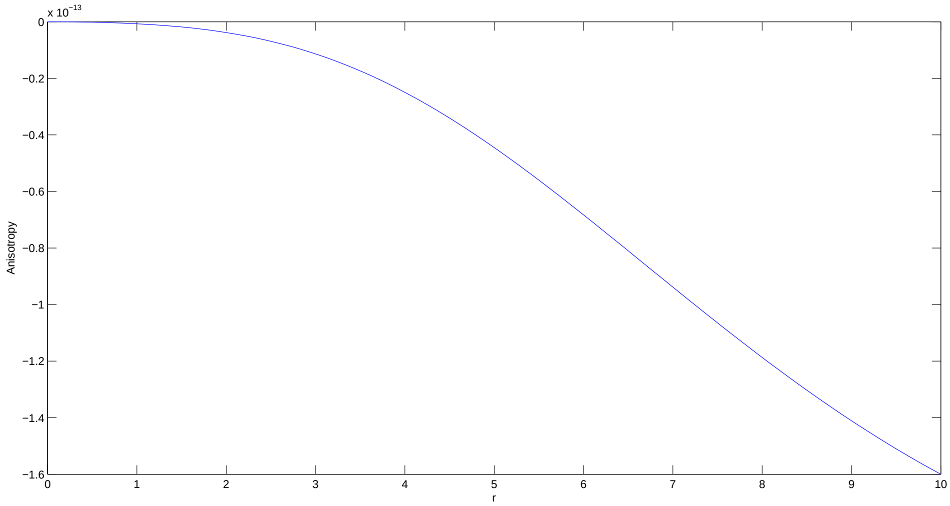

Figure 6.4: Variation of anisotropic parameter $S(r)$ against the radial parameter r.

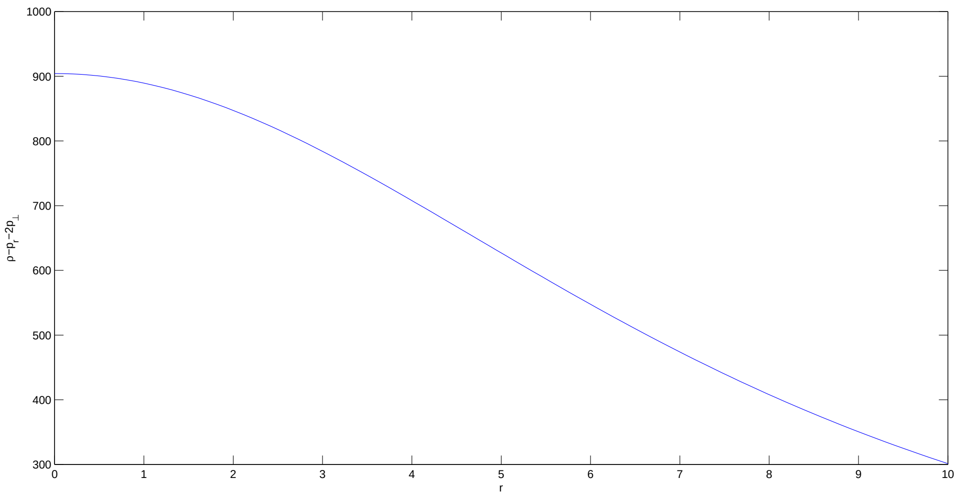

Figure 6.5: Variation of $\rho - p_r - 2p_\perp$ against the radial parameter r.

The figures clearly indicate that the physical parameters are well-behaved and all the regularity conditions discussed above are satisfied at all interior points of the star. Moreover, the assumed parameters generate a stable configuration as shown in Figure 6.6.

