Nonlinear Fuzzy Optimization Problems and their Applications

Thesis Submitted By

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Towards the Partial Fulfillment for the Degree of

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Guided By

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Dedicated to

My Parents

and

My dear brother...

Declaration

I hereby declare that:

(i) the thesis comprises only my original work towards the Ph.D. except where indicated,

(ii) due acknowledgment has been made in the text to all other materials used,

(iii) this work has not formed the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Certificate

This is to certify that Ms Umme salma M. Pirzada has worked under my guidance to prepare the thesis entitled "Nonlinear Fuzzy Optimization Problems and their Applications" which is being submitted herewith towards the requirement for the degree of Doctor of Philosophy in Applied Mathematics.

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This thesis entitled "Nonlinear Fuzzy Optimization Problems and their Applications" submitted by Umme salma M. Pirzada in Applied Mathematics is hereby approved for the degree of Doctor Of Philosophy.

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Contents

Chapter 1

Introduction

Contents

1.1 Introduction and motivation

Soft computing is tolerant of imprecision, uncertainty, partial truth and approximation. The principal components of soft computing are Fuzzy Systems, Neural Networks, Genetic Algorithms, Probabilistic Reasoning etc. Fuzzy Systems are based on Fuzzy theory, which plays a leading role in soft computing. This stems from the fact that human reasoning is not crisp and admits degrees. Fuzzy theory was introduced by L. A. Zadeh [83] from university of California, Berkeley , U.S.A. in 1965. He says that fuzzy theory is not a single theory, but it is a process of "fuzzification" as a methodology to generalize any specific theory from a crisp to fuzzy form. Fuzzy theory is important in almost all fields of mathematics. It is studied by many authors over the years.

Crisp optimization techniques have been successfully applied for years to solve problems with a well-defined structure using precise mathematics. Unfortunately, real world situations are often not precise. There exist various types of uncertainties in social, industrial and economic systems, such as randomness of occurrence of events, imprecision and ambiguity of system data, linguistic vagueness etc; such uncertainties arise due to errors of measurement, deficiency in history and statistical data, incomplete knowledge expression, the subjectivity and preferences of human judgment etc (see ref. [63]). As pointed out by

Zimmermann [88], these kinds of uncertainties can be categorized as stochastic uncertainty or fuzziness.

Stochastic uncertainty relates to the uncertainty of occurrence of phenomena or events. Systems with this type of uncertainty are the so called stochastic systems, for which the stochastic optimization techniques using probability theory can be applied. In other situations, when the information is vague or when the information could not be described and defined well due to limited knowledge and deficiency in its understanding, then stochastic uncertainties can not be used to model the system and one has to look for techniques based on fuzzy theory. A system with vague and ambiguous information is a so-called soft system in which the structure is ill-defined. It can not be formulated and effectively tackled by traditional mathematics-based optimization techniques nor probability-based stochastic optimization approaches. Thus , fuzzy optimization techniques provide a useful and efficient tool for optimizing systems under fuzzy environments.

The studies on the theory and methodology of the fuzzy optimization have been active since the concept of fuzzy decision and the decision model under fuzzy environments were proposed by Bellman and Zadeh in 1970's [4]. After this, various models and approaches to fuzzy linear programming [21, 20], fuzzy multi-objective programming [59, 60], fuzzy integer programming [67], fuzzy dynamic programming [34] and fuzzy non-linear programming (see ref. [42], [71],[72] and [73]) have been developed over the years.

In order to properly formulate the fuzzy optimization problem, it is necessary to define appropriate mathematical structure on the set of fuzzy numbers which represent imprecise quantities in the fuzzy systems. We consider algebraic operations and metric structure on set of fuzzy numbers as given in [39] and [80] respectively. We use concepts of differential and integral calculus of fuzzy numbers as discussed in [50] and [28]. Further order relations on fuzzy numbers can be defined in a variety of ways. We use two such order relations as discussed in [52] and [58]. In the present research work, we have developed necessary and sufficient optimality conditions of fuzzy valued functions defined on $\mathbb{R}^n$ with and without constraints, with respect to various order relationships. Appropriate illustrations have been discussed in order to justify our results. Further gradient based numerical methods for fuzzy optimization problems have been proposed.

1.2 Modeling through fuzzy optimization

Descriptions of the objective function and of the constraints in a optimization problem usually include some parameters. For example, in problems of resources allocation such parameters may represent economic parameters like costs of various types of production, labor costs, shipment costs, etc. The nature of these parameters depends, of course, on the detailization accepted for the model representation, and their values are considered as data that should be exogenously used for the analysis.

Clearly, the values of such parameters depend on multiple factors not included into the formulation of the problem. Trying to make the model more representative, we often include the corresponding complex relations into it, causing the model to become more cumbersome and analytically unsolvable. Moreover, it can happen that such attempts to increase the precision of the model will be of no practical value due to the impossibility of measuring the parameters accurately. On the other hand, the model with some fixed values of its parameters may be too crude, since these values are often chosen in a quite an arbitrary way.

An intermediate approach is based on introduction into the model the means of a more adequate representation of expert's understanding of the nature of the parameters in the form of fuzzy sets of their possible values. The resultant model, although not taking into account many details of the real system in question could be a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. In this way we obtain a new type of optimization problems containing fuzzy parameters. Treating such problems requires the application of fuzzy-set-theoretic tools in a logically consistent manner. Such treatment forms an essence of fuzzy optimization problems.

The use of fuzzy optimization models does not only avoid unrealistic modeling, but also offers a chance for reducing information costs. Fuzzy optimization problems and related problems have been extensively analyzed and many papers have been published displaying a variety of formulations and approaches. Most approaches to fuzzy optimization problems are based on the straightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function. This approach has been mentioned by Bellman and Zadeh already in their paper [4] published in the early seventies. Later on many papers have been devoted to the problem of optimization with fuzzy parameters, which we discussed in details in next Section.

1.3 Literature survey

Bellman and Zadeh [4] (1970) inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. The earliest interesting work in this direction was initiated by Rödder and Zimmermann [56]

(1977) and Zimmermann [85] (1976) , [86] (1978) , [87] (1985) who applied fuzzy set theory to the linear programming problems and linear multi-objective programming problems by using the aspiration level approach. The collection of papers on fuzzy optimization edited by Słowiński [65] (1998) and Delgado et al. [14](1994) gives the main stream of this topic. Insightful surveys on the advancement of fuzzy optimization can be found in Kacprzyk [35](1987) , Luhandjula [46](1989), Fedrizzi [22](1991) and Lai and Hwang ([40] (1992) and [41] (1994)).

On the other hand, the book edited by Słowiński and Teghem [66](1990) gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramík [33](2000) also gives a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

Fuzzy optimization problems have been studied in a variety of ways which we have discussed briefly in the first section. For example, Robert Fuller [24] has studied the stability of the solution of fuzzy linear programming problems. Stephan Dempe and Tatiana Starostina [13] have studied linear programming problems with fuzzy coefficients in the objective functions. The basic introduction to the main models and methods in fuzzy linear programming is presented by Cadenas J.M. and Verdegay J.L. (2009) in [8].

Our main focus is on nonlinear fuzzy optimization problems. Different aspects of these problems have been studied by many researchers. We refer here some of the recent work that has been done in this direction. Fuzzy mathematical programming using unified approach has been studied by Ramik I. and Vlach M. (2002) in [53]. Lodwick W.A and Bachman K.A. (2005) have studied large scale fuzzy and possibilistic optimization problems in [43]. Distinctions and relationships between Fuzzy and Possibilistic Optimization have been studied by Lodwick W.A. et al (2007), (2009) in [44] and [45] respectively. Buckley J.J. and Abdalla A. (2009) have considered Monte Carlo methods in fuzzy queuing theory in [7]. The technique for solving fuzzy optimization problems using embedding theorem was proposed in Wu [77](2004) which also introduced the concept of an (α, β)-optimal solution. The solution concepts of fuzzy optimization problems based on convex cones (ordering cone) was also proposed in Wu [75](2003). Hsien-Chung Wu [79] (2007) has proved the Kuhn-Tucker like sufficient optimality conditions for fuzzy optimization problems using the concept of generalized convexity. Hsien-Chung Wu [80] (2008) has proved the Kuhn-Tucker optimality conditions for fuzzy optimization problems using integral approach. He has also discussed the duality theory in fuzzy optimization problems [78] and saddle point optimality conditions for fuzzy optimization problems [76]. Fuzzy nonlinear optimization problem for linear fuzzy real number system has been studied in [23] (2009) where they have considered fuzzy-valued functions whose domain and range are fuzzy numbers and solved the unconstrained and constrained fuzzy optimization problems.

Fuzzy quadratic programming has been studied in [49] (2008).

In the current work, taking motivation from two papers of Wu ([80] and [79]), we establish Kuhn Tucker like optimality conditions for general nonlinear fuzzy optimization problems with fuzzy-valued functions having real domain.We also propose a gradient based nonlinear optimization method for the same problem. We use different ranking methods for defining order relation on the set of fuzzy numbers.

1.4 Layout of thesis

The thesis is organized in the following manner :

The first chapter includes introduction and motivation of our work and the literature survey. Second chapter contains basic concepts of fuzzy numbers, their arithmetics and order relation on them. It also includes concepts of fuzzy-valued function, its continuity, H-differentiability, integrability and convexity that we use in our research work.

Chapter three deals with the optimal solution of nonlinear unconstrained fuzzy optimization problem under the concept of parametric total order relation defined on fuzzy numbers having specific L-shape membership function.

In chapter four, we find the non-dominated solution of nonlinear unconstrained and constrained fuzzy optimization problems (FOP) under the concept of partial order relation defined on the set of fuzzy numbers.

In chapter five, we prove the sufficient optimality conditions for obtaining non-dominated solution of a constrained fuzzy optimization problem under convexity and weaker convexity-pseudoconvexity and quasiconvexity of a fuzzy-valued objective function and fuzzy constraints.

In chapter six, we establish Newton's method for solving single-variable and multi-variable unconstrained fuzzy optimization problem. We use H-differentiability of fuzzy-valued functions to prove the results. We also show the convergence criteria of proposed methods. Summary following Chapter seven summarises the current research work.

List of publications and references are given at the end.

Chapter 2

Basic terminologies

Contents

2.1 Basic concepts

In this section, we present some basic concepts regarding fuzzy sets which are used in this research work.

2.1.1 Fuzzy sets and their operations

Sets are one of the most fundamental concepts in mathematics. A set is a collection of well-defined objects. For instance, if we consider a set containing a group of young people, it is not cleared whether a person having age above fifty belongs to the set. So this is, actually example of "not a set" or "not a crisp set". In a crisp set, there is sharp boundary between membership and non membership. But many times, there is ambiguity about whether the element belongs to the set or not. This motivates introduction to fuzzy sets. The concept of a fuzzy set was introduced by L.A. Zadeh [83] in the 1960's. The concept of Fuzzy sets is used to distinguish the elements of the universe of discourse in terms of a property whose perception is not crisp and hence the members cannot be classified into two crisp classes one having those members which satisfies the property and the other having those members which does not satisfy the property. Instead, the distinction is made on the bases of the degree to which a particular element of the universe possess the property. Thus, a fuzzy set is characterized by a membership function which is a mapping from the universe of discourse into the unit interval. Mathematically, it can be defined as follows:

Definition 2.1.1. *$\tilde{A}$ is a fuzzy subset of a universal set X, defined by its membership function $\mu_{\tilde{A}} : X \to [0, 1]$. For each $x \in X$, $\mu_{\tilde{A}}(x)$ is interpreted as the degree to which x is a member of fuzzy set $\tilde{A}$ where 1 represents full membership and 0 represents complete non membership.*

Remark 2.1.1. *The membership function of the given set is also denoted by the symbol $\tilde{A}(x)$ instead of $\mu_{\tilde{A}}(x)$.*

We give here two examples of fuzzy sets.

Example 2.1.1. *Let X be a set of real numbers $\mathbb{R}$, and let $\tilde{A}$ be a fuzzy set of numbers which are near to 0. Then one can write the membership function as*

$$\tilde{A}(x) = \begin{cases} (x+1) & if \quad -1 \leq x < 0 \\ (x-1) & if \quad 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

Thus membership grades are like $\mu_{\tilde{A}}(0) = 1$, $\mu_{\tilde{A}}(0.1) = 0.9$, $\mu_{\tilde{A}}(-0.2) = 0.8$, $\mu_{\tilde{A}}(1) = 0$ etc.

The following is another example of a fuzzy set.

Example 2.1.2. *Let X be a set of age of the persons from a particular class and let $\tilde{A}$ be fuzzy set defining the concept " set of young persons" then the grade of membership values can be $\mu_{\tilde{A}}(20) = 1$, $\mu_{\tilde{A}}(30) = 0.8$, $\mu_{\tilde{A}}(45) = 0.5$, $\mu_{\tilde{A}}(60) = 0.1$.*

Fuzzy sets can be differentiated on the basis of the universal set on which they are defined. A fuzzy set defined on a continuous universal set is define by continuous membership function. For the discrete or finite universal set $X = \{x_1, x_2, ..., x_n\}$, a fuzzy set $\tilde{A}$ defined on X, can be represented as

$$\tilde{A} = \tilde{A}(x_1)/x_1 + \tilde{A}(x_2)/x_2 + ... + \tilde{A}(x_n)/x_n$$

Example 2.1.3. *Let $X = \{0, 1, 2, 3, 4\}$ and let $\tilde{A}$ be a fuzzy set defining the concept "numbers much greater than 1" then it can be expressed as*

$$0/0 + 0/1 + 0.2/2 + 0.4/3 + 0.6/4$$

There is one special type of fuzzy set called normal fuzzy set which is defined as follows.

Definition 2.1.2. *A fuzzy set $\tilde{A}$ on the universe X, is said to be normal if there exists $x \in X$ such that $\tilde{A}(x) = 1$.*

For example, the fuzzy sets defined in Example 2.1.1 and Example 2.1.2 are normal fuzzy sets while the fuzzy set in Example 2.1.3 is not a normal fuzzy set.

The three basic set theoretic operations on crisp sets, namely, Complement, Intersection and Union, can be generalized to fuzzy sets in more than one way. Here, we define one of the generalizations, referred as standard fuzzy set operations, that has a special significance in fuzzy set theory. (See reference [29]).

Definition 2.1.3. *The standard fuzzy complement, $\bar{\tilde{A}}$, of fuzzy set $\tilde{A}$ with respect to the universal set X is defined by the membership function ,*

$$\bar{\tilde{A}}(x) = 1 - \tilde{A}(x),$$

for all $x \in X$.

Definition 2.1.4. *Given two fuzzy sets, $\tilde{A}$ and $\tilde{B}$, their standard fuzzy intersection, $\widetilde{A \cap B}$, and standard fuzzy union, $\widetilde{A \cup B}$, are respectively defined by the membership functions*

$$(\widetilde{A \cap B})(x) = \min[\tilde{A}(x), \tilde{B}(x)],$$

$$(\widetilde{A \cup B})(x) = \max[\tilde{A}(x), \tilde{B}(x)],$$

for all $x \in X$.

Example 2.1.4. *Consider universal set $X = \{a, b, c\}$ and two fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined on X as*

$$\tilde{A} = 0.1 \ / \ a \ + \ 0.6 \ / \ b \ + \ 1 \ / \ c$$

$$\tilde{B} = 0.5 \ / \ a \ + \ 0.9 \ / \ b \ + \ 0.3 \ / \ c,$$

then their standard fuzzy union, intersection and complements are given as follows.

$$\widetilde{A \cap B} = 0.5 \ / \ a \ + \ 0.9 \ / \ b \ + \ 1 \ / \ c,$$

$$\widetilde{A \cup B} = 0.1 \ / \ a \ + \ 0.6 \ / \ b \ + \ 0.3 \ / \ c,$$

$$\bar{\tilde{A}} = 0.9 \ / \ a \ + \ 0.4 \ / \ b \ + \ 0 \ / \ c,$$

$$\bar{\tilde{B}} = 0.5 \ / \ a \ + \ 0.1 \ / \ b \ + \ 0.7 \ / \ c.$$

2.1.2 α-level sets and their properties

Now we define one of the important concept of fuzzy set theory called α-level sets.

Definition 2.1.5. *For a given fuzzy set $\tilde{A}$ defined on universal set X and any number $\alpha \in [0, 1]$, the α-level set of $\tilde{A}$, is the crisp set*

$$\tilde{A}_\alpha = \{x / \mu_{\tilde{A}}(x) \geq \alpha\}$$

Example 2.1.5. *For the fuzzy set given in Example 2.1.1, the α-level set of $\tilde{A}$, for $\alpha = 0.1$ is $\tilde{A}_{0.1} = \{x / \mu_{\tilde{A}}(x) \geq 0.1\} = [-0.9, 0.9]$.*

Now we define convexity of fuzzy set, which is based on its α-level sets.

Definition 2.1.6. *A fuzzy set defined on universal set* $\mathbb{R}$ *is said to be convex if all its* α*-level sets are convex, for* $\alpha \in (0, 1]$.

Example 2.1.6. *Consider a fuzzy set* $\tilde{B}$ *defined on* $\mathbb{R}$ *with the following membership function*

$$\tilde{B}(x) = \begin{cases} \sin x & if \quad 0 < x < \pi \\ 0 & otherwise \end{cases}$$

Its α*-level sets are* $\tilde{B}_\alpha = [0 + \sin^{-1}\alpha, \pi - \sin^{-1}\alpha]$ *for* $\alpha \in (0, 1]$. *Clearly, all the* α*-level sets are convex for* $\alpha \in (0, 1]$. *Therefore, the given fuzzy set* $\tilde{B}$ *is convex.*

A useful Theorem about convexity of fuzzy sets given in [29].

Theorem 2.1.1. *A fuzzy set* $\tilde{A}$ *on* $\mathbb{R}$ *is convex if and only if*

$$\tilde{A}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\tilde{A}(x_1), \tilde{A}(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}$ *and all* $\lambda \in [0, 1]$.

Remark 2.1.2. *It can be observed from the above theorem that if the given fuzzy set is convex, it does not mean that its membership function is also convex. In fact, the membership function is quasi-concave.*

The capability of α-level sets of a fuzzy set is to represent the fuzzy set in terms of its α-level sets. That is, each fuzzy set can be uniquely represented by the family of all its α-level sets. These representations allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts. Klir and Yaun [29] have referred this representation as a decomposition of fuzzy set $\tilde{A}$. We state here, the first decomposition theorem of it.

Theorem 2.1.2. *For every fuzzy set* $\tilde{A}$ *on* X,

$$\tilde{A} = \cup_{\alpha \in [0,1]} \alpha \cdot \tilde{A}_\alpha$$

where $\tilde{A}_\alpha$ *is* α*-level set of* $\tilde{A}$ *and* $\cup$ *denotes the standard fuzzy union.*

2.1.3 Extension principle of Zadeh

Zadeh introduced extension principle for fuzzy sets in his paper [84]. The extension principle for fuzzy sets is in essence a basic identity which allows the domain of the definition of a mapping or a relation to be extended from points in X to fuzzy subsets of X. More specifically, suppose that f is a mapping from X to Y, and $\tilde{A}$ is a fuzzy subset of X, then the extension principle asserts that $f(\tilde{A})$ is a fuzzy set on Y and

$$f(\tilde{A}) = \sup_{x/f(x)=y} \tilde{A}(x)$$

Let $X, Y, Z \subseteq \mathbb{R}$ and f be a crisp function $f : X \times Y \to Z$. Assume $\tilde{A}$ and $\tilde{B}$ are two fuzzy subsets on X and Y respectively. By the extension principle, we can use the crisp function f to induce a fuzzy function

$$F : F(X) \times F(Y) \to F(Z).$$

That is to say, $F(\tilde{A}, \tilde{B})$ is a fuzzy subset of Z with membership function

$$F(\tilde{A}, \tilde{B})(z) = \begin{cases} \sup_{f(x,y)=z}\{\min\{\tilde{A}(x), \tilde{B}(y)\}\}, & f^{-1}(z) \neq \phi \\ 0 & , f^{-1}(z) = \phi \end{cases}$$

where $f^{-1}(z) = \{(x, y) \in X \times Y : f(x, y) = z \in Z\}$. Such a function F is called a fuzzy function induced by the extension principle.

Example 2.1.7. *Let $X = Y = \{1, 2, ..., 10\}$ be universal sets and let "Approximately 2 " and "Approximately 6 " be fuzzy sets defined on X and Y respectively as*

$$\tilde{A} = \textit{Approximately } 2 = 1/2 + 0.6/1 + 0.8/3$$

$$\tilde{B} = \textit{Approximately } 6 = 1/6 + 0.8/5 + 0.7/7$$

and crisp function $f(x, y)$ be arithmetic product $(\times)$ of x and y. Here we apply the extension principle defined above, to fuzzify the given crisp function.

$$\begin{aligned} (\tilde{A} \times \tilde{B}) &= (1/2 + 0.6/1 + 0.8/3) \times (1/6 + 0.8/5 + 0.7/7) \\ &= 0.6/5 + 0.6/6 + 0.6/7 + 0.8/10 + 1/12 + 0.7/14 + 0.8/15 + 0.8/18 \\ &\quad +0.7/21 \end{aligned}$$

2.2 Fuzzy numbers and their arithmetic

This section starts with one simple concept of classical calculus called upper semi-continuity of a real-valued function which we use in the definition of fuzzy numbers.

2.2.1 Upper semi-continuity of a real-valued function

The definition of upper semi-continuity of a real-valued function is given as follows.

Definition 2.2.1. *An extended real-valued function f is upper semi-continuous at a point x_0 if, roughly speaking, the function values for arguments near x_0 are either close to $f(x_0)$ or less than $f(x_0)$. Mathematically, Suppose X is a topological space, x_0 is a point in X and $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is an extended real-valued function. We say that f is upper semi-continuous at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \leq f(x_0) + \epsilon$ for all $x \in U$. Equivalently, this can be expressed as*

$$\limsup_{x \to x_0} f(x) \leq f(x_0)$$

where $\limsup$ *is the limit superior (of the function f at point x_0).*

Example 2.2.1. *Consider the function f, piecewise defined by $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. This function is upper semi-continuous at $x_0 = 0$.*

Remark 2.2.1. *A function may be upper or lower semi-continuous without being either left or right continuous. For example, the function*

$$f(x) = \begin{cases} 1, & x < 1 \\ 2, & x = 1 \\ 1/2, & x > 1 \end{cases}$$

is upper semi-continuous at $x = 1$ although not left or right continuous. The limit from the left is equal to 1 and the limit from the right is equal to 1/2, both of which are different from the function value of 2.

2.2.2 Fuzzy numbers and their arithmetic

Classical optimization is influenced by calculus and order structure defined on real numbers. To deal with fuzzy optimization problems , we need fuzzy calculus and order structure on fuzzy numbers. Here we state basic terminologies about fuzzy numbers and fuzzy-valued functions.

In many scientific areas, such as systems analysis and operations research, a model has to be set up using data which is only approximately known. Fuzzy set theory, introduced by Zadeh in (1965) [83], makes this possible. Fuzzy numerical data can be represented by means of fuzzy subsets of the real line, known as fuzzy numbers. Dubois and Prade introduced the notion of fuzzy numbers in their paper [18] (1978) and established some of their basic properties. Goetschel and Voxman in [32] introduced new equivalent definition of fuzzy numbers using the parametric representation (α-level set presentation). Out of many ways of defining fuzzy numbers, we state the following.

Definition 2.2.2. *[28] Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \to [0, 1]$ be a fuzzy set. We say that $\tilde{a}$ is a fuzzy number if it satisfies the following properties:*

(i) *$\tilde{a}$ is normal;*

(ii) *$\tilde{a}$ is fuzzy convex;*

(iii) *$\tilde{a}(x)$ is upper semi-continuous on $\mathbb{R}$, that is, for each $\alpha \in (0, 1]$, the α-level set of $\tilde{a}$ is a closed subset of $\mathbb{R}$;*

(iv) *The 0-level set of $\tilde{a}$, defined as $\tilde{a}_0 = cl\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$ forms a compact set,*

where cl denotes closure of a set. The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. Any real number r can be regarded as a fuzzy number, $\tilde{r}$ such that $\tilde{r}(z) = 1$ if $z = r$ and $\tilde{r}(z) = 0$ if $z \neq r$. By definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in [0, 1]$, $\tilde{a}_\alpha$ is compact convex subset of $\mathbb{R}$, and hence a closed bounded interval in $\mathbb{R}$. We write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. $\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-level sets by a well-known decomposition theorem (ref. Theorem 2.1.2).

Example 2.2.2. *Example 2.1.6 is a fuzzy number.*

The following Theorem of Goetschel and Voxman [32], shows the characterization of a fuzzy number in terms of its α-level sets.

Proposition 2.2.1. *([32], Theorem 1.1) For $\tilde{a} \in F(\mathbb{R})$, define two functions $\tilde{a}^L_\alpha$, $\tilde{a}^U_\alpha$: $[0, 1] \to \mathbb{R}$ given by $\tilde{a}^L_\alpha = \tilde{a}^L(\alpha)$ and $\tilde{a}^U_\alpha = \tilde{a}^U(\alpha)$. Then*

(i) *$\tilde{a}^L_\alpha$ is bounded left continuous non-decreasing function on (0,1];*

(ii) *$\tilde{a}^U_\alpha$ is bounded left continuous non-increasing function on (0,1];*

(iii) $\tilde{a}^L_\alpha$ *and* $\tilde{a}^U_\alpha$ *are right continuous at* $\alpha = 0$;

(iv) $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$.

Moreover, if the pair of functions $\tilde{a}^L_\alpha$ *and* $\tilde{a}^U_\alpha$ *satisfy the conditions (i)-(iv), then there exists a unique* $\tilde{a} \in F(\mathbb{R})$ *such that* $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$*, for each* $\alpha \in [0, 1]$.

Definition 2.2.3. *[39] Applying the Zadeh's extension principle, the addition and scalar multiplication on* $\mathbb{R}$ *are extended to those on* $F(\mathbb{R})$ *as follows: For* $\tilde{a}, \tilde{b} \in F(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$,

$$(\tilde{a} \oplus \tilde{b})(z) = \sup_{\{x,y \in \mathbb{R}/z=x+y\}} \{\min\{\tilde{a}(x), \tilde{b}(y)\}\}$$

$$(\lambda \odot \tilde{a})(z) = \begin{cases} \tilde{a}(z/\lambda) \quad if \quad \lambda \neq 0 \\ 0 \quad if \quad \lambda = 0 \end{cases}$$

for all $z \in \mathbb{R}$*. It is clear that for any* $\tilde{a}, \tilde{b} \in F(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$, $\tilde{a} \oplus \tilde{b}$ *and* $\lambda \odot \tilde{a}$ *are fuzzy numbers and by using interval arithmetic , we can show that*

$$\begin{aligned} (\tilde{a} \oplus \tilde{b})_\alpha &= \tilde{a}_\alpha + \tilde{b}_\alpha \\ (\lambda \odot \tilde{a})_\alpha &= \lambda \cdot \tilde{a}_\alpha, \ for\ all\ \alpha \in \ [0, 1]. \end{aligned}$$

Example 2.2.3. *Let* $X_1 = X_2 = \{1, 2, ..., 10\}$ *be universal sets and let "Approximately 2 " and "Approximately 6 " be fuzzy numbers defined on* X_1 *and* X_2 *respectively as*

$$\tilde{A}_1 = \textit{Approximately } 2 = 1/2 + 0.6/1 + 0.8/3$$

$$\tilde{A}_2 = \textit{Approximately } 6 = 1/6 + 0.8/5 + 0.7/7$$

using the extension principle of zadeh, we can obtain addition of $\tilde{A}_1$ *and* $\tilde{A}_2$ *and scalar multiplication of* $\tilde{A}_1$ *with a scalar* $\lambda \in \mathbb{R}$ *in the following way.*

$$\begin{aligned} (\tilde{A}_1 \oplus \tilde{A}_2) &= (1/2 + 0.6/1 + 0.8/3) \oplus (1/6 + 0.8/5 + 0.7/7) \\ &= 1/8 + 0.8/7 + 0.7/9 + 0.6/7 + 0.6/6 + 0.6/8 + 0.8/9 + 0.8/8 \\ &\quad +0.7/10 \\ &= 0.6/6 + 0.8/7 + 1/8 + 0.8/9 + 0.7/10 \end{aligned}$$

Take $\lambda = 2$,

$$\begin{aligned}(\lambda \odot \tilde{A}_1) &= 2 \odot (1/2 + 0.6/1 + 0.8/3) \\ &= 1/1 + 0.6/0.5 + 0.8/(1.5)\end{aligned}$$

To get fast computation formulas for the operations of fuzzy numbers, Dubois and Prade introduced the concept of L-R fuzzy numbers [18] as follows:

Definition 2.2.4. *[39] Let* $L, R : [0, \infty) \to [0, 1]$ *be two non increasing and non-constant (shape) functions with* $L(0) = R(0) = 1$ *and* $L(z_0) = R(z_0) = 0$ *for some* $z_0 > 0$. *A fuzzy number* $\tilde{a}$ *is called a L-R fuzzy number if there exist real numbers* $m, n (m \leq n)$, $\alpha, \beta (\alpha, \beta > 0)$ *such that*

$$\tilde{a}(z) = \begin{cases} L(\frac{m-z}{\alpha}) & for \quad z \leq m \\ \quad 1 & for \quad m \leq z \leq n \\ R(\frac{z-n}{\beta}) & for \quad z > n \end{cases}$$

where α, β *are the left and right spreads, respectively. L-R fuzzy numbers include the triangular and trapezoidal fuzzy numbers. In particular, the symmetric L-R fuzzy number is called a L- fuzzy number. The set of L-fuzzy numbers on* $\mathbb{R}$ *is denoted by* $F_L(\mathbb{R})$.

Definition 2.2.5. *For any real number* r, *a triangular fuzzy number* $\tilde{r}$ *is defined as*

$$\tilde{r}(z) = \begin{cases} \frac{(z-r^L)}{(r-r^L)} & if \quad r^L \leq z \leq r \\ \frac{(r^U-z)}{(r^U-r)} & if \quad r < z \leq r^U \\ 0 & \quad otherwise \end{cases}$$

which is denoted by $\tilde{a} = (r^L, r, r^U)$. *The* α*-level set of* $\tilde{r}$ *is*

$$\tilde{r}_\alpha = [(1-\alpha)r^L + \alpha r, (1-\alpha)r^U + \alpha r].$$

Example 2.2.4. *A fuzzy number* $\tilde{r}$ *defining the concept "about 3" can be represented by following triangular membership function:*

$$\tilde{r}(x) = \begin{cases} \frac{(x-1)}{2} & if \quad 1 \leq x \leq 3 \\ \frac{(6-x)}{3} & if \quad 3 < x \leq 6 \\ 0 & \quad otherwise \end{cases}$$

It can be denoted as $\tilde{r} = (1, 3, 6)$.

Remark 2.2.2. *If we consider a triangular fuzzy number with equal left and right spread values then it can be an example of L-fuzzy number.*

2.3 Fuzzy differential calculus

Differentiability for fuzzy-valued functions has been studied by several mathematicians in a variety of ways. For example, Puri and Ralescu [50] have introduced the concept of Hukuhara differentiability of fuzzy-valued functions in (1983). After that, many researchers have used this derivative as applications to fuzzy differential equations, including Ding and Kandel [16], Kaleva [36, 37] and Seikklala [62] and fuzzy optimization problems also. Recently more work has been done regarding new concepts of H-differentiability. Like, Bede and Gal (2005) have introduced a more general definition of derivative for fuzzy-valued functions called weakly and strongly generalized differentials in [2]. Bede et al (2007) have studied first order linear fuzzy differential equations using generalized differentiability- strongly generalized differentiability in their paper [3]. Chalco-Cano et al [10] (2008) and [11](2009) have also studied fuzzy differential equations using the concept of generalized H-differentiability. In the current work, we use first order Hukuhara derivatives of fuzzy-valued functions as given in [50] and further define the second order Hukuhara differentiability of fuzzy-valued functions. Using these concepts, we establish the first and second order necessary and sufficient conditions for optimality of nonlinear fuzzy optimization problems and verify the results with appropriate illustrations.

We discuss here fuzzy differential calculus starting with the definition of a fuzzy-valued function.

Definition 2.3.1. *[77] Let V be a real vector space and $F(\mathbb{R})$ be a set of fuzzy numbers. Then a function $\tilde{f} : V \to F(\mathbb{R})$ is called fuzzy-valued function defined on V.*
Corresponding to such a function $\tilde{f}$ and $\alpha \in [0, 1]$, we define two real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ on V as $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for all $x \in V$.

Example 2.3.1. *Let $\tilde{f} : \mathbb{R} \to F(\mathbb{R})$ be defined by $\tilde{f}(x) = \tilde{a} \odot x$, where $\tilde{a} = (1, 2, 3)$ is a triangular fuzzy number, then $\tilde{f}$ is a fuzzy-valued function. The two real valued functions associate with this functions are $\tilde{f}^L_\alpha(x) = (1 + \alpha)x$ and $\tilde{f}^U_\alpha(x) = (3 - \alpha)x$, for all $\alpha \in [0, 1]$ and $x \in \mathbb{R}$.*

To proceed further we need to define Hausdorff metric on fuzzy numbers.

Definition 2.3.2. *[80] Let $A, B \subseteq \mathbb{R}^n$. The Hausdorff metric d_H is defined by*

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{y\in B} \inf_{x\in A} \|x - y\|\}.$$

*Then the **metric** d_F on $F(\mathbb{R})$ is defined as*

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \le \alpha \le 1} \{d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)\},$$

for all $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. Since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are closed bounded intervals in $\mathbb{R}$,

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \le \alpha \le 1} \max\{|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha|, |\tilde{a}^U_\alpha - \tilde{b}^U_\alpha|\}.$$

2.3.1 Continuity of a fuzzy-valued function

Continuity a of fuzzy-valued function is the basic concept in fuzzy mathematics. Various authors have studied continuity of fuzzy-valued functions defined through the supremum metric on fuzzy numbers (for details, see [15], [27], [32] and [69]). We define continuity of a fuzzy-valued function given in [15].

Definition 2.3.3. *Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. We say that $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $\delta = \delta(c, \epsilon) > 0$ such that*

$$d_F(\tilde{f}(x), \tilde{f}(c)) < \epsilon$$

for all $x \in \mathbb{R}^n$ with $\|x - c\| < \delta$. That is,

$$\lim_{x \to c} \tilde{f}(x) = \tilde{f}(c).$$

Example 2.3.2. *A fuzzy-valued function given in Example 2.3.1 is the continuous fuzzy-valued function.*

We prove the following proposition.

Proposition 2.3.1. *[76] Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. If $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$, then functions $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are continuous at c for all $\alpha \in [0, 1]$.*

Proof. The result follows using the definitions of continuity of fuzzy-valued function $\tilde{f}$ and metric on fuzzy numbers. □

2.3.2 Hukuhara differentiability of a fuzzy valued function on $\mathbb{R}$

To define fuzzy differentiability, first we define Hukuhara difference (H-difference) of two fuzzy numbers.

Definition 2.3.4. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$. Then $\tilde{c}$ is called Hukuhara difference of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a} \ominus_H \tilde{b}$.*

H-differentiability (Hukuhara differentiability) of single-variable fuzzy-valued function due to M.L. Puri and D.A. Ralescu [50] is as follows :

Definition 2.3.5. *Let X be a subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is said to be H-differentiable at $x^0 \in X$ if there exists a fuzzy number $D\tilde{f}(x^0)$ such that the limits (with respect to metric d_F)*

$$\lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0 + h) \ominus_H \tilde{f}(x^0)], \quad \textit{and} \quad \lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0) \ominus_H \tilde{f}(x^0 - h)]$$

both exist and are equal to $D\tilde{f}(x^0)$. In this case, $D\tilde{f}(x^0)$ is called the H-derivative of $\tilde{f}$ at x^0. If $\tilde{f}$ is H-differentiable at all $x \in X$, we call $\tilde{f}$ is H-differentiable over X.