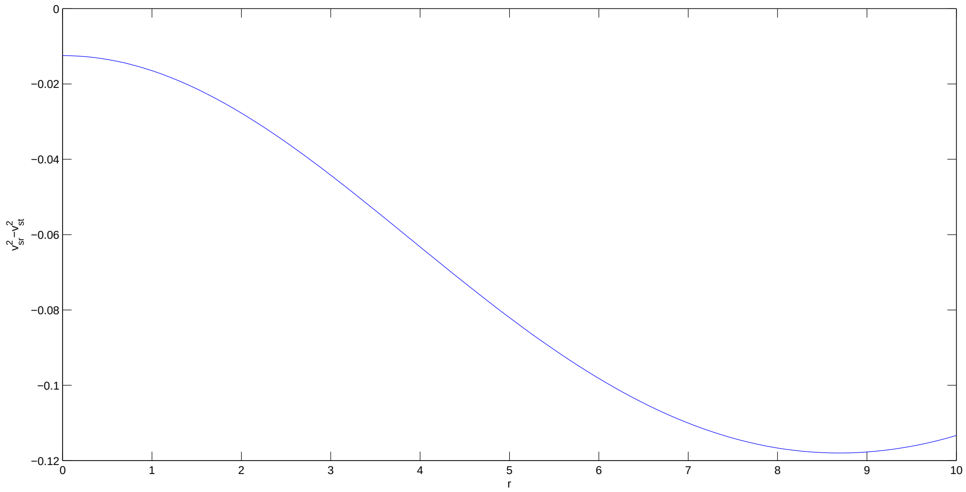

Figure 6.6: Variation of $v_{sr}^2 - v_{sp}^2$ against the radial parameter r.

Having derived a physically acceptable model, question to be asked is, what kind of material composition can be predicted for the stellar configuration admissible in this model? In other words, what would be the equation of state corresponding to the material compositions of the configurations constructed from the model? Though construction of equation of state is essentially governed by the physical laws of the system, one can parametrically relate energy-density and the radial pressure from the mathematical model which may be useful in predicting the composition of the system. Making use of equations (6.3.2) and (6.3.5), we have plotted variation of the radial pressure against the energy-density as shown by the solid curve in Figure 6.7. Our intention now is to prescribe an approximate equation of state which can produce similar kind of curve. Though, in principle, a barotropic equation of state ($p_r = p_r(\rho)$) can be generated from equations (6.3.2) and (6.3.5) by eliminating r, however we have tried curve fitting approach to find equation of state, we have tried linear equation of state $p_r = \rho_0 + \alpha\rho$ and quadratic equation of state $p_r = \rho_0 + \alpha\rho + \beta\rho^2$, where ρ_0, α and β are constants. We found that linear equation of state has norm of residuals 0.021983 and quadratic equation of state has norm of residuals 0.0027629. Hence we consider that the relevant equation of state has the form

$$p_r = \rho_0 + \alpha\rho + \beta\rho^2, \tag{6.4.1}$$

where ρ_0, α and β are constants. We make use of this equation of state to plot $\rho\ Vs\ p_r$ (dashed curve) in Figure 6.7, which turns out to be almost identical to the curve generated from the analytic model if we set $\rho_0 = -0.36$, $\alpha = 9.6 \times 10^{-5}$ and $\beta = 7.2 \times 10^{-8}$. Though this has been shown to be true for a particular choice (case VI), it can be shown that the model admits the quadratic equation of state (6.4.1) for different choices of the parameters as well.

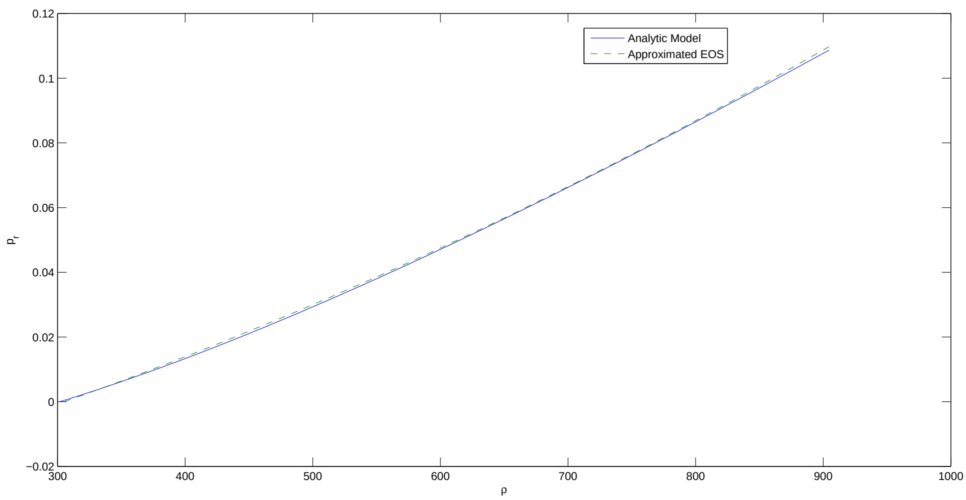

Figure 6.7: Variation of radial pressure against density.