Remark 2.3.1. *Above definition implies that if a fuzzy-valued function is H-differentiable at a point $x^0 \in X$, then the Hukuhara differences $\tilde{f}(x^0+h) \ominus_H \tilde{f}(x^0)$ and $\tilde{f}(x^0) \ominus_H \tilde{f}(x^0-h)$ for any $h \in (x^0 - h, x^0 + h)$ both exist. However, if for some fuzzy-valued functions the H-differences may not exist for $h > 0$ or if H-differences exist but limit does not exist then function is not H-differentiable. The following example illustrates the fact.*

Example 2.3.3. *(From [15]). Let $\tilde{f} : (0, 2\pi) \to F(\mathbb{R})$ be defined on level sets by*

$$[\tilde{f}(x)]_\alpha = (1 - \alpha)(2 + \sin(x))[-1, 1],$$

for $\alpha \in [0, 1]$. At $x^0 = \pi/2$, $\tilde{f}^L_\alpha(\pi/2 + h) - \tilde{f}^L_\alpha(\pi/2)$ and $\tilde{f}^U_\alpha(\pi/2 + h) - \tilde{f}^U_\alpha(\pi/2)$ are as follows.

$$\begin{aligned} \tilde{c}^L_\alpha &= \tilde{f}^L_\alpha(\pi/2 + h) - \tilde{f}^L_\alpha(\pi/2) \\ &= (1 + \alpha) - (1 + \alpha)\sin(\pi/2 + h) \end{aligned}$$

and

$$\begin{aligned} \tilde{c}^U_\alpha &= \tilde{f}^U_\alpha(\pi/2 + h) - \tilde{f}^U_\alpha(\pi/2) \\ &= -(1 + \alpha) + (1 + \alpha)\sin(\pi/2 + h) \end{aligned}$$

It can easily be verified that $\tilde{c}^L_\alpha \nleq \tilde{c}^U_\alpha$, for any $h > 0$ and $\alpha \in [0, 1]$. That is, there exists no $\tilde{c}$ such that $\tilde{c} \oplus \tilde{f}(\pi/2) = \tilde{f}(\pi/2 + h)$. That is, H-difference does not exist. Therefore, given function is not H-differentiable at $x^0 = \pi/2$.

We prove following proposition regarding differentiability of $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$.

Proposition 2.3.2. *Let X be a subset of $\mathbb{R}$. Let $\tilde{f} : X \to F(\mathbb{R})$ be a H-differentiable fuzzy-valued function at x^0 with H-derivative $D\tilde{f}(x^0)$. Denote $\tilde{f}_\alpha(x) = [\tilde{f}^L_\alpha(x), \tilde{f}^U_\alpha(x)]$ then $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are differentiable at x^0, for all $\alpha \in [0,1]$. Moreover, we have $(D\tilde{f})_\alpha(x^0) = [D(\tilde{f}^L_\alpha)(x^0), D(\tilde{f}^U_\alpha)(x^0)]$.*

Proof. For $h > 0$, since $[\tilde{f}(x^0+h) \ominus_H \tilde{f}(x^0)]_\alpha = [\tilde{f}^L_\alpha(x^0+h) - \tilde{f}^L_\alpha(x^0), \tilde{f}^U_\alpha(x^0+h) - \tilde{f}^U_\alpha(x^0)]$ and similarly for $[\tilde{f}(x^0) \ominus_H \tilde{f}(x^0 - h)]_\alpha$. Dividing by h and taking limits as h tends to 0^+ proves the result. □

Second order H-differentiability of a fuzzy-valued function is given as,

Definition 2.3.6. *If a fuzzy valued function $\tilde{f} : X \to F(\mathbb{R}), X \subset \mathbb{R}$ has a H-derivative $D\tilde{f}$ on X and if $D\tilde{f}$ is itself H-differentiable, we denote the H-derivative of $D\tilde{f}$ by $D^2\tilde{f}$ and call $D^2\tilde{f}$ the second H-derivative of $\tilde{f}$.*

Proposition 2.3.3. *Let $\tilde{f} : X \to F_L(\mathbb{R})$ be H-differentiable with H-derivative $D\tilde{f}$ on X and let $D\tilde{f} : X \to F_L(\mathbb{R})$ be also H-differentiable at x with H-derivative $D^2\tilde{f}(x)$. Then $D\tilde{f}^L_\alpha(x)$ and $D\tilde{f}^U_\alpha(x)$ are also differentiable at x, for all $\alpha \in [0,1]$. Also, we have $(D^2\tilde{f})_\alpha(x) = [D^2(\tilde{f}^L_\alpha)(x), D^2(\tilde{f}^U_\alpha)(x)]$.*

Similarly, we define $C^n([a,b], F(\mathbb{R}))$, $n \geq 1$ as the space of n-times continuously H-differentiable functions from $[a,b] \subseteq \mathbb{R}$ into $F(\mathbb{R})$. In [28], it has been proved that

$$(\tilde{f}^{(i)}(x))_\alpha = [(\tilde{f}^{(i)}(x))^L_\alpha, (\tilde{f}^{(i)}(x))^U_\alpha],$$

for $i = 1, 2, ..., n$ (i indicates order of derivative) and in particular we have

$$(\tilde{f}^{(i)}(x))^L_\alpha = (\tilde{f}^L_\alpha(x))^{(i)}$$

and

$$(\tilde{f}^{(i)}(x))^U_\alpha = (\tilde{f}^U_\alpha(x))^{(i)}$$

for $\tilde{f} \in C^n([a,b], F(\mathbb{R}))$ and all $\alpha \in [0,1]$.

2.3.3 Hukuhara differentiability of a fuzzy-valued function on $\mathbb{R}^n$

We proceed to state H-differentiability of a multi-variable fuzzy-valued function.

Definition 2.3.7. *[80] Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$ and let $\bar{x}^0 = (x^0_1, ..., x^0_n) \in X$ be fixed.*

We say that $\tilde{f}$ has the i^{th} partial H-derivative $D_i\tilde{f}(\bar{x}^0)$ at $\bar{x}^0$ if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(x_1^0, .., x_{i-1}^0, x_i, x_{i+1}^0, .., x_n^0)$ is H-differentiable at x_i^0 with H-derivative $D_i\tilde{f}(x^0)$. We also write $D_i\tilde{f}(\bar{x}^0)$ as $(\partial\tilde{f}/\partial x_i)(\bar{x}^0)$.

Definition 2.3.8. *We say that $\tilde{f}$ is H-differentiable at $\bar{x}^0$ if one of the partial H-derivatives $\partial\tilde{f}/\partial x_1, ..., \partial\tilde{f}/\partial x_n$ exists at $\bar{x}^0$ and the remaining n-1 partial H-derivatives exist on some neighborhoods of $\bar{x}^0$ and are continuous at $\bar{x}^0$ (in the sense of continuity of fuzzy-valued functions).*

The gradient of $\tilde{f}$ at $\bar{x}^0$ is denoted by

$$\nabla\tilde{f}(\bar{x}^0) = (D_1\tilde{f}(\bar{x}^0), ..., D_n\tilde{f}(\bar{x}^0)),$$

and it defines a fuzzy-valued function from X to $F^n(\mathbb{R}) = F(\mathbb{R}) \times .... \times F(\mathbb{R})$ (n times), where each $D_i\tilde{f}(\bar{x}^0)$ is a fuzzy number for $i = 1, ..., n$. The α-level set of $\nabla\tilde{f}(\bar{x}^0)$ is defined and denoted by

$$(\nabla\tilde{f}(\bar{x}^0))_\alpha = (D_1\tilde{f}(\bar{x}^0))_\alpha \times ((D_2\tilde{f}(\bar{x}^0))_\alpha \times ... \times (D_n\tilde{f}(\bar{x}^0))_\alpha,$$

where

$$(D_i\tilde{f}(\bar{x}^0))_\alpha = [D_i\tilde{f}_\alpha^L(\bar{x}^0), D_i\tilde{f}_\alpha^U(\bar{x}^0)],$$

$i = 1, ..., n$.
We say that $\tilde{f}$ is H-differentiable on X if it is H-differentiable at every $\bar{x} \in X$.

Proposition 2.3.4. *Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable on X. Then $\tilde{f}_\alpha^L(\bar{x})$ and $\tilde{f}_\alpha^U(\bar{x})$ are also differentiable on X, for all $\alpha \in [0, 1]$. Moreover, for each $\bar{x} \in X$, $(D_i\tilde{f}(\bar{x}))_\alpha = [D_i\tilde{f}_\alpha^L(\bar{x}), D_i\tilde{f}_\alpha^U(\bar{x})]$, $i = 1, ..., n$.*

Proof. The result follows from Propositions 2.3.1 and 2.3.2. □

Definition 2.3.9. *We say that $\tilde{f}$ is continuously H-differentiable at $\bar{x}^0$ if all of the partial H-derivatives $\partial\tilde{f}(\bar{x})/\partial x_i$, $i = 1, .., n$, exist on some neighborhoods of $\bar{x}^0$ and are continuous at $\bar{x}^0$ (in the sense of fuzzy-valued function).*
We say that $\tilde{f}$ is continuously H-differentiable on X if it is continuously H-differentiable at every $\bar{x}^0 \in X$.

Proposition 2.3.5. *Let $\tilde{f} : X \to F(\mathbb{R})$ be continuously H-differentiable on X. Then $\tilde{f}_\alpha^L(\bar{x})$ and $\tilde{f}_\alpha^U(\bar{x})$ are also continuously differentiable on X, for all $\alpha \in [0, 1]$.*

Proof. Followed by Propositions 2.3.1 and 2.3.4. □

Definition 2.3.10. *Let X be an open set and $\tilde{f} : X \to F(\mathbb{R}), X \subset \mathbb{R}^n$ be a fuzzy-valued function. Suppose now that there is $\bar{x}^0 \in X$ such that gradient of $\tilde{f}$, $\nabla \tilde{f}$, is itself H-differentiable at $\bar{x}^0$, that is, for each i, the function $D_i\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at $\bar{x}^0$. Denote the partial H-derivative of $D_i\tilde{f}$ in the direction of $\bar{e}_j$ at $\bar{x}^0$ by*

$$D^2_{ij}\tilde{f} \quad or \quad \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_i \partial x_j}, \quad if \quad i \neq j,$$

and

$$D^2_{ii}\tilde{f} \quad or \quad \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_i^2}, \quad if \quad i = j.$$

If these $n \times n$ H-derivatives $[D^2_{ij}\tilde{f}], i, j = 1, ..., n$ exist, then we say that $\tilde{f}$ is twice H-differentiable at $\bar{x}^0$, with the matrix of second H-derivatives $\nabla^2 \tilde{f}(\bar{x}^0)$ which can be called as the fuzzy Hessian matrix and is denoted by

$$\nabla^2 \tilde{f}(\bar{x}^0) = \begin{pmatrix} \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_1^2} & ... & \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_1 \partial x_n} \\ ... & ... & ... \\ \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_n^2} \end{pmatrix}$$

where $\frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_i \partial x_j} \in F(\mathbb{R})$, $i, j = 1, ..., n$.
If $\tilde{f}$ is twice H-differentiable at each $\bar{x}$ in X, we say that $\tilde{f}$ is twice H-differentiable on X, and if for each $i, j = 1, ..., n$, the cross-partial H-derivative $\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}$ is continuous function from X to $F(\mathbb{R})$, we say that $\tilde{f}$ is twice continuously H-differentiable on X.

2.3.4 Definite and semidefinite fuzzy matrix

First we define definiteness and semidefiniteness of a real matrix.

Definition 2.3.11. *Let A be any $n \times n$ symmetric matrix. Then A is said to be*

(i) *positive definite if we have $x^t \cdot A \cdot x > 0$ for all $x \in \mathbb{R}^n$,$x \neq 0$.*

(ii) *positive semidefinite if we have $x^t \cdot A \cdot x \geq 0$ for all $x \in \mathbb{R}^n$,$x \neq 0$.*

(iii) *negative definite if we have $x^t \cdot A \cdot x < 0$ for all $x \in \mathbb{R}^n$,$x \neq 0$.*

(iv) *negative semidefinite if we have $x^t \cdot A \cdot x \leq 0$ for all $x \in \mathbb{R}^n$,$x \neq 0$.*

Example 2.3.4.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is a positive definite matrix.

Example 2.3.5.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This is a positive semidefinite matrix.

Example 2.3.6.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is an indefinite matrix.

Here we state one important result on the basis of which we can check the definiteness of a given symmetric matrix.

Theorem 2.3.1. ***Sylvester Criteria***
An $n \times n$ symmetric matrix A is

(1) *negative definite if and only if $(-1)^k \cdot |A_k| > 0$ for all $k \in \{1, ..., n\}$*

(2) *positive definite if and only if $|A_k| > 0$ for all $k \in \{1, ..., n\}$.*

Now we define definiteness and semidefiniteness of fuzzy matrix.

Definition 2.3.12. *Let $\tilde{A} = (\tilde{a}_{ij})$, $i, j = 1, ..., n$ be a fuzzy matrix. That is, all the elements $(\tilde{a}_{ij})$ in the fuzzy matrix $\tilde{A}$, are fuzzy numbers defined on $\mathbb{R}$. There are associated two real matrices called α-level matrices , $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$, $\alpha \in [0, 1]$ which are given as follows:*

$$\tilde{A}^L_\alpha = \begin{pmatrix} (\tilde{a}_{11})^L_\alpha & ... & (\tilde{a}_{1n})^L_\alpha \\ ... & .... & .... \\ (\tilde{a}_{n1})^L_\alpha & ... & (\tilde{a}_{nn})^L_\alpha \end{pmatrix}$$

and

$$\tilde{A}^U_\alpha = \begin{pmatrix} (\tilde{a}_{11})^U_\alpha & ... & (\tilde{a}_{1n})^U_\alpha \\ ... & .... & .... \\ (\tilde{a}_{n1})^U_\alpha & ... & (\tilde{a}_{nn})^U_\alpha \end{pmatrix}$$

Then $\tilde{A}$ is said to be

(i) *positive definite fuzzy matrix if the α-level matrices $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$ are positive definite real matrices, for all $\alpha \in [0, 1]$,*

(ii) *positive semidefinite fuzzy matrix if the α-level matrices $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$ are positive semidefinite real matrices, for all $\alpha \in [0, 1]$.*

Example 2.3.7. *Consider the fuzzy matrix*

$$\tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$$

where $\tilde{a} = (1,2,4)$ *and* $\tilde{0} = (0,0,0)$ *are fuzzy numbers. Then we obtain two* α*-level matrices for* $\tilde{A}$*.*

$$\tilde{A}^L_\alpha = \begin{pmatrix} (1+\alpha) & 0 \\ 0 & (1+\alpha) \end{pmatrix}$$

and

$$\tilde{A}^U_\alpha = \begin{pmatrix} (4-2\alpha) & 0 \\ 0 & (4-2\alpha) \end{pmatrix}$$

for all $\alpha \in [0,1]$*. Clearly these matrices are positive definite for all* α*. Therefore, the given fuzzy matrix* $\tilde{A}$ *is positive definite.*

Example 2.3.8. *Now consider the fuzzy matrix*

$$\tilde{B} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$$

where $\tilde{a} = (0,2,4)$ *and* $\tilde{0} = (0,0,0)$ *are fuzzy numbers. Then we obtain two* α*-level matrices for* $\tilde{B}$*.*

$$\tilde{B}^L_\alpha = \begin{pmatrix} (2\alpha) & 0 \\ 0 & (2\alpha) \end{pmatrix}$$

and

$$\tilde{B}^U_\alpha = \begin{pmatrix} (4-2\alpha) & 0 \\ 0 & (4-2\alpha) \end{pmatrix}$$

for all $\alpha \in [0,1]$*. These matrices are positive definite, for all* α *except* $\alpha = 0$*. For* $\alpha = 0$*, the first matrix is positive semidefinite. Therefore, the given fuzzy matrix* $\tilde{B}$ *is positive semidefinite.*

2.4 Fuzzy Riemann integrability

The concept of fuzzy integral was introduced by Sugeno [68] (1974). After that, many formulations of fuzzy integrals have been developed. For instance, Dubois and Prade [19]

(1980) considered a certain type of fuzzy-valued function and defined the integral of such a function using the extension principle. While, Puri and Ralescu (1986) have defined integral of fuzzy-valued functions levelwise in their paper [51]. Goetschel and and Voxman [32] (1986) defined differentiation and integration of fuzzy-valued functions in ways that parallel closely the corresponding definitions for real differentiation and integration. Sims and Wang [64] gave a good review of literature in this subject area. In this work, we use fuzzy Riemann integral of a fuzzy-valued function given in [32].

We cite the definition of fuzzy Riemann integrability from [28].

Definition 2.4.1. *[28] Let $\tilde{f} : [a, b] \to F(\mathbb{R})$. We say that $\tilde{f}$ is Fuzzy-Riemann (F-R) integrable to $\tilde{I} \in F(\mathbb{R})$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any division $P = \{[u, v] : \xi\}$ of $[a, b]$ with the norm $\Delta(P) < \delta$, we have*

$$d_F(\sum_P{}^* (v - u) \odot \tilde{f}(\xi), \tilde{I}) < \epsilon$$

where $\sum^*$ denotes the fuzzy summation. We choose to write

$$\tilde{I} = (FR) \int_a^b \tilde{f}(x)dx.$$

On the basis of Fuzzy Riemann integrability, we state here Fuzzy Taylor's formula for single-variable and multi variable fuzzy-valued functions.

Theorem 2.4.1. *[28] Let $\tilde{f} \in C^n([a, b], F(\mathbb{R}))$, $n \geq 1$, $[t_0, t_1] \subseteq \mathbb{R}$. Then*

$$\begin{aligned} \tilde{f}(t_1) &= \tilde{f}(t_0) \oplus (t_1 - t_0) \odot \tilde{f}'(t_0) \oplus ... \oplus \frac{(t_1 - t_0)^{n-1}}{(n-1)!} \odot \tilde{f}^{(n-1)}(t_0) \\ &\oplus \frac{1}{(n-1)!} \odot (FR) \int_{t0}^{t_1} (t_1 - t) \odot \tilde{f}^{(n)}(t)dt. \end{aligned}$$

The fuzzy integral remainder is a continuous function in t.

Theorem 2.4.2. *[28] Let U be an open convex subset of $\mathbb{R}^n, n \in \mathbb{N}$ and $\tilde{f} : U \to F(\mathbb{R})$ be a continuous fuzzy-valued function. Assume that all fuzzy H-partial derivatives of $\tilde{f}$ up to order $m \in \mathbb{N}$ exist and are continuous. Let $z = (z_1, z_2, ..., z_n)$, $x_0 = (x_{01}, ..., x_{0n}) \in U$ such that $z_i \geq x_{0i}$, $i = 1, ..., n$. Let $0 \leq t \leq 1$, we define $x_i = x_{0i} + t(z_i - x_{0i})$, $i = 1, ..., n$ and $\tilde{g}_z(t) = \tilde{f}(x_0 + t(z - x_0))$. (clearly $x_0 + t(z - x_0) \in U$). Then for $N = 1, ..., m$, we* *obtain*

$$\tilde{g}_z^{(N)}(t) = \Big[\Big(\sum_{i=1}^{n}(z_i - x_{0i}) \odot \frac{\partial}{\partial x_i}\Big)^N \tilde{f}\Big](x_1, ..., x_n).$$

Furthermore the following fuzzy multivariate Taylor formula holds

$$\tilde{f}(z) = \tilde{f}(x_0) \oplus \sum_{N=1}^{m-1} \frac{\tilde{g}_z^{(N)}(0)}{N!} \oplus R_m(0,1),$$

where

$$R_m(0,1) = \frac{1}{(m-1)!} \odot (FR)\int_0^1 (1-s)^{m-1} \odot \tilde{g}_z^{(m)}(s)ds.$$

2.5 Order relations on fuzzy numbers

Another important concept used in our research work is of order structure on fuzzy numbers. Order structures play a very important role in fuzzy optimization problems. Many methods of ordering fuzzy numbers have been proposed in the literature (see ,e.g, Bortolan and Degani [5]).

2.5.1 A partial order relation: Fuzzy-max order

Among the various ordering methods for fuzzy numbers the commonly used one is a partial order relation called the fuzzy-max order, introduced by Ramík and Rimanek [52], which is defined as follows:

Definition 2.5.1. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers in $F(\mathbb{R})$ and let $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\tilde{b}_\alpha = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$ be two closed intervals in $\mathbb{R}$, $\alpha \in [0,1]$. We define $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U$ for all $\alpha \in [0,1]$.*

It is well known that the order relation $\preceq$ satisfies the axioms of a partial order relation on the family $F(\mathbb{R})$. Using a partial order relation on fuzzy numbers, we can define $\tilde{a} \prec \tilde{b}$ in different ways:

(i) $\tilde{a} \prec \tilde{b}$ if and only if $\tilde{a} \preceq \tilde{b}$ and there exists an $\alpha_0 \in [0,1]$ such that $\tilde{a}_{\alpha_0}^L < \tilde{b}_{\alpha_0}^L$ or $\tilde{a}_{\alpha_0}^U < \tilde{b}_{\alpha_0}^U$

(ii) $\tilde{a} \prec \tilde{b}$ if and only if

$$\begin{cases} \tilde{a}_\alpha^L < \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U \end{cases} \text{or} \begin{cases} \tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U < \tilde{b}_\alpha^U \end{cases} \text{or} \begin{cases} \tilde{a}_\alpha^L < \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U < \tilde{b}_\alpha^U \end{cases} \qquad \text{for all } \alpha \in [0,1].$$

We use this partial order relation to define an optimal solution for a fuzzy optimization problem and establish the optimality conditions based on this order relation.

2.5.2 A parametric total order relation

Another order relation is a parametric total order relation defined on L-fuzzy numbers. This order relation "$\preceq_\lambda$" , with parameter $\lambda \in [0, 1]$, on set of L-fuzzy numbers, introduced by S. Saito and H. Ishii [58] and is defined as follows:

Definition 2.5.2. *For any $\tilde{a}, \tilde{b} \in F_L(\mathbb{R})$, we say that $\tilde{a} \preceq_\lambda \tilde{b}$, where "$\preceq_\lambda$" is a parametric order relation on $F_L(\mathbb{R})$, for $0 \leq \lambda \leq 1$ if only one of the following inequalities hold:*

(i) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$ *for* $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$

(ii) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L \leq \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$ *for* $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{b}_1^L - \tilde{b}_0^L$

It can be easily proved that "$\preceq_\lambda$" for any fixed $\lambda \in [0, 1]$ is a total order relation on $F_L(\mathbb{R})$. $\tilde{a} \succeq_\lambda \tilde{b}$ is defined by $\tilde{b} \preceq_\lambda \tilde{a}$.

Proposition 2.5.1. *Prove that $\preceq_\lambda$ is a total order relation on $F_L(\mathbb{R})$, $F_L(\mathbb{R})$ is a set of L-fuzzy numbers defined on $\mathbb{R}$.*

Proof. First we prove that "$\preceq_\lambda$" is a partial order relation on $F_L(\mathbb{R})$.

First property is **Reflexivity**:
For $\tilde{a}_1^L - \tilde{a}_0^L = \tilde{a}_1^L - \tilde{a}_0^L$, we have $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L = \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$. Therefore, $\tilde{a} \preceq_\lambda \tilde{a}$, for a fixed $\lambda \in [0, 1]$.

Second property is **Anti-symmetry**:
If $\tilde{a} \preceq_\lambda \tilde{b}$ and $\tilde{b} \preceq_\lambda \tilde{a}$, then $\tilde{a} = \tilde{b}$.

Since $\tilde{a} \preceq_\lambda \tilde{b}$, by definition, only one of the following inequalities holds, for a fixed $\lambda \in [0, 1]$:

(i) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$, for $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$

(ii) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L \leq \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$, for $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{b}_1^L - \tilde{b}_0^L$

and $\tilde{b} \preceq_\lambda \tilde{a}$, implies only one of the following inequalities holds:

(i') $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L < \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$, for $\tilde{b}_1^L - \tilde{b}_0^L < \tilde{a}_1^L - \tilde{a}_0^L$

(ii') $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L \leq \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$, for $\tilde{b}_1^L - \tilde{b}_0^L \geq \tilde{a}_1^L - \tilde{a}_0^L$

We can see here **(i)** and **(i')** do not hold simultaneously. Similarly, **(i)** and **(ii')** and **(ii)** and **(i')** also do not hold. Hence the only possible case is:
(ii) and **(ii')** hold. In this case, we must have $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L = \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$.
Therefore, $\tilde{a} = \tilde{b}$.

Now we prove third property, **Transitivity**: If $\tilde{a} \preceq_\lambda \tilde{b}$ and $\tilde{b} \preceq_\lambda \tilde{c}$, then $\tilde{a} \preceq_\lambda \tilde{c}$.

Since $\tilde{a} \preceq_\lambda \tilde{b}$, by definition, only one of the following inequalities hold, for a fixed $\lambda \in [0, 1]$:

(i) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$, for $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$

(ii) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L \leq \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$, for $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{b}_1^L - \tilde{b}_0^L$

and $\tilde{b} \preceq_\lambda \tilde{c}$, implies only one of the following inequalities hold:

(i') $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L < \lambda[\tilde{c}_1^L - \tilde{c}_0^L] + \tilde{c}_1^L$, for $\tilde{b}_1^L - \tilde{b}_0^L < \tilde{c}_1^L - \tilde{c}_0^L$

(ii') $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L \leq \lambda[\tilde{c}_1^L - \tilde{c}_0^L] + \tilde{c}_1^L$, for $\tilde{b}_1^L - \tilde{b}_0^L \geq \tilde{c}_1^L - \tilde{c}_0^L$

Now for the case **(i)** and **(i')** hold, we have
$\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{c}_1^L - \tilde{c}_0^L] + \tilde{c}_1^L$, for $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{c}_1^L - \tilde{c}_0^L$.

For the case **(i)** and **(ii')** hold,that is, for $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$ and $\tilde{b}_1^L - \tilde{b}_0^L \geq \tilde{c}_1^L - \tilde{c}_0^L$,

$\tilde{a} \preceq_\lambda \tilde{b}$ implies $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$ and

$\tilde{b} \preceq_\lambda \tilde{c}$ implies $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L \leq \lambda[\tilde{c}_1^L - \tilde{c}_0^L] + \tilde{c}_1^L$.

This implies, $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{c}_1^L - \tilde{c}_0^L] + \tilde{c}_1^L$, either $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{c}_1^L - \tilde{c}_0^L$ or $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{c}_1^L - \tilde{c}_0^L$.

Therefore, $\tilde{a} \preceq_\lambda \tilde{c}$. We can prove the inequality easily, for the case **(i)** and **(i')** hold.

Therefore, "$\preceq_\lambda$" is a partial order relation on $F_L(\mathbb{R})$.

Now we prove that "$\preceq_\lambda$" is a total order relation on $F_L(\mathbb{R})$. That is, if $\tilde{a} \npreceq_\lambda \tilde{b}$ then $\tilde{b} \preceq_\lambda \tilde{a}$. Since $\tilde{a} \npreceq_\lambda \tilde{b}$, by definition

case i: $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$ but $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L \geq \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$.

case ii: Suppose $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{b}_1^L - \tilde{b}_0^L$ but $\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L < \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$.

Then, from **case ii**,

$\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L < \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$ when $\tilde{b}_1^L - \tilde{b}_0^L < \tilde{a}_1^L - \tilde{a}_0^L$.

And by taking **case i** and **ii** together, we have

$\lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L \leq \lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L$ when $\tilde{b}_1^L - \tilde{b}_0^L \geq \tilde{a}_1^L - \tilde{a}_0^L$.

Therefore, for both the cases we have $\tilde{b} \preceq_\lambda \tilde{a}$, for a fixed $\lambda \in [0, 1]$. □

Different values of λ will represent different practical situations for comparison of two L-fuzzy numbers. Here it can also be noticed that when we define a parametric total order relation "$\preceq_\lambda$" on $F_L(\mathbb{R})$, shape function L must be fixed for all the L-fuzzy numbers. Using this order relation , we define an optimal solution of L-fuzzy optimization problems and prove the necessary and sufficient optimality conditions for the same.

2.6 Generalized convexity of a fuzzy valued function

First we list some of the basic concepts about the convexity of crisp sets and functions.

2.6.1 Convex sets and functions

Here we begin with the definition of an Affine set.

Definition 2.6.1. *A set $C \subseteq \mathbb{R}^n$ is affine if the line through any two distinct points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$. In other words, C contains the linear combination of any two points in C, provided the coefficients in the linear combination sum to one.*

Example 2.6.1. *For example, solution set of linear equations $\{x|Ax = b\}$.*

Now we define a convex set as follows.

Definition 2.6.2. *Any subset C of $\mathbb{R}^n$ is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have*

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Example 2.6.2. *For example, a solid cube is convex, but anything that is hollow or has a dent in it, for example, a crescent shape, is not convex.*

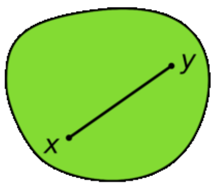

Figure 2.1: Convex set

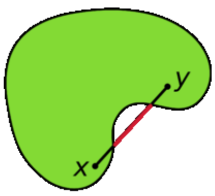

Figure 2.2: Non-Convex set

Some important examples are given as follows:

- The empty set ϕ, any single point (i.e., singleton) $\{x_0\}$, and the whole space $\mathbb{R}^n$ are affine (hence, convex) subsets of $\mathbb{R}^n$.

- Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point).
- Any subspace is affine, and a convex cone (hence convex).

Now we define convexity of a real-valued function.

Definition 2.6.3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if domain of f is a convex set and if for all x, $y \in$ domain of f , and θ with $0 < \theta < 1$, we have*

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y). \tag{2.6.1}$$

A function f is strictly convex if strict inequality holds in (2.6.1) whenever $x \neq y$ and $0 < \theta < 1$. We say f is concave if $-f$ is convex, and strictly concave if $-f$ is strictly convex.

We give some examples of convex and concave functions:

- Exponential function. e^{ax} is convex on $\mathbb{R}$, for any $a \in \mathbb{R}$.
- Powers. x^a is convex on $\mathbb{R}_+$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- Powers of absolute value. $|x|^p$, for $p \geq 1$, is convex on $\mathbb{R}$.
- Logarithm. $\log x$ is concave on $\mathbb{R}_+$.

Here $\mathbb{R}_+$ denotes the set of positive numbers. The following is another example:

Example 2.6.3. *Norms. If $f : \mathbb{R}^n \to \mathbb{R}$ is a norm, and $0 \leq \theta \leq 1$, then*

$$f(\theta x + (1-\theta)y) \leq f(\theta x) + f((1-\theta)y) = \theta f(x) + (1-\theta)f(y)$$

The inequality follows from the triangle inequality, and the equality follows from homogeneity of a norm.

The following theorem states an property of differentiable convex real-valued functions.

Theorem 2.6.1. *[47] Let $f : T \subseteq \mathbb{R}^n \to \mathbb{R}$, T is convex set. If f is differentiable at $\bar{x}^0 \in T$, then $f(x)$ is convex at $\bar{x} = \bar{x}^0$ if and only if*

$$(\nabla f(\bar{x}^0))^t(\bar{x} - \bar{x}^0) \leq f(\bar{x}) - f(\bar{x}^0),$$

for all $\bar{x} \in T$.

2.6.2 Convexity of a fuzzy-valued function

Convex analysis is one of the most important areas in fuzzy mathematics. Nanda and Kar [48] introduced the concept of convexity for fuzzy-valued functions. Yan-Xu [81] has discussed convexity and quasiconvexity of fuzzy-valued functions. Syau [70] has studied new concepts of pseudoconvexity, invexity and pseudoinvexity for fuzzy-valued functions of several variables. Convexity and Lipschitz continuity of fuzzy-valued functions are studied by Furukawa [27]. Here we consider the convexity of a fuzzy-valued function in terms of fuzzy-max order.

Convexity of a fuzzy-valued function is defined as follows:

Definition 2.6.4. *Let $\tilde{f} : T \subseteq \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function and T be a convex set. We say that $\tilde{f}$ is convex at $\bar{x}_0 \in T$ if*

$$\tilde{f}(\lambda\bar{x}_0 + (1-\lambda)\bar{x}) \preceq (\lambda \odot \tilde{f}(\bar{x}_0) \oplus ((1-\lambda) \odot \tilde{f}(\bar{x})))$$

for each $\lambda \in (0,1)$ and each $\bar{x} \in T$.

The following Proposition holds for convexity of a fuzzy-valued function.

Proposition 2.6.1. *[76] Let $\tilde{f} : T \subseteq \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function and T be a convex set. Then $\tilde{f}$ is convex at $\bar{x}_0 \in T$ if and only if $\tilde{f}^L_\alpha(\bar{x})$ and $\tilde{f}^U_\alpha(\bar{x})$ are convex at $\bar{x}_0$ for all $\alpha \in [0,1]$.*

Proof. The result can be proved easily using the concepts of arithmetic operations and partial order relation of fuzzy numbers. □

Example 2.6.4. *The fuzzy-valued function $\tilde{f}(x_1,x_2) = (1,2,3) \odot x_1^2 \oplus (2,3,4) \odot x_1^2$, where $(1,2,3)$ and $(2,3,4)$ are triangular fuzzy numbers, is a convex fuzzy-valued function, since its α-level functions $\tilde{f}^L_\alpha(x_1,x_2) = (1+\alpha)x_1^2+(2+\alpha)x_2^2$ and $\tilde{f}^U_\alpha(x_1,x_2) = (3-\alpha)x_1^2+(4-\alpha)x_2^2$ are convex functions, for all α.*

2.6.3 Quasiconvexity and pseudoconvexity of a fuzzy-valued function

Convexity carries powerful implications for optimization theory. However, from the point of view of applications, convexity is quite restrictive as an assumption. For instance, such a commonly used utility function as the Cobb-Douglus function

$$u(x_1, x_2, ..., x_n) = x_1^{t_1} \cdot ...x_n^{t_n}$$

is not concave unless $\sum_{i=1}^{n} t_i \leq 1$. So we define here the less restrictive assumptions of quasiconvexity and pseudoconvexity of a real-valued function. First we define quasiconvexity of a real-valued function.

Definition 2.6.5. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex if its domain ($dom(f)$) and all its sublevel sets*

$$\{x \in dom(f) | f(x) \leq \gamma\},$$

$\gamma \in \mathbb{R}$, are convex. A function is quasiconcave if its every superlevel set

$$\{x \in dom(f) | f(x) \geq \gamma\}$$

is convex. A function that is both quasiconvex and quasiconcave is called quasilinear.

Example 2.6.5. **(1)** $\log x$ *on $\mathbb{R}_+$ is quasiconvex.*

(2) *$f : \mathbb{R}^2 \to \mathbb{R}$ with domain $\mathbb{R}^2_+$ and $f(x_1, x_2) = x_1 \cdot x_2$ is quasiconcave function, since the superlevel sets*

$$\{x \in \mathbb{R}^2 | x_1 \cdot x_2 \geq \gamma\}$$

are convex sets for γ.

Here we state some of the basic properties of quasiconvex functions.

(1) There exists a strong relationship between the value of a function at two points x and y, and the value of the function at its convex combination $\lambda x + (1 - \lambda)y$.

Theorem 2.6.2. *[54] A function $f : T \subseteq \mathbb{R}^n \to \mathbb{R}$, T is a convex set, is quasiconcave on T if and only if for all $x, y \in T$, and for all $\lambda \in (0, 1)$,*

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}. \tag{2.6.2}$$

The function f is quasiconvex on T if and only if for all $x, y \in T$ and for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}. \tag{2.6.3}$$

We can define strict quasiconcavity (strict quasiconvexity) of f by defining strict inequalities in (2.6.2) and (2.6.3) respectively with $x, y \in T$ and $x \neq y$.