6.5 Discussion

Making use of paraboloidal spacetime metric, we have generated exact solution of Einstein's field equations representing a static spherically symmetric anisotropic stellar configuration. Bounds on the model parameters have been obtained on physical grounds and it has been shown that model is stable for $\frac{1}{3} < p_0 < 0.3944$. In this model $\frac{p_0}{R^2}$ denotes the central density and, therefore, the bound indicates that for a given radius or mass arbitrary choice of the central density is not permissible in this model. We have shown that the model admits an equation of state which is quadratic in nature. Mathematically, this may be understood in the following manner. The ansatz (6.3.1), together with the assumption (6.3.5), generates an anisotropic stellar model whose composition may be described by the equation of state of the form (6.4.1). In [25], [60], quadratic equation of state have been assumed a priori to obtain exact solutions of Einstein's field equations. We have shown that such an assumption is consistent with an analytical model which has been constructed by making use of paraboloidal spacetime metric. In cosmology, a non-linear quadratic equation of state has been shown to be relevant for the descreption of dark energy and dark matter [3]. What type of matter can generate such an equation of state in the ultra-high density regime of an astrophysical object is an open question.

Summary

The goal of this thesis was to generate superdense star models on geometrically significant spacetimes. The nonlinear Einstein's field equations describing interior of the superdense stars were linearized by applying suitable transformation. We have constructed core-envelope models of massive stars on pseudo spheroidal spacetime, the model describing collapse of the radiating star on pseudo spheroidal spacetime. We have also examined stability of stars on paraboloidal spacetime and constructed two core-envelope models of stars on paraboloidal spacetime. Anisotropic model of superdense stars on paraboloidal spacetime is also constructed, it is shown that the model admits quadratic equation of state. All the models described in the thesis satisfy necessary physical plausibility conditions.

We now present overview of results obtained during the course of research:

- The objective of chapter 2 was to study core-envelope models of massive stars on pseudo spheroidal spacetime. We have considered isotropic pressure in the core, while the surrounding envelope has anisotropic pressure. The density profile is continuous even at the core boundary. The scheme given by Tikekar [88] is used to obtain the mass and the size of the star. The surface density is taken as $\rho(a) = 2 \times 10^{14}\ gm/cm^3$, and density variation parameter is defined as $\lambda = \frac{\rho(a)}{\rho(0)}$. The model admits high degree of density variation from centre to boundary. It is observed that as the density variation parameter λ increases, the radius of the star increases and the thickness of the envelope decreases. The core radius is found to be $b = \sqrt{2}R$ and for positivity of tangential pressure $p_\perp$ it is required that $\frac{a^2}{R^2} > 2$, where a is the radius of the star. This requirement restrict the value of density variation parameter $\lambda = \frac{\rho(a)}{\rho(0)} \leq 0.093$.

- In chapter 3, we have studied non-adiabatic gravitaitonal collapse of spherical distribution of matter having radial heat flux on pseudo spheroidal spacetime. The spherical distribution of matter is divided into two regions: core having anisotropic pressure distribution and envelope having isotropic pressure distribution. The exterior spacetime is taken as Vaidya metric. First and second fundamental forms are matched to guarantee the continuity of metric coefficients across the boundary of collapsing sphere. It is observed that the total luminosity for an observer at rest at infinity $L_\infty \to 0$ as $re^{\mu/2} \to 2m$. That is when $re^{\mu/2} = 2m$, the boundary redshift becomes infinity. It is observed that for density variation parameter $\lambda = 0.05$, the polytropic index γ at the centre is less than $\frac{4}{3}$ and at the boundary is higher than $\frac{4}{3}$ during initial stage of collapse. Hence the central region is unstable. Assuming evolution of heat flow is governed by Maxwell-Cattaneo trasport equation and following Martínez [62] we have drived equation governing temperature profile for the model under consideration.

- In Chapter 4 we investigate stability of superdense star on paraboloidal spacetime. A sufficient condition for dynamic stability of a spherical distribution of matter under small radial adiabatic perturbations have been developed by Chandrasekhar [9]. The stability of models of stars on paraboloidal spacetime is investigated by integrating Chandrasekhar's pulsation equation and it is found that the models with $0.26 < m < 0.36$ will be stable under radial modes of pulsation. The static paraboloidal spacetime metric for its spatial section $t = constant$ thus admits the possibility of describing spacetime of superdense star in equilibrium.

- Two core-envelope models with the feature core consisting of isotropic fluid and envelope consisting of anisotropic fluid distribution on the background of paraboloidal spactime have been reported in Chapter 5. The nonlinear equation governing anisotropy in the envelope is converted to second order linear variable coefficient differential equation by applying suitable transformation. Tikekar's [88] scheme is used to compute mass and size of the star. Following Following Sharma *et al.* [81], we choose central density as $\rho(0) = 4.68 \times 10^{15}\ gmcm^{-3}$. For both the models thickness of envelope increases as $\frac{m}{a}$ increases. A noteworthy feature of these models is, they admits thin envelope, hence is significant in the study of glitches and star quakes.