(2) Every concave function is quasiconcave function and every convex function is quasiconvex function. But converse is not true in general. For instance, $f(x) = x^3$ for all $x \in \mathbb{R}$, is quasiconcave but not concave and quasiconvex but not convex.

(3) Quasiconvex and quasiconcave functions are not necessarily continuous in the interior of their domains.

(4) Quasiconcave functions can have local maximum that are not global maximum, and quasiconvex functions can have local minimum that are not global minimum.

(5) First-order conditions are not sufficient to identify even local optimum under quasiconvexity.

The following example illustrates the above three points.

Example 2.6.6. *[54] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^3, & x \in [0, 1] \\ 1, & x \in (1, 2] \\ x^3, & x > 2. \end{cases}$$

It is easily checked that f is both quasiconvex and quasiconcave on $\mathbb{R}$. Clearly, f has a discontinuity at $x = 2$. Moreover, f is constant on the open interval $(1, 2)$, so every point in this interval is a local maximum of f as well as a local minimum of f. However, no point in $(0, 1)$ is either a global maximum or a global minimum. Finally, $f'(0) = 0$, although 0 is evidently neither a local maximum, nor a local minimum.

In some of the sources, definition of quasiconvexity of a real-valued function is given as

Definition 2.6.6. *[1, 47] $f : T \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be quasiconvex at $\bar{x}_0 \in T$, with T an arbitrary set, if and only if*

$$f(\bar{x}) \leq f(\bar{x}_0) \implies f(\lambda\bar{x} + (1 - \lambda)\bar{x}_0) \leq f(\bar{x}_0),$$

for all $\bar{x} \in T$ and $\lambda \in [0, 1]$.

$f(\bar{x})$ is said to be quasiconvex on T if and only if it is quasiconvex at each $\bar{x} \in T$. Furthermore, if $f(\bar{x})$ is differentiable at $\bar{x}_0 \in T$, $f(\bar{x})$ is quasiconvex at $\bar{x} = \bar{x}_0$ if and only if

$$f(\bar{x}) \leq f(\bar{x}_0) \implies (\nabla f(\bar{x}_0))^t(\bar{x} - \bar{x}_0) \leq 0, \ for\ all\ \bar{x} \in T.$$

A pseudoconvex function is a function that behaves like a convex function with respect to finding its local minimum, but need not actually be convex. Informally, a differentiable function is pseudoconvex if it is increasing in any direction where it has a positive directional derivative. Formally, the definition is given as follows.

Definition 2.6.7. *A function $f(\bar{x})$ defined on an open set $T \subseteq \mathbb{R}^n$, is said to be pseudoconvex at $\bar{x}_0 \in T$ (on T), if and only if it is differentiable at $\bar{x}_0$ (at each point of T) with*

$$(\nabla f(\bar{x}_0))^t(\bar{x} - \bar{x}_0) \geq 0 \implies f(\bar{x}) \geq f(\bar{x}_0);$$

or equivalently

$$f(\bar{x}) < f(\bar{x}_0) \implies (\nabla f(\bar{x}_0))^t(\bar{x} - \bar{x}_0) < 0,$$

for all $\bar{x} \in T$ (for all $\bar{x}_0 \in T$).

Example 2.6.7. *$f(x) = x^3 + x$ defined on $\mathbb{R}$ is pseudoconvex function for all $x \in \mathbb{R}$ as*

$$Df(x_2)(x_1 - x_2) \geq 0 \implies (3x_2^2 + 1)(x_1 - x_2) \geq 0$$

implies $(x_1 - x_2) \geq 0$ implies $f(x_1) \geq f(x_2)$.

Example 2.6.8. *$f(x) = x^3 - x$ defined on $\mathbb{R}$ is pseudoconvex function for all $x \in \mathbb{R}$ by similar arguments as in above example.*

Example 2.6.9. *Consider the function $f : S \to \mathbb{R}$, $S = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$, unit square in $\mathbb{R}^2$ and $f(x_1, x_2) = -x_1^2 - x_1$. Consider arbitrary $x = [x_1, x_2], y = [y_1, y_2] \in S$. If we have*

$$(\nabla f(x))^t(y - x) \geq 0 \implies (-2x_1 - 1)(y_1 - x_1) \geq 0 \implies y_1 \leq x_1$$

This implies $f(y) \geq f(x)$. Hence f is pseudoconvex function for all $x \in S$.

Definition 2.6.8. *[1, 47] A function $f(\bar{x})$, defined on an open set $T(\subseteq \mathbb{R}^n)$, is said to be strictly pseudoconvex at $\bar{x}_0 \in T$ (on T) if and only if it is differentiable at $\bar{x}_0$ (at each point $\bar{x} \in T$) with*

$$(\nabla f(\bar{x}_0))^t(\bar{x} - \bar{x}_0) \geq 0 \implies f(\bar{x}) > f(\bar{x}_0);$$

or equivalently

$$f(\bar{x}) \leq f(\bar{x}_0) \implies (\nabla f(\bar{x}_0))^t(\bar{x} - \bar{x}_0) < 0,$$

for each $\bar{x}$ ($\neq \bar{x}_0$) $\in T$ (for each $\bar{x} \in T$).

Example 2.6.10. *$f(x) = -x^2 - x$, $0 \leq x \leq 1$ is strictly pseudoconvex function as for each $x_1, x_2 \in [0, 1]$ with $x_1 \neq x_2$ we have,*

$$Df(x_1)(x_2 - x_1) \geq 0 \implies (-2x_1 - 1)(x_2 - x_1) \geq 0$$

implies $x_2 < x_1$*. This implies* $f(x_2) > f(x_1)$*.*

Quasiconvexity and pseudoconvexity of fuzzy-valued functions are define as

Definition 2.6.9. *Let* $\tilde{f} : T \to F(\mathbb{R})$ *be a fuzzy-valued function defined on an arbitrary set* $T \subseteq \mathbb{R}^n$*. We say that* $\tilde{f}$ *is qausiconvex at* $\bar{x}_0$ *if and only if the real-valued functions* $\tilde{f}_\alpha^L$ *and* $\tilde{f}_\alpha^U$ *are quasiconvex at* $\bar{x}_0$*, for all* $\alpha \in [0, 1]$*.*

Example 2.6.11. *Consider a fuzzy-valued function* $\tilde{f}(x) = (1, 2, 3) \odot x^3$ *defined on* $\mathbb{R}$*. It can be easily verified that* $\tilde{f}$ *is the quasiconvex function since* $\tilde{f}_\alpha^L(x) = (1 + \alpha)x^3$ *and* $\tilde{f}_\alpha^U(x) = (3 - \alpha)x^3$ *are quasiconvex functions, for all* $\alpha \in [0, 1]$*.*

Definition 2.6.10. *Let* $\tilde{f} : T \to F(\mathbb{R})$ *be a fuzzy-valued function defined on an open set* $T \subseteq \mathbb{R}^n$*. We say that* $\tilde{f}$ *is pseudoconvex (strictly pseudoconvex) at* $\bar{x}_0$ *if and only if the real-valued functions* $\tilde{f}_\alpha^L$ *and* $\tilde{f}_\alpha^U$ *are pseudoconvex (strictly pseudoconvex) at* $\bar{x}_0$*, for all* $\alpha \in [0, 1]$*.*

Example 2.6.12. *Consider a fuzzy-valued function* $\tilde{f}(x) = (1, 2, 3) \odot x \oplus (2, 3, 4) \odot x^3$ *defined on* $\mathbb{R}$*. It can be easily verified that* $\tilde{f}$ *is the pseudoconvex function since* $\tilde{f}_\alpha^L(x) = (1 + \alpha)x + (2 + \alpha)x^3$ *and* $\tilde{f}_\alpha^U(x) = (3 - \alpha)x + (4 - \alpha)x^3$ *are pseudoconvex functions, for all* $\alpha \in [0, 1]$*.*

Using H-differentiability of a fuzzy-valued function and a parametric total order relation on set of fuzzy numbers, we prove the necessary and sufficient optimality conditions for unconstrained single and multi variable fuzzy optimization problems, in chapter three. In chapter four, we solve the unconstrained and constrained fuzzy optimization problems using partial order relation-fuzzy max order on the set of fuzzy numbers. Under the concepts of convexity and generalized convexity of fuzzy-valued functions, we derive sufficient optimality conditions for constrained fuzzy optimization problems, in chapter five. In chapter six, we propose the Newton's method for solving unconstrained single and multi variable fuzzy optimization problems.

Chapter 3

Unconstrained L-fuzzy optimization problems

Contents

3.1 Introduction

Many authors have extensively studied the problem of ranking of fuzzy numbers in reference to the optimization problems, e.g. [5], [9], [17], [31], [38], [82]. In general, fuzzy numbers are not comparable in a natural way. However, some authors have defined different ranking methods in terms of the parametric total order relations. For instance, Furukawa [25] introduced a parametric total order relation on the class of symmetric fuzzy numbers generated by a shape function. He applied it as a criterion of minimization to solve a fuzzy shortest route problem. Furukawa defined two types of parametric order relations on the class of symmetric fuzzy numbers generated by a shape function in his paper [26](1997) and used it to find the solution of a fuzzy optimization problem.

S. Saito and H. Ishii have introduced a parametric total order relation "$\preceq_\lambda$", $\lambda \in [0, 1]$ in their paper [58] (2001). They have solved fuzzy optimization problems having fuzzy-valued functions which have both domain and range as L-fuzzy numbers. In the current work, we use the same parametric total order relation, "$\preceq_\lambda$", $\lambda \in [0, 1]$, to obtain an optimum solution for nonlinear unconstrained single and multi-variable L-fuzzy optimization problems. We obtain the necessary and sufficient optimality conditions using the concept of Hukuhara differentiability of a fuzzy-valued function. Appropriate examples are provided to illustrate the results.

3.2 Single-variable L-fuzzy optimization problem

In this section, we consider a fuzzy-valued function defined on real domain whose values are L-fuzzy numbers and obtain the necessary and sufficient conditions for optimality for the same.

3.2.1 Problem definition

Let X be an open subset of $\mathbb{R}$, $F_L(\mathbb{R})$ be the set of all L-fuzzy numbers and $\tilde{f} : X \to F_L(\mathbb{R})$ be any function. We consider a following unconstrained single-variable L-fuzzy optimization problem (USFOP):

$$\begin{aligned} &Minimize\ \tilde{f}(x), \\ &Subject\ to\ x \in X \end{aligned}$$

We define the local and global optimum solution of (USFOP) using the parametric total order relation "$\preceq_\lambda$", $\lambda \in [0,1]$ is fixed. Here we recall the definition of "$\preceq_\lambda$", $\lambda \in [0,1]$.

Definition 3.2.1. *For any $\tilde{a}, \tilde{b} \in F_L(\mathbb{R})$, we say that $\tilde{a} \preceq_\lambda \tilde{b}$, where "$\preceq_\lambda$" is a parametric order relation on $F_L(\mathbb{R})$, for $0 \leq \lambda \leq 1$, if only one of the following inequalities hold:*

(i) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L < \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$ *for* $\tilde{a}_1^L - \tilde{a}_0^L < \tilde{b}_1^L - \tilde{b}_0^L$

(ii) $\lambda[\tilde{a}_1^L - \tilde{a}_0^L] + \tilde{a}_1^L \leq \lambda[\tilde{b}_1^L - \tilde{b}_0^L] + \tilde{b}_1^L$ *for* $\tilde{a}_1^L - \tilde{a}_0^L \geq \tilde{b}_1^L - \tilde{b}_0^L$

$\tilde{a} \succeq_\lambda \tilde{b}$ *is defined by* $\tilde{b} \preceq_\lambda \tilde{a}$.

Definition 3.2.2. *Let X be an open subset of $\mathbb{R}$, and $\tilde{f}$ be a fuzzy-valued function defined on X, $x^* \in X$ is said to be*

(i) *A local minimizer (maximizer) of $\tilde{f}$ if there exists a $\delta > 0$ such that $\tilde{f}(x^*) \preceq_\lambda \tilde{f}(x)$ $(\tilde{f}(x) \preceq_\lambda \tilde{f}(x^*))$ for all $x \in (x^* - \delta, x^* + \delta)$.*

(ii) *A global minimizer (maximizer) of $\tilde{f}$ if $\tilde{f}(x^*) \preceq_\lambda \tilde{f}(x)$ $(\tilde{f}(x) \preceq_\lambda \tilde{f}(x^*))$ for all $x \in X$.*

(iii) *An extremizer if it is either a minimizer or a maximizer.*

3.2.2 Necessary condition for optimality

Here we present the necessary condition for optimality for the (USFOP).

Theorem 3.2.1. *Let X be an open subset of $\mathbb{R}$ and $x^* \in X$ be a local extremizer of $\tilde{f} : X \to F_L(\mathbb{R})$. If $\tilde{f}$ is H-differentiable at x^*. Then $\lambda[D\tilde{f}_1^L(x^*) - D\tilde{f}_0^L(x^*)] + D\tilde{f}_1^L(x^*) = 0$, $D\tilde{f}_\alpha^L(x)$ is derivative of $\tilde{f}_\alpha^L(x)$ for each $\alpha \in [0,1]$.*

Proof. Since x^* a local minimum of $\tilde{f}$, by definition, there exists a $\delta > 0$ such that $(x^* - \delta, x^* + \delta) \subset X$ and $\tilde{f}(x^*) \preceq_\lambda \tilde{f}(x)$, for all $x \in (x^* - \delta, x^* + \delta)$.
By definition of total order relation "$\preceq_\lambda$", only one of the following inequalities hold for each $x \in (x^* - \delta, x^* + \delta)$.

(i) $\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) < \lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x)$

for $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) < \tilde{f}_1^L(x) - \tilde{f}_0^L(x)$,

(ii) $\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) \le \lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x)$
for $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) \ge \tilde{f}_1^L(x) - \tilde{f}_0^L(x)$.

Now, for each $x \in (x^* - \delta, x^* + \delta)$, either $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) < \tilde{f}_1^L(x) - \tilde{f}_0^L(x)$ or

$\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) \ge \tilde{f}_1^L(x) - \tilde{f}_0^L(x)$.

Therefore, for all $x \in (x^* - \delta, x^* + \delta)$,

either $\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) < \lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x)$

or $\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) \le \lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x)$.

Therefore, for all $h \in [0, \delta)$,

$\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) < \ or\ \le \ \lambda[\tilde{f}_1^L(x^* + h) - \tilde{f}_0^L(x^* + h)] + \tilde{f}_1^L(x^* + h)$,

That is, $\lambda(\tilde{f}_1^L(x^*+h) - \tilde{f}_1^L(x^*)) - \lambda(\tilde{f}_0^L(x^*+h) - \tilde{f}_0^L(x^*)) + \tilde{f}_1^L(x^*+h) - \tilde{f}_1^L(x^*) > \ or\ \ge 0$.

Since $\tilde{f}$ is H-differentiable at x^*, by Proposition 2.3.2, $\tilde{f}_\alpha^L$ is also differentiable at x^* for each $\alpha \in [0, 1]$.
Dividing above inequality by $h > 0$ and taking limit as $h \to 0^+$, we get

$$\lambda[D(\tilde{f}_1^L)(x^*) - D(\tilde{f}_0^L)(x^*)] + D(\tilde{f}_1^L)(x^*) \ge 0 \tag{3.2.1}$$

Similarly, we can prove that for all $h \in [0, \delta)$,

$\lambda(\tilde{f}_1^L(x^* - h) - \tilde{f}_1^L(x^*)) - \lambda(\tilde{f}_0^L(x^* - h) - \tilde{f}_0^L(x^*)) + \tilde{f}_1^L(x^* - h) - \tilde{f}_1^L(x^*) > \ or\ \ge 0$.

Dividing this inequality by -h and taking limit as $h \to 0^+$, we get,

$$\lambda[D\tilde{f}_1^L(x^*) - D\tilde{f}_0^L(x^*)] + D\tilde{f}_1^L(x^*) \le 0 \tag{3.2.2}$$

Therefore, by (3.2.1) and (3.2.2) the result holds.

□

3.2.3 Sufficient condition for optimality

Now we present the sufficient condition for optimality for the (USFOP).

Theorem 3.2.2. *Let X be an open subset of $\mathbb{R}$ and $\tilde{f} : X \to F_L(\mathbb{R})$ be twice continuously H-differentiable fuzzy-valued function. If x^* in X is such that*

(1) $\lambda[D\tilde{f}_1^L(x^*) - D\tilde{f}_0^L(x^*)] + D\tilde{f}_1^L(x^*) = 0$ *and*

(2) $\lambda[D^2\tilde{f}_1^L(x) - D^2\tilde{f}_0^L(x)] + D^2\tilde{f}_1^L(x) > 0$, *for all* $x \in X$.

Then x^* is a global minimizer of $\tilde{f}$.

Proof. To prove that x^* is a global minimizer of $\tilde{f}$, we must prove that $\tilde{f}(x^*) \preceq_\lambda \tilde{f}(x)$ for all $x \in X$. That is, for all $x \in X$, we have to prove that, if $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) \geq \tilde{f}_1^L(x) - \tilde{f}_0^L(x)$ then

$\lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) \leq \lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x)$ else the later inequality is strict.

Since, $\tilde{f}$ is twice continuously H-differentiable function, by Proposition 2.3.3, $\tilde{f}_\alpha^L$ is twice continuously differentiable real-valued function, for each $\alpha \in [0, 1]$.

Therefore, by Taylor's Theorem, for $x \neq x^*$ and $0 < \tau < 1$,

$$\tilde{f}_\alpha^L(x) = \tilde{f}_\alpha^L(x^*) + D\tilde{f}_\alpha^L(x^*)(x - x^*) + \frac{1}{2}D^2\tilde{f}_\alpha^L(z)(x - x^*)^2,$$

where $z = x^* + \tau(x - x^*)$ for all $\alpha \in [0, 1]$.

Therefore, by using this equation for $\alpha = 0, \alpha = 1$ and using hypothesis (1), we have,

$$\begin{aligned}\lambda[\tilde{f}_1^L(x) - \tilde{f}_0^L(x)] + \tilde{f}_1^L(x) &= \lambda[\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)] + \tilde{f}_1^L(x^*) \\ &+ \{\lambda[D^2\tilde{f}_1^L(z) - D^2\tilde{f}_0^L(z)] + D^2\tilde{f}_1^L(z)\}\frac{1}{2}(x - x^*)^2 \qquad (3.2.3)\end{aligned}$$

Now, by using hypothesis (2) from (3.2.3),

$[\lambda(\tilde{f}_1^L(x) - \tilde{f}_0^L(x)) + \tilde{f}_1^L(x)] > [\lambda(\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*)) + \tilde{f}_1^L(x^*)]$.

Therefore $\tilde{f}(x^*) \preceq_\lambda \tilde{f}(x)$ for all $x \in X$. That is, x^* is a global minimizer of $\tilde{f}$ on X.

□

3.2.4 Illustration

Example 3.2.1.

$$Minimize \quad \tilde{f}(x) = \tilde{a} \odot x \oplus \tilde{b} \odot x^2$$
$$Subject \ \ to \ \ x \in \mathbb{R}.$$

where $\tilde{a}$ *and* $\tilde{b}$ *are triangular fuzzy numbers denoted by* $\tilde{a} = (1, 2, 3)$ *and* $\tilde{b} = (0, 1, 2)$ *which are defined on* $\mathbb{R}$ *as*

$$\tilde{a}(r) = \begin{cases} (r-1), & if \ 1 \leq r \leq 2, \\ (3-r), & if \ 2 < r \leq 3, \\ 0 & otherwise \end{cases} \qquad \tilde{b}(r) = \begin{cases} r, & if \ 0 \leq r \leq 1, \\ 2-r, & if \ 1 < r \leq 2, \\ 0 & otherwise \end{cases}$$

The α*-level set of* $\tilde{f}(x)$ *is then* $\tilde{f}_\alpha(x) = [(1+\alpha)x + \alpha x^2, (3-\alpha)x + (2-\alpha)x^2]$ *and*

the α*-level set of* $D\tilde{f}(x)$ *is* $D\tilde{f}_\alpha(x) = [(1+\alpha) + 2\alpha x, (3-\alpha) + 2(2-\alpha)x]$.

By applying Necessary condition: $\lambda[D\tilde{f}_1^L(x^*) - D\tilde{f}_0^L(x^*)] + D\tilde{f}_1^L(x^*) = 0$,

we get $x^* = -(\lambda+2)/2(\lambda+1)$.

Now by verifying Sufficient condition:

$$\lambda[D^2\tilde{f}_1^L(x) - D^2\tilde{f}_0^L(x)] + D^2\tilde{f}_1^L(x) = 2\lambda + 2 > 0$$

where $D^2\tilde{f}_\alpha(x) = [2\alpha, 2(2-\alpha)]$ *and* $\lambda \in [0, 1]$.

Therefore, we say that $x^* = -(\lambda+2)/2(\lambda+1)$ *is global minimum of the given fuzzy-*

valued function for a fixed value of $\lambda \in [0,1]$. *The following table shows that if we take different values* λ *from 0 to 1, we get different minimum point for a given function. Here this parameter* λ *works like a weight, we can adjust its value according to the practical situation arise in the problem.*

λ	x^*	$\tilde{f}(x^*)$
0	-1	$(-3, -1, 1)$
0.2	-0.9167	$(-2.7500, -0.9931, 0.7639)$
0.6	-0.8125	$(-2.4375, -0.9648, 0.5078)$
1.0	-0.75	$(-2.3824, -0.9576, 0.4671)$

Now we give one more example as a case study to illustrate the results.

Example 3.2.2. *[55] The horsepower generated by a pelton wheel is proportional to* $u(V - u)$, *where* u *is the velocity of the wheel, which is a variable, and* V *is the velocity of the jet, which is fixed. It is desired to find the velocity of the pelton wheel at which its efficiency is maximum.*

Here, if we take the velocity V *of the jet as a fixed real constant. Then the problem is modeled into a crisp optimization problem. However, the velocity of the jet may be about* V *units and more realistic model would treat this velocity as fuzzy number, say* $\tilde{V}$.

To be more precise, we take $\tilde{V}$ *as a triangular fuzzy number* $(V - 1, V, V + 1)$. *Then, new fuzzy optimization problem is to maximize* $\tilde{f}(u) = u \odot \tilde{V} - u^2$ *where* $\tilde{f}$ *is a function from* $\mathbb{R}$ *to* $F_L(\mathbb{R})$.

By applying the necessary condition: $\lambda[D\tilde{f}_1^L(u) - D\tilde{f}_0^L(u)] + D\tilde{f}_1^L(u) = 0$, *where* $\tilde{f}_\alpha^L(u) = u(V - 1 + \alpha) - u^2$ *and* $D\tilde{f}_\alpha^L(u) = (V - 1 + \alpha) - 2u$, *for* $\alpha \in [0,1]$. *We get parametric a stationary value* $u = \frac{\lambda+V}{2}$.

Now by verifying the sufficient condition: we find $\lambda[D^2\tilde{f}_1^L(u) - D^2\tilde{f}_0^L(u)] + D^2\tilde{f}_1^L(u) = -2 < 0$, *therefore,* $u = \frac{\lambda+V}{2}$ *is a parametric global maximizer of* $\tilde{f}$ *with respect to the total-order "*$\preceq_\lambda$*" on* $F_L(\mathbb{R})$.

3.3 Multi-variable L-fuzzy optimization problem

This section considers an unconstrained multi-variable L-fuzzy optimization problem. We define an optimum solution for the problem using the parametric total order relation "$\preceq_\lambda$", for $\lambda \in [0,1]$. We establish the first and second order necessary and sufficient optimality conditions for a L-fuzzy optimization problem. We use the Hukuhara differentiability of a fuzzy-valued function to prove the results. We provide suitable examples to illustrate the results.

3.3.1 Problem definition

Here we consider an unconstrained multi-variable L-fuzzy optimization problem (UMFOP).

$$\begin{gathered} Minimize\ \tilde{f}(\bar{x}), \\ Subject\ to\ \bar{x} \in X \end{gathered}$$

where $X \subseteq \mathbb{R}^n$ is an open set and $\tilde{f} : X \to F_L(\mathbb{R})$ is a fuzzy-valued function. We define a local optimum solution of the (UMFOP) using the parametric total order relation " $\preceq_\lambda$ ", $\lambda \in [0,1]$.

Definition 3.3.1. *Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ and $\lambda \in [0,1]$ be fixed. Then*

(1) *a point $\bar{x}^* \in X$ is called a local minimum (maximum) of $\tilde{f}$ with respect to parametric total order relation "$\preceq_\lambda$" , if there exists $r > 0$ such that $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ ($\tilde{f}(\bar{x}^*) \succeq_\lambda \tilde{f}(\bar{x})$), for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.*

(2) *a point $\bar{x}^* \in X$ is called a strict local minimum (maximum) of $\tilde{f}$ with respect to parametric total order relation "$\preceq_\lambda$", if there exists $r > 0$ such that $\tilde{f}(\bar{x}^*) \prec_\lambda \tilde{f}(\bar{x})$ ($\tilde{f}(\bar{x}^*) \succ_\lambda \tilde{f}(\bar{x})$), for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.*

3.3.2 First-order condition

Now we present first-order necessary condition (FONC) for optimality of (UMFOP).

Theorem 3.3.1. ***(FONC)*** *Suppose $\bar{x}^* \in intX = \{\bar{x} \in X\ /\ there\ exists\ r > 0\ such\ that\ B(\bar{x}^*; r) \subset X\} \subseteq \mathbb{R}^n$ be a local minimizer of $\tilde{f} : X \to F_L(\mathbb{R})$ with respect to parametric total order relation "$\preceq_\lambda$". Suppose also that $\tilde{f}$ is H-differentiable at $\bar{x}^*$. Then $\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0$.*

Proof. Since $\bar{x}^* \in intX$ a local minimum of $\tilde{f}$ on X, by definition, we have $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.

That is, only one of the following inequalities hold:

(i) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) < \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$

(ii) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) \geq \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$.

Therefore, for any $\bar{x} \in X \cap B(\bar{x}^*; r)$ we have either

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}),$$

or

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}).$$

Let $\bar{e}_i = [0, ..., 1, ...0]^T$ be a unit vector 1 in the i^{th} location. Then $(\bar{x}^* + h\bar{e}_i)$ with $h > 0$ will represent a perturbation of magnitude h in $\bar{x}^*$ in the direction $\bar{e}_i$.

Let $\bar{x} = \bar{x}^* + h\bar{e}_i$, where $h < r$, then we have

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \ or \ \leq \ \lambda[\tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_0^L(\bar{x}^* + h)] + \tilde{f}_1^L(\bar{x}^* + h)$$

Similarly, for $\bar{x} = \bar{x}^* - h\bar{e}_i$, we have

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \ or \ \leq \ \lambda[\tilde{f}_1^L(\bar{x}^* - h) - \tilde{f}_0^L(\bar{x}^* - h)] + \tilde{f}_1^L(\bar{x}^* - h)$$

for sufficiently small h. That is,

$$\lambda\Big(\tilde{f}_1^L(\bar{x}^*+h)-\tilde{f}_1^L(\bar{x}^*)\Big)-\lambda\Big(\tilde{f}_0^L(\bar{x}^*+h)-\tilde{f}_0^L(\bar{x}^*)\Big)+\tilde{f}_1^L(\bar{x}^*+h)-\tilde{f}_1^L(\bar{x}^*) > \ or \ \geq 0, \quad (3.3.1)$$

$$\lambda\Big(\tilde{f}_1^L(\bar{x}^*-h)-\tilde{f}_1^L(\bar{x}^*)\Big)-\lambda\Big(\tilde{f}_0^L(\bar{x}^*-h)-\tilde{f}_0^L(\bar{x}^*)\Big)+\tilde{f}_1^L(\bar{x}^*-h)-\tilde{f}_1^L(\bar{x}^*) > \ or \ \geq 0. \quad (3.3.2)$$

Since $\tilde{f}$ is H-differentiable at $\bar{x}^*$, by Proposition 2.3.2, $\tilde{f}_\alpha^L$ is also differentiable at $\bar{x}^*$ for all $\alpha \in [0, 1]$. Dividing the inequalities (3.3.1) and (3.3.2) by h and $-h$ respectively and taking limit as $h \to 0$, we get

$$\lambda[D_i\tilde{f}_1^L(\bar{x}^*) - D_i\tilde{f}_0^L(\bar{x}^*)] + D_i\tilde{f}_1^L(\bar{x}^*) \geq 0,$$

for all $i = 1, ..., n$. Therefore,

$$\lambda[\nabla\tilde{f}_1^L(\bar{x}^*) - \nabla\tilde{f}_0^L(\bar{x}^*)] + \nabla\tilde{f}_1^L(\bar{x}^*) \geq 0$$

$$\lambda[\nabla\tilde{f}_1^L(\bar{x}^*) - \nabla\tilde{f}_0^L(\bar{x}^*)] + \nabla\tilde{f}_1^L(\bar{x}^*) \leq 0$$

which gives

$$\lambda[\nabla\tilde{f}_1^L(\bar{x}^*) - \nabla\tilde{f}_0^L(\bar{x}^*)] + \nabla\tilde{f}_1^L(\bar{x}^*) = 0.$$

□

3.3.3 Second-order conditions

We start by discussing the second-order necessary conditions (SONC) for optimality of a fuzzy-valued function defined on $\mathbb{R}^n$.

Theorem 3.3.2. ***(SONC)*** *Suppose $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ be a continuously H-differentiable fuzzy-valued function, and $\bar{x}^*$ in X is a point in the interior of X.*

(i) *If $\tilde{f}$ has a local minimum at $\bar{x}^*$, then $\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*)$ is positive semidefinite.*

(ii) *If $\tilde{f}$ has a local maximum at $\bar{x}^*$, then $\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*)$ is negative semidefinite.*

We adopt a two step procedure to prove this theorem. We first prove this result for the case where n = 1 i.e., ($X \subseteq \mathbb{R}$) and then we use this result to prove the general case.

Proof. Case 1: n = 1
When $n = 1$, $\tilde{f} : X \subseteq \mathbb{R} \to F_L(\mathbb{R})$ and $\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*)$ is real number. We have to prove that

$$\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*) \geq 0.$$

Since $\tilde{f}$ has a local minimum at $\bar{x}^*$, by definition, we have $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and $r > 0$. That is, only one of the following inequalities hold:

(i) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) < \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$

(ii) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) \geq \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and $0 \leq \lambda \leq 1$.

Consider Taylor's series expansion of $\tilde{f}_\alpha^L$ at $\bar{x}^*$ for sufficiently small h such that $\bar{x}^* + h \in B(\bar{x}^*; r)$ and

$$\tilde{f}_\alpha^L(\bar{x}^* + h) = \tilde{f}_\alpha^L(\bar{x}^*) + hD\tilde{f}_\alpha^L(\bar{x}^*) + \frac{1}{2}h^2 D^2 \tilde{f}_\alpha^L(\bar{x}^*) + O(h^3)$$

Using this,

$$\begin{aligned}
0 \leq \Big\{\lambda[\tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_0^L(\bar{x}^* + h)] + \tilde{f}_1^L(\bar{x}^* + h)\Big\} \quad = \quad & \Big\{\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)\Big\} + \\
& +h\Big\{\lambda[D\tilde{f}_1^L(\bar{x}^*) - D\tilde{f}_0^L(\bar{x}^*)] + \\
& D\tilde{f}_1^L(\bar{x}^*)\Big\} + \\
& \frac{h^2}{2}\Big\{\lambda[D^2\tilde{f}_1^L(\bar{x}^*) - D^2\tilde{f}_0^L(\bar{x}^*)] + \\
& D^2\tilde{f}_1^L(\bar{x}^*)\Big\} \\
& +O(h^3)
\end{aligned}$$

At local minimum, $\lambda[D\tilde{f}_1^L(\bar{x}^*) - D\tilde{f}_0^L(\bar{x}^*)] + D\tilde{f}_1^L(\bar{x}^*) = 0$. Thus ,

$$\begin{aligned}
\Big\{\lambda[\tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_0^L(\bar{x}^* + h)] + \tilde{f}_1^L(\bar{x}^* + h)\Big\} \quad - \quad & \Big\{\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)\Big\} \\
= \quad & \frac{h^2}{2}\Big\{\lambda[D^2\tilde{f}_1^L(\bar{x}^*) - D^2\tilde{f}_0^L(\bar{x}^*)] + \\
& D^2\tilde{f}_1^L(\bar{x}^*)\Big\} \\
& +O(h^3)
\end{aligned}$$

Upon choosing h sufficiently small, we can ensure that the term

$$\frac{h^2}{2}\Big\{\lambda[D^2\tilde{f}_1^L(\bar{x}^*) - D^2\tilde{f}_0^L(\bar{x}^*)] + D^2\tilde{f}_1^L(\bar{x}^*)\Big\}$$

dominates the remainder term $O(h^3)$. Thus at a local minimum, we have

$$\lambda[D^2\tilde{f}_1^L(\bar{x}^*) - D^2\tilde{f}_0^L(\bar{x}^*)] + D^2\tilde{f}_1^L(\bar{x}^*) \geq 0.$$

Case 2 : $n > 1$

We prove part 1. Let $\bar{x}^*$ be a local minimum of $\tilde{f}$ on X. We have to show that for any $z \in \mathbb{R}^n$, $z \neq 0$, we have $z^{'}Az \geq 0$, where

$$A = \lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*).$$

Pick any $z \in \mathbb{R}^n$, define the fuzzy-valued function $\tilde{g} : \mathbb{R} \to F(\mathbb{R})$ by $\tilde{g}(t) = \tilde{f}(\bar{x}^* + tz)$.

Note that $\tilde{g}(0) = \tilde{f}(\bar{x}^*)$. For $|t|$ sufficiently small, $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x}^* + tz)$, since $\tilde{f}(\bar{x})$ has a local minimum at $\bar{x}^*$.

It follows that there exists a $\epsilon > 0$ such that $\tilde{g}(0) \preceq_\lambda \tilde{g}(t)$ for all $t \in (-\epsilon, \epsilon)$. That is, 0 is a local minimum of $\tilde{g}$.

By case 1, therefore we must have

$$\lambda[D^2\tilde{g}_1^L(0) - D^2\tilde{g}_0^L(0)] + D^2\tilde{g}_1^L(0) \geq 0.$$

On the other hand, it follows from the definition of $\tilde{g}$, that $\tilde{g}$ is twice continuously H-differentiable, as $\tilde{g}(t) = \tilde{f}(x^* + tz)$. Therefore, $\tilde{g}_\alpha^L(t) = \tilde{f}_\alpha^L(x^* + tz)$ and $D^2\tilde{g}_\alpha^L(t) = z^{'}\nabla^2\tilde{f}_\alpha^L(x^* + tz)z$. That is, $D^2\tilde{g}_\alpha^L(0) = z^{'}\nabla^2\tilde{f}_\alpha^L(x^*)z$. Therefore,

$$\lambda[D^2\tilde{g}_1^L(0) - D^2\tilde{g}_0^L(0)] + D^2\tilde{g}_1^L(0) = z^{'}Az$$

where $A = \lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*) - \nabla^2\tilde{f}_0^L(\bar{x}^*)] + \nabla^2\tilde{f}_1^L(\bar{x}^*)$, so that

$$z^{'}Az = \lambda[D^2\tilde{g}_1^L(0) - D^2\tilde{g}_0^L(0)] + D^2\tilde{g}_1^L(0) \geq 0,$$

as desired. This completes the proof of Part 1. Part 2 is proved similarly. □

Now we prove the second-order sufficient conditions (SOSC) for $\bar{x}^*$ to be a strict local minimizer (maximizer) of (UMFOP). To prove the theorem, we need the following result from calculus called, **Rayleigh's Inequality** (Ref. [12], pp. 34).