- In chapter 6 a class of solution describing the interior of a static spherically symmetric compact anisotropic star based on paraboloidal spacetime is reported. We have obtained bounds on the model parameters. We have made a choice on radial pressure in such a way that the field equations are integrable and radial pressure decreases from centre to boundary. We found that radius of the star is same as the curvature parameter R. The centre pressure is found to be $\frac{p_0}{R}$. The anisotropic parameter S is vanishes at the centre, which also vanishes at the boundary for particular choice of $p_0 = 1$. The method of least square is applied to obtain equation of state for the model under consideration. We found that norm of residuals is less for quadratic equation of state compair to linear equation of state. Hence, the model admits an equation of state which is quadratic in nature. We have used Herrera's [35] overtuning method to prove the stability of the compact anisotropic star model.

Appendix-A

The Nonvanishing Components of Einstein's Tensor

[I] The nonvanishing components of Einstein's tensor for spacetime metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(dr^2 + \sin^2\theta d\phi^2\right)$$

$$G_0^0 = e^{\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2},$$

$$G_1^1 = -e^{\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2},$$

$$G_2^2 = -e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{(\nu' - \lambda')}{2r}\right].$$

$$G_3^3 = G_2^2.$$

[II] The nonvanishing components of Einstein's tensor for spacetime metric

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\mu(t)+\lambda(r)}dr^2 - r^2e^{\mu(t)}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$G_0^0 = -e^{-(\mu+\lambda)}\left(\frac{\lambda'}{r} - \frac{1}{r}\right) - \frac{e^{-\mu}}{r^2} - \frac{3}{4}\dot{\mu}^2e^{-\nu},$$

$$G_1^1 = e^{-(\mu+\lambda)}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{e^{-\mu}}{r^2} - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right),$$

$$G_2^2 = e^{-(\mu+\lambda)}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r} - \frac{\nu'\lambda'}{4} - \frac{\lambda'}{2r}\right] - e^{-\nu}\left(\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\dot{\mu}\dot{\nu}}{2}\right).$$

$$G_3^3 = G_2^2,$$

$$G_0^1 = \frac{1}{2}e^{-(\mu+\lambda)}\dot{\mu}\nu'.$$

where dot denote differentiation with respect to t and prime denote differentiation with respect to r.

Appendix-B

The Nonvanishing Components of intrinsic curvature and extrinsic curvature used in Chapter 3

The nonvanishing components of intrinsic curvature κ_{ij}^- to surface Σ are

$$\kappa_{\tau\tau}^- = \left(-\frac{1}{2}\frac{\nu'}{e^{(\mu+\lambda)/2}}\right)_\Sigma,$$

$$\kappa_{\theta\theta}^- = \left(re^{(\mu-\lambda)/2}\right)_\Sigma.$$

The nonvanishing components of extrinsic curvature κ_{ij}^+ to surface Σ are

$$\kappa_{\tau\tau}^+ = \left(\frac{\ddot{\upsilon}}{\dot{\upsilon}} - \dot{\upsilon}\frac{m}{y^2}\right)_\Sigma,$$

$$\kappa_{\theta\theta}^+ = \left[\dot{\upsilon}\left(1 - \frac{2m}{y}\right)y + \dot{y}y\right]_\Sigma.$$

Publications

The following are the research publications resulted from the present thesis work.

- **V. O. Thomas, B. S. Ratanpal and P. C. Vinodkumar,** *Core-envelope Models of Superdense Star with Anisotropic Envelope*, International Journal of Modern Physics D, Vol. 14, No. 1, pp. 85 - 96 (2005).
- **V. O. Thomas and B. S. Ratanpal,** *Non-adiabatic Gravitaitonal Collapse with Anisotropic Core*, International Journal of Modern Physics D, Vol. 16, No. 9, pp. 1479 - 1495 (2007).
- **B. S. Ratanpal, V. O. Thomas and S. Ramamohan,** *On Models of Stars on Paraboloidal Spacetime*, Bulletin of Marathwada Mathematical Society. (Communicated)
- **R. Sharma and B. S. Ratanpal,** *Relativistic Stellar Model Admitting a Quadratic Equation of State*, International Journal of Modern Physics D. (Communicated)

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