Theorem 3.3.3. *If an $n \times n$ matrix P is real symmetric positive definite, then*

$$\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2,$$

where $\lambda_{\min}(P)$ denotes the smallest eigenvalue of P, and $\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

Theorem 3.3.4. ***(SOSC)*** *Suppose $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ is a twice continuously H-differentiable function.*

1. *If $\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0$ and $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is positive definite, then $\bar{x}^*$ is a strict local minimum of $\tilde{f}$ on X.*
2. *If $\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0$ and $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is negative definite, then $\bar{x}^*$ is a strict local maximum of $\tilde{f}$ on X.*

Proof. We prove Part 1, Part 2 is proved similarly. Here we have to prove that $\bar{x}^*$ is a strict local minimum of $\tilde{f}$ on X. By definition, we have to show

$$\tilde{f}(\bar{x}^*) \prec_\lambda \tilde{f}(\bar{x})$$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and for fixed $\lambda \in [0, 1]$. That is,

$$\tilde{f}(\bar{x}) \not\preceq_\lambda \tilde{f}(\bar{x}^*)$$

That means, we have to show that the following inequalities fail simultaneously.

(i) $\lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}) < \lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)$
for $\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x}) < \tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)$

(ii) $\lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}) \leq \lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)$
for $\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x}) \geq \tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and for fixed $\lambda \in [0, 1]$.

Let $\tilde{H}^L(\bar{x}^*) = \lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$. Using assumption 2, and Rayleigh's inequality (Theorem 3.3.3 of this Section), it follows that if $\bar{d} \neq 0$ then

$$\lambda_{\min}(\tilde{H}^L(\bar{x}^*))\|\bar{d}\|^2 \leq \bar{d}^T \tilde{H}^L(\bar{x}^*)\bar{d}$$

where $\lambda_{\min}(\tilde{H}^L(\bar{x}^*))$ is the smallest eigen value of $\tilde{H}^L(\bar{x}^*)$. By Taylor's theorem and assumption 1,

$$\begin{aligned}\left\{\lambda[\tilde{f}_1^L(\bar{x}^*+d)-\tilde{f}_0^L(\bar{x}^*+d)]+\tilde{f}_1^L(\bar{x}^*+d)\right\} &- \left\{\lambda[\tilde{f}_1^L(\bar{x}^*)-\tilde{f}_0^L(\bar{x}^*)]+\tilde{f}_1^L(\bar{x}^*)\right\}\\ &= \frac{1}{2}\bar{d}^T\tilde{H}^L(\bar{x}^*)\bar{d}+O(\|\bar{d}\|^2)\\ &\geq \frac{\lambda_{\min}(\tilde{H}^L(\bar{x}^*))}{2}\|\bar{d}\|^2+O(\|\bar{d}\|^2)\end{aligned}$$

Hence for all $\bar{d}$ such that $\|\bar{d}\|$ is sufficiently small,

$$\{\lambda[\tilde{f}_1^L(\bar{x}^*+d)-\tilde{f}_0^L(\bar{x}^*+d)]+\tilde{f}_1^L(\bar{x}^*+d)\} > \{\lambda[\tilde{f}_1^L(\bar{x}^*)-\tilde{f}_0^L(\bar{x}^*)]+\tilde{f}_1^L(\bar{x}^*)\}.$$

This inequality fails the above two inequalities (i) and (ii) simultaneously. Therefore we say that $\tilde{f}$ has a strict local minimizer at $\bar{x}^*$ with respect to the total order relation "$\preceq_\lambda$". $\square$

3.3.4 Illustrations

Example 3.3.1.

$$Minimize\ \ \tilde{f}(x_1,x_2)=(\tilde{1}\odot x_1^2)\oplus(\widetilde{0.5}\odot x_2^2)\oplus(\tilde{3}\odot x_2)\oplus\widetilde{4.5},\quad x_1,\ x_2\in\mathbb{R}$$

where $\tilde{1}=(0,1,2)$, $\widetilde{0.5}=(0.4,0.5,0.6)$, $\tilde{3}=(2,3,4)$ *and* $\widetilde{4.5}=(3.5,4.5,5.5)$ *are triangular fuzzy numbers defined on* $\mathbb{R}$ *and with respect to the total order relation "*$\preceq_\lambda$*" for some fixed* $\lambda\in[0,1]$.

Here $\tilde{f}_\alpha^L(x_1,x_2)=\alpha x_1^2+(0.4+\alpha 0.1)x_2^2+(2+\alpha)x_2+(3.5+\alpha)$,

$$\nabla\tilde{f}_\alpha^L=\begin{pmatrix}2\alpha x_1\\ 2(0.4+0.1\alpha)x_2+(2+\alpha)\end{pmatrix}.$$

By first order necessary condition $\lambda[\nabla\tilde{f}_1^L(\bar{x})-\nabla\tilde{f}_0^L(\bar{x})]+\nabla\tilde{f}_1^L(\bar{x})=0$.

That is

$$\begin{aligned}\lambda 2x_1+2x_1&=0\\ \lambda(0.2x_2+1)+x_2+3&=0\end{aligned}$$

Solving these equations, we get parametric solution

$$\bar{x}^* = \Big(0, -\frac{(\lambda+3)}{0.2\lambda+1}\Big)$$

Now

$$\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}) - \nabla^2 \tilde{f}_0^L(\bar{x})] + \nabla^2 \tilde{f}_1^L(\bar{x}) = \begin{pmatrix} 2\lambda+2 & 0 \\ 0 & 0.2\lambda+1 \end{pmatrix}$$

where

$$\nabla^2 \tilde{f}_\alpha^L(\bar{x}) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2(0.4+0.1\alpha) \end{pmatrix}.$$

Since this matrix is positive definite for each $\lambda \in [0,1]$*, the point* $\bar{x}^* = \Big(0, -\frac{(\lambda+3)}{0.2\lambda+1}\Big)$ *satisfies the* ***SOSC*** *(Note that* ***FONC*** *and* ***SONC*** *are also satisfied). So it is a strict local minimizer of given fuzzy-valued function.*

Example 3.3.2.

$$\textit{Minimize} \quad \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus ((\widetilde{-1}) \odot x_2^2), \quad x_1, x_2 \in \mathbb{R}$$

where $\tilde{1} = (0,1,2)$ *and* $(\widetilde{-1}) = (-2,-1,0)$ *are triangular fuzzy numbers defined on* $\mathbb{R}$ *and with respect to total order relation "*$\preceq_\lambda$*" for some fixed* $\lambda \in [0,1]$.

Here $\tilde{f}_\alpha^L(x_1, x_2) = \alpha x_1^2 + (-2+\alpha)x_2^2$,

$$\nabla \tilde{f}_\alpha^L = \begin{pmatrix} 2\alpha x_1 \\ 2(-2+\alpha)x_2 \end{pmatrix}.$$

By first order necessary condition $\lambda[\nabla \tilde{f}_1^L(\bar{x}) - \nabla \tilde{f}_0^L(\bar{x})] + \nabla \tilde{f}_1^L(\bar{x}) = 0.$

That is

$$\lambda 2x_1 + 2x_1 = 0$$
$$\lambda 2x_2 - 2x_2 = 0.$$

Solving these equations, we get the solution

$$\bar{x}^* = (0,0).$$

We evaluate

$$\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}) - \nabla^2 \tilde{f}_0^L(\bar{x})] + \nabla^2 \tilde{f}_1^L(\bar{x}) = \begin{pmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda - 2 \end{pmatrix}$$

where

$$\nabla^2 \tilde{f}_\alpha^L(\bar{x}) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2(-2+\alpha) \end{pmatrix}.$$

The point $\bar{x}^* = (0,0)$ *satisfies the* ***FONC*** *but* ***SONC*** *is not satisfied, since* $2\lambda+2 > 0$ *but* $(2\lambda+2)(2\lambda-2) \leq 0$*, for all* $\lambda \in [0,1]$*. Moreover,* ***SOSC*** *is not satisfied as* $(2\lambda+2)(2\lambda-2) \leq 0$*, for all* $\lambda \in [0,1]$*. Therefore,* $\bar{x}^* = (0,0)$ *is not optimum point for given fuzzy-valued function, if* $\lambda \in [0,1)$*.*

3.4 Conclusions

In this chapter, we have developed the first and second order, necessary and sufficient optimality conditions for unconstrained fuzzy optimization problems. We have defined a parametric total order relation "$\preceq_\lambda$" for $\lambda \in [0,1]$, on set of fuzzy numbers having particular L-shape membership function. Though this order relation is restricted to a sub-set of fuzzy numbers and also depends upon the parameter value λ, it is still useful for comparing fuzzy numbers in a natural way which is not possible for the set of all fuzzy numbers. Further, the value of λ can be altered according to the particular situation arising in the modelling of a problem and thus it gives flexibility in comparison of L-fuzzy numbers. It is interesting to note that if instead of fuzzy-valued functions we restrict our attention to real-valued functions, then the total order relation "$\preceq_\lambda$" for $\lambda \in [0,1]$, reduces to the usual total order relation "$\preceq$" on $\mathbb{R}$ and the optimality conditions reduce to the usual optimality conditions for real-valued functions defined on $\mathbb{R}^n$.

Chapter 4

Fuzzy optimization problems

Contents

4.1 Introduction

The non-dominated solution of a nonlinear constrained fuzzy optimization problem having real constraints has been proposed by Wu [76], who also derived the sufficient optimality conditions for it. We generalize the results of Wu for fuzzy optimization problems having fuzzy-valued constraints.

Before considering constrained fuzzy optimization problems, we establish the first and second order necessary and sufficient optimality conditions for unconstrained fuzzy optimization problems under the partial order relation on the set of fuzzy numbers. We give appropriate illustrations to explain the proposed results, for both unconstrained and constrained fuzzy optimization problems.

4.2 Pre-requisites

In order to define the Kuhn-Tucker like optimality conditions for nonlinear fuzzy optimization problems, we need to provide some properties of fuzzy-valued functions. We start by stating the following two Propositions from Real Analysis.

Proposition 4.2.1. *[57] Let ϕ be a real-valued function of two variables defined on $I \times [a, b]$, where I is an interval in $\mathbb{R}$. Suppose that the following conditions are satisfied:*

(i) *For every $x \in I$, the real-valued function $h(y) = \phi(x, y)$ is Riemann integrable on $[a, b]$. In this case, we write $f(x) = \displaystyle\int_a^b \phi(x, y)\, dy$;*

(ii) *Let $x^0 \in int(I)$, the interior of I then for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\left| \frac{\partial \phi}{\partial x}(x, y) - \frac{\partial \phi}{\partial x}(x^0, y) \right| < \epsilon$$

for all $y \in [a, b]$ and all $x \in (x^0 - \delta, x^0 + \delta)$.

Then $\frac{\partial \phi}{\partial x}(x^0, y)$ is Riemann integrable on $[a, b]$, $f'(x^0)$ exists, and

$$f'(x^0) = \int_a^b \frac{\partial \phi}{\partial x}(x^0, y)\, dy.$$

Proposition 4.2.2. *[6] Every function monotonic on an interval is Riemann integrable there.*

Let $\tilde{f} : X \to F(\mathbb{R})$ be a fuzzy-valued function defined on subset X of $\mathbb{R}^n$. Then for each $\alpha \in [0, 1]$, $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are real-valued functions defined on X. For any fixed $\bar{x}^0 \in X$, we have the corresponding real-valued functions $\tilde{f}_\alpha^L(\bar{x}^0)$ and $\tilde{f}_\alpha^U(\bar{x}^0)$ defined on $[0, 1]$ can be regarded as functions of variable $\alpha \in [0, 1]$. By Proposition 2.2.1, these functions are monotonic on $[0, 1]$ and hence, are Riemann integrable by Proposition 4.2.2. So, we define new functions F^L and F^U as follows

$$F^L(\bar{x}) = \int_0^1 \tilde{f}^L_\alpha(\bar{x})d\alpha \ \ and \ \ F^U(\bar{x}) = \int_0^1 \tilde{f}^U_\alpha(\bar{x})d\alpha \tag{4.2.1}$$

for every $\bar{x} \in X$. Then we have following useful proposition.

Proposition 4.2.3. *[76] Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$. If $\tilde{f}$ is continuously H-differentiable on some neighborhood of $\bar{x}^0 = (\bar{x}^0_1, \bar{x}^0_2, ..., \bar{x}^0_n)$. Then the real-valued functions F^L and F^U defined in (4.2.1) are continuously differentiable at $\bar{x}^0$ and*

$$\frac{\partial F^L}{\partial x_i}(\bar{x}^0) = \int_0^1 \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(\bar{x}^0)\, d\alpha \ \ and \ \ \frac{\partial F^U}{\partial x_i}(\bar{x}^0) = \int_0^1 \frac{\partial \tilde{f}^U_\alpha}{\partial x_i}(\bar{x}^0)\, d\alpha$$

for all $i = 1, .., n$.

Proof. We need to show that the partial derivatives $\frac{\partial F^L}{\partial x_i}$ and $\frac{\partial F^U}{\partial x_i}$ exist on some neighborhood of $\bar{x}^0$ and are continuous at $\bar{x}^0$, for all $i = 1, .., n$. From (4.2.1), we say that condition (1) in Proposition 4.2.1 is satisfied.

Since $\tilde{f}$ is continuously H-differentiable on some neighborhood of $\bar{x}^0$. By definition, all the partial H-derivatives $\frac{\partial \tilde{f}}{\partial x_i}$, $i = 1, ..., n$ exists on some neighborhood of $\bar{x}^0$, and are continuous at $\bar{x}^0$. Therefore, by definition of continuity of a fuzzy-valued function, we have

for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\bar{x} - \bar{x}^0\| < \delta$ implies

$$d_F\Big(\frac{\partial \tilde{f}(\bar{x})}{\partial x_i}, \frac{\partial \tilde{f}(\bar{x}^0)}{\partial x_i}\Big) < \epsilon, \tag{4.2.2}$$

for each $i = 1, ..., n$. Now by applying the definition of metric d_F,

$$\begin{aligned} d_F\Big(\frac{\partial \tilde{f}(\bar{x})}{\partial x_i}, \frac{\partial \tilde{f}(\bar{x}^0)}{\partial x_i}\Big) &= \sup_{\alpha\in[0,1]} \Big\{ d_H\Big(\Big(\frac{\partial \tilde{f}(\bar{x})}{\partial x_i}\Big)_\alpha, \Big(\frac{\partial \tilde{f}\big(\bar{x}^0\big)}{\partial x_i}\Big)_\alpha\Big)\Big\} \\ &= \sup_{\alpha\in[0,1]} \Big\{ \max\Big\{ \Big|\Big(\frac{\partial \tilde{f}(\bar{x})}{\partial x_i}\Big)^L_\alpha - \Big(\frac{\partial \tilde{f}(\bar{x}^0)}{\partial x_i}\Big)^L_\alpha\Big|, \\ &\quad \Big|\Big(\frac{\partial \tilde{f}(\bar{x})}{\partial x_i}\Big)^U_\alpha - \Big(\frac{\partial \tilde{f}(\bar{x}^0)}{\partial x_i}\Big)^U_\alpha\Big|\Big\}\Big\} \end{aligned}$$

Therefore, by (4.2.2), we have

$$\Big|\Big(\frac{\partial\tilde{f}(\bar{x})}{\partial x_i}\Big)^L_\alpha - \Big(\frac{\partial\tilde{f}(\bar{x}^0)}{\partial x_i}\Big)^L_\alpha\Big| < \epsilon$$

and

$$\Big|\Big(\frac{\partial\tilde{f}(\bar{x})}{\partial x_i}\Big)^U_\alpha - \Big(\frac{\partial\tilde{f}(\bar{x}^0)}{\partial x_i}\Big)^U_\alpha\Big| < \epsilon,$$

for all $\alpha \in [0,1]$ and $i = 1, ..., n$. Since

$$\Big(\frac{\partial\tilde{f}(\bar{x})}{\partial x_i}\Big)^L_\alpha = \frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}) \textit{ and } \Big(\frac{\partial\tilde{f}(\bar{x})}{\partial x_i}\Big)^U_\alpha = \frac{\partial\tilde{f}^U_\alpha}{\partial x_i}(\bar{x}),$$

for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\bar{x} - \bar{x}^0\| < \delta \;\; \textit{implies} \;\; \left|\frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}) - \frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}^0)\right| < \epsilon,$$

for all $\alpha \in [0,1]$ and $i = 1, ..., n$. That is, condition (2) of Proposition 4.2.1 is also satisfied. Therefore, by Proposition 4.2.1

$$\frac{\partial F^L}{\partial x_i}(\bar{x}^0) = \int_0^1 \frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}^0)\, d\alpha \;\; \textit{and} \;\; \frac{\partial F^U}{\partial x_i}(\bar{x}^0) = \int_0^1 \frac{\partial\tilde{f}^U_\alpha}{\partial x_i}(\bar{x}^0)\, d\alpha \tag{4.2.3}$$

for all $i = 1, .., n$.

So if $\|\bar{x} - \bar{x}^0\| < \delta$ then using (4.2.3), we have

$$\begin{aligned} \left|\frac{\partial F^L}{\partial x_i}(\bar{x}) - \frac{\partial F^L}{\partial x_i}(\bar{x}^0)\right| &= \left|\int_0^1 \Big[\frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}) - \frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}^0)\Big]\, d\alpha\right| \\ &\leq \int_0^1 \left|\frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}) - \frac{\partial\tilde{f}^L_\alpha}{\partial x_i}(\bar{x}^0)\right| d\alpha < \epsilon, \end{aligned}$$

for all $i = 1, ..., n$. Therefore, $\frac{\partial F^L}{\partial x_i}$ is continuous, for all $i = 1, ..., n$. Similarly, we can prove continuity of $\frac{\partial F^U}{\partial x_i}$. Hence complete the proof. □

Notations:

(1) We write gradient of $\tilde{f} : X \subset \mathbb{R}^n \to F(\mathbb{R})$ using its α-level sets as follows:

$$\nabla\tilde{f}^L_\alpha(\bar{x}) = \Big(\frac{\partial\tilde{f}^L_\alpha}{\partial x_1}(\bar{x}), ..., \frac{\partial\tilde{f}^L_\alpha}{\partial x_n}(\bar{x})\Big)^t$$

and

$$\nabla \tilde{f}_\alpha^U(\bar{x}) = \Big(\frac{\partial \tilde{f}_\alpha^U}{\partial x_1}(\bar{x}), ..., \frac{\partial \tilde{f}_\alpha^U}{\partial x_n}(\bar{x})\Big)^t.$$

Moreover,

$$\int_0^1 \nabla \tilde{f}_\alpha^L(\bar{x})d\alpha = \Big(\int_0^1 \frac{\partial \tilde{f}_\alpha^L}{\partial x_1}(\bar{x})d\alpha, ..., \int_0^1 \frac{\partial \tilde{f}_\alpha^L}{\partial x_n}(\bar{x})d\alpha\Big)^t$$

and

$$\int_0^1 \nabla \tilde{f}_\alpha^U(\bar{x})d\alpha = \Big(\int_0^1 \frac{\partial \tilde{f}_\alpha^U}{\partial x_1}(\bar{x})d\alpha, ..., \int_0^1 \frac{\partial \tilde{f}_\alpha^U}{\partial x_n}(\bar{x})d\alpha\Big)^t.$$

(2) The fuzzy matrix of second order partial derivatives of $\tilde{f}$ is given as follows:

$$\nabla^2 \tilde{f}(\bar{x}^0) = \begin{pmatrix} \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_1^2} & ... & \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_1 \partial x_n} \\ ... & ... & ... \\ \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_n^2} \end{pmatrix}$$

where $\frac{\partial^2 \tilde{f}(\bar{x}^0)}{\partial x_i \partial x_j} \in F(\mathbb{R})$, $i, j = 1, ..., n$. Then α-level matrices of above matrix are

$$\nabla^2 \tilde{f}_\alpha^L(\bar{x}^0) = \begin{pmatrix} \frac{\partial^2 \tilde{f}_\alpha^L(\bar{x}^0)}{\partial x_1^2} & ... & \frac{\partial^2 \tilde{f}_\alpha^L(\bar{x}^0)}{\partial x_1 \partial x_n} \\ ... & ... & ... \\ \frac{\partial^2 \tilde{f}_\alpha^L(\bar{x}^0)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 \tilde{f}_\alpha^L(\bar{x}^0)}{\partial x_n^2} \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}_\alpha^U(\bar{x}^0) = \begin{pmatrix} \frac{\partial^2 \tilde{f}_\alpha^U(\bar{x}^0)}{\partial x_1^2} & ... & \frac{\partial^2 \tilde{f}_\alpha^U(\bar{x}^0)}{\partial x_1 \partial x_n} \\ ... & ... & ... \\ \frac{\partial^2 \tilde{f}_\alpha^U(\bar{x}^0)}{\partial x_n \partial x_1} & ... & \frac{\partial^2 \tilde{f}_\alpha^U(\bar{x}^0)}{\partial x_n^2} \end{pmatrix}$$

We would like to discuss one important point here as a remark.

Remark 4.2.1. *Optimization problems mainly depend upon order relations. In crisp optimization problems, we deal with real numbers which are linearly ordered and there is no problem regarding order relation. But while dealing with fuzzy optimization problems, the main issue is the definition of a order relation on the set of fuzzy numbers. There are many different ways to define order relations on fuzzy numbers. Some of them are partial orders and some are total order relations. In the current section, we consider the fuzzy optimization problem with partial order relation of so called "fuzzy max-order", which has been defined in Chapter 2. In this order relation, we have defined the strict inequality between two fuzzy numbers in two different ways. For unconstrained and constrained fuzzy optimization problems, we use different strict inequalities.*

4.3 Unconstrained fuzzy optimization problem

Here we consider the unconstrained fuzzy optimization problem and prove the first and second order necessary and sufficient optimality conditions for the same.

First we formulate the problem definition and its solution.

Let $T \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f}$ be fuzzy-valued function defined on T. Consider the following nonlinear unconstrained fuzzy optimization problem (FOP1).

$$Minimize \;\; \tilde{f}(x) = \tilde{f}(x_1, .., x_n)$$
$$Subject \;\; to \;\; \bar{x} \in T$$

We recall the definition of partial order relation on fuzzy numbers from Preliminaries.

Definition 4.3.1. *For* $\tilde{a}$, $\tilde{b}$ *in* $F(\mathbb{R})$, *we say that* $\tilde{a} \preceq \tilde{b}$ *if and only if* $\tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha$ *and* $\tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha$ *for all* $\alpha \in [0, 1]$,

where $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ *and* $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ *are* α*-level sets of given fuzzy numbers. Moreover, we define the strict inequality between two fuzzy numbers as*

$\tilde{a} \prec \tilde{b}$ *if and only if* $\tilde{a} \preceq \tilde{b}$ *and there exists an* $\alpha_0 \in [0, 1]$ *such that* $\tilde{a}^L_{\alpha_0} < \tilde{b}^L_{\alpha_0}$ *or* $\tilde{a}^U_{\alpha_0} < \tilde{b}^U_{\alpha_0}$.

Now we define comparable fuzzy numbers as follows.

Definition 4.3.2. *For* $\tilde{a}$, $\tilde{b}$ *in* $F(\mathbb{R})$, *we say that* $\tilde{a}$ *and* $\tilde{b}$ *are comparable if either* $\tilde{a} \preceq \tilde{b}$ *or* $\tilde{b} \preceq \tilde{a}$.

A locally non-dominated solution of (FOP1) is given as follows.

Definition 4.3.3. *Let* T *be an open subset of* $\mathbb{R}^n$. *A point* $\bar{x}^0 \in T$ *is a locally non-dominated solution of (FOP1) if there exists no* $\bar{x}^1 (\neq \bar{x}^0) \in N_\epsilon(\bar{x}^0) \cap T$ *such that* $\tilde{f}(\bar{x}^1) \preceq \tilde{f}(\bar{x}^0)$, *where* $N_\epsilon(\bar{x}^0)$ *is a* ϵ*-neighborhood of* $\bar{x}^0$.

4.3.1 Necessary and sufficient optimality conditions

The first and second order necessary and sufficient optimality conditions for real unconstrained optimization problem, given in [12], are as follows.

Theorem 4.3.1. *Let T be an open subset of $\mathbb{R}^n$.*

(i) ***(FONC)*** *Let f continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla f(x^*) = 0$.*

(ii) ***(SONC)*** *Let f twice continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla^2 f(x^*)$ is positive semidefinite.*

(iii) ***(SOSC)*** *Let f twice continuously differentiable function on T. Suppose that*

1. *$\nabla f(x^*) = 0$ and*
2. *$\nabla^2 f(x^*)$ is positive definite.*

Then x^* is a strict local minimizer of f.

We prove here necessary and sufficient optimality conditions for obtaining the locally non-dominated solution of (FOP1). We need the following Theorem of classical optimization theory given in [1].

Theorem 4.3.2. *[1] Suppose that $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\bar{x}$. If there is a vector $\bar{d}$ such that $\nabla f(\bar{x})^T \cdot \bar{d} < 0$, then there exists a $\delta > 0$ such that $f(\bar{x} + \lambda \bar{d}) < f(\bar{x})$ for each $\lambda \in (0, \delta)$, so that $\bar{d}$ is a descent direction of f at $\bar{x}$.*

The first order necessary condition is as follows.

Theorem 4.3.3. *Suppose $\tilde{f} : T \to F(\mathbb{R})$ is continuously H-differentiable fuzzy-valued function, T is an open subset of $\mathbb{R}^n$. If $\bar{x}^0 \in T$ is a locally non-dominated solution of (FOP1) and for any direction $\bar{d}$ and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ and $\tilde{f}(\bar{x}^0)$ are comparable, then $\nabla \tilde{f}(\bar{x}^0) = \tilde{0}$.*

Proof. Suppose that

$$\nabla \tilde{f}(\bar{x}^0) \neq \tilde{0},$$

then there exists $\alpha_0 \in [0, 1]$ such that

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \neq 0$$

or

$$\nabla \tilde{f}^U_{\alpha_0}(\bar{x}^0) \neq 0.$$

Without loss of generality suppose that

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \neq 0.$$

Let $\bar{d} = -\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0)$. Then we get

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \cdot \bar{d} = -\|\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0)\|^2 < 0.$$

By Theorem 4.3.2, there is a $\delta > 0$ such that

$$\tilde{f}^L_{\alpha_0}(\bar{x}^0 + \lambda \bar{d}) < \tilde{f}^L_{\alpha_0}(\bar{x}^0) \tag{4.3.1}$$

for $\lambda \in (0, \delta)$. Now by assumption of the theorem,

for any direction $\bar{d}$ and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ and $\tilde{f}(\bar{x}^0)$ are comparable.

Thus, either $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d}) \preceq \tilde{f}(\bar{x}^0)$ or $\tilde{f}(\bar{x}^0) \preceq \tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$. But from (4.3.1), we must have

$$\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d}) \prec \tilde{f}(\bar{x}^0).$$

Which contradicts to our assumption that $\bar{x}^0$ is a non-dominated solution. Therefore,

$$\nabla \tilde{f}(\bar{x}^0) = \tilde{0}.$$

$\square$

Remark 4.3.1.

$$\nabla \tilde{f}(\bar{x}^0) = \tilde{0}$$

implies

$$\nabla \tilde{f}^L_{\alpha}(\bar{x}^0) = 0 \textit{ and } \nabla \tilde{f}^U_{\alpha}(\bar{x}^0) = 0$$

for all $\alpha \in [0, 1]$. This implies

$$\int_0^1 \nabla \tilde{f}_\alpha^L(\bar{x}^0) \cdot d\alpha = 0 \textit{ and } \int_0^1 \nabla \tilde{f}_\alpha^U(\bar{x}^0) \cdot d\alpha = 0$$

That is

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = 0$$

Next, we prove second order necessary condition.

Theorem 4.3.4. *Let $\tilde{f}$ be a twice continuously H-differentiable fuzzy-valued function defined on $T \subseteq \mathbb{R}^n$. If $\bar{x}^0$ is a locally non-dominated solution of (FOP1) and for any direction $\bar{d}$ and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ and $\tilde{f}(\bar{x}^0)$ are comparable then $\nabla^2 \tilde{f}(\bar{x}^0)$ is positive semidefinite fuzzy matrix.*

Proof. We prove the result by contradiction. Suppose $\nabla^2 \tilde{f}(\bar{x}^0)$ is not a positive semidefinite fuzzy matrix. Then by definition, there exists some $\alpha_0 \in [0, 1]$, such that either

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^L(\bar{x}) \cdot \bar{d}_0 < 0$$

or

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^U(\bar{x}) \cdot \bar{d}_0 < 0,$$

for some direction $\bar{d}_0$. Without loss of generality, we assume that

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^L(\bar{x}) \cdot \bar{d}_0 < 0 \tag{4.3.2}$$

Now let $\bar{x}(\beta) = \bar{x}^0 + \beta\bar{d}$ and define the composite function

$$\phi_\alpha(\beta) = \tilde{f}_\alpha^L(\bar{x}^0 + \beta\bar{d}),$$

for all $\alpha \in [0, 1]$.

Since $\tilde{f}$ is twice continuously H-differentiable fuzzy-valued function on T. By Proposition 2.3.1 and 2.3.3, $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are also twice continuously differentiable functions on T, for all $\alpha \in [0, 1]$. Then by Taylor's theorem,

$$\phi_\alpha(\beta) = \phi_\alpha(0) + \phi'_\alpha(0) \cdot \beta + \phi''_\alpha(0) \cdot \frac{\beta^2}{2} + O(\beta^2),$$

for all $\alpha \in [0,1]$. Now since $\bar{x}^0$ is a locally non-dominated solution of (FOP1) then by Theorem 4.3.3,

$$\phi'_\alpha(0) = \bar{d} \cdot \nabla \tilde{f}^L_\alpha(\bar{x}^0) = 0,$$

for all $\alpha \in [0,1]$. Therefore,

$$\phi_\alpha(\beta) - \phi_\alpha(0) = \phi''_\alpha(0) \cdot \frac{\beta^2}{2} + O(\beta^2).$$

Since $\phi''_\alpha(0) = \bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d}$,

$$\phi_\alpha(\beta) - \phi_\alpha(0) = (\bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d})\frac{\beta^2}{2} + O(\beta^2).$$

Taking $\alpha = \alpha_0$ and $\bar{d} = \bar{d}_0$, from (4.3.2) and for sufficiently small β,

$$\phi_{\alpha_0}(\beta) - \phi_{\alpha_0}(0) < 0.$$

That is,

$$\tilde{f}^L_{\alpha_0}(\bar{x}^0 + \beta \bar{d}_0) < \tilde{f}^L_{\alpha_0}(\bar{x}^0) \tag{4.3.3}$$

Now β is chosen in such a way that $\tilde{f}(\bar{x}^0 + \beta \bar{d}_0)$ and $\tilde{f}(\bar{x}^0)$ are comparable. That is either $\tilde{f}(\bar{x}^0 + \beta \bar{d}_0) \preceq \tilde{f}(\bar{x}^0)$ or $\tilde{f}(\bar{x}^0 + \beta \bar{d}_0) \succeq \tilde{f}(\bar{x}^0)$. But $\tilde{f}(\bar{x}^0 + \beta \bar{d}_0) \succeq \tilde{f}(\bar{x}^0)$ not possible because of (4.3.3). Therefore,

$$\tilde{f}(\bar{x}^0 + \beta \bar{d}_0) \preceq \tilde{f}(\bar{x}^0)$$

which contradicts the assumption that , $\bar{x}^0$ is a locally non-dominated solution. Therefore, $\nabla^2 \tilde{f}(\bar{x}^0)$ is a positive semidefinite fuzzy matrix. $\square$

Now, we prove second-order sufficient condition.

Theorem 4.3.5. *Let $\tilde{f}$ be a twice continuously H-differentiable function on $T \subseteq \mathbb{R}^n$. Suppose that*

1. $\nabla \tilde{f}(\bar{x}^0) = \tilde{0}$

2. $\nabla^2 \tilde{f}(\bar{x}^0)$ *is positive definite fuzzy matrix.*

Then, $\bar{x}^0$ is locally non-dominated solution of (FOP1).

Proof. We prove this result by contradiction. Suppose $\bar{x}^0 \in T$ is not a locally non-dominated solution of (FOP1). Then, for any $\epsilon > 0$ there exists $\bar{x}^1 (\neq \bar{x}^0) \in N_\epsilon(\bar{x}^0) \cap T$ such that $\tilde{f}(\bar{x}^1) \preceq \tilde{f}(\bar{x}^0)$. That is., there exists $\bar{x}^1 \in N_\epsilon(\bar{x}^0) \cap T$ such that

$$\tilde{f}(\bar{x}^1)^L_\alpha \leq \tilde{f}(\bar{x}^0)^L_\alpha \text{ and } \tilde{f}(\bar{x}^1)^U_\alpha \leq \tilde{f}(\bar{x}^0)^U_\alpha \tag{4.3.4}$$

for all $\alpha \in [0, 1]$. Now since $\tilde{f}$ is the twice continuously H-differentiable function, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also twice continuously differentiable functions, for all $\alpha \in [0.1]$. Using assumption 2 and Rayleigh's inequality (refer Theorem 3.3.3 of Chapter 3),

it follows that if $\bar{d} \neq 0$, then

$$0 < \lambda_{\min}(\nabla^2 \tilde{f}^L_\alpha(\bar{x}^0)) \|\bar{d}\|^2 \leq \bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d}.$$

By Taylor's theorem and assumption 1,

$$\begin{aligned}
\tilde{f}^L_\alpha(\bar{x}^0 + \bar{d}) - \tilde{f}^L_\alpha(\bar{x}^0) &= \frac{1}{2} \bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d} + O(\|\bar{d}\|^2) \\
&\geq \frac{\lambda_{\min}(\nabla^2 \tilde{f}^L_\alpha(\bar{x}^0))}{2} \|\bar{d}\|^2 + O(\|\bar{d}\|^2) \\
&> 0,
\end{aligned}$$

for all $\bar{d}$ such that $\|\bar{d}\|$ is sufficiently small. Now choose $\bar{x}^1$ so close to $\bar{x}^0$ so that $\bar{d} = \bar{x}^1 - \bar{x}^0$ is sufficiently small and hence,

$$\tilde{f}^L_\alpha(\bar{x}^1) - \tilde{f}^L_\alpha(\bar{x}^0) = \tilde{f}^L_\alpha(\bar{x}^0 + \bar{d}) - \tilde{f}^L_\alpha(\bar{x}^0) > 0$$

That is,

$$\tilde{f}^L_\alpha(\bar{x}^1) > \tilde{f}^L_\alpha(\bar{x}^0)$$

This gives contradiction to inequality (4.3.4). Hence proved the result. □

We consider two examples to illustrate the above results.

Example 4.3.1.

$$\mathit{Minimize} \quad \tilde{f}(x_1, x_2)$$

$$\mathit{Subject\ \ to} \quad \bar{x} = (x_1, x_2) \in \mathbb{R}^2,$$

where $\tilde{f} : \mathbb{R}^2 \to F(\mathbb{R})$ *be defined by* $\tilde{f}(x_1, x_2) = (1, 2, 4) \odot x_1^2 \oplus (1, 2, 4) \odot x_2^2 \oplus (1, 3, 5)$, $(1, 2, 4)$ *and* $(1, 3, 5)$ *are triangular fuzzy numbers.*

By the first order necessary condition, we have

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = 0$$

Here, $\tilde{f}_\alpha^L(x_1, x_2) = (1+\alpha)x_1^2 + (1+\alpha)x_2^2 + (1+2\alpha)$ *and*

$\tilde{f}_\alpha^U(x_1, x_2) = (4-2\alpha)x_1^2 + (4-2\alpha)x_2^2 + (5-2\alpha)$. *Therefore,*

$$\nabla f_\alpha^L(x_1, x_2) = \begin{pmatrix} 2(1+\alpha)x_1 \\ 2(1+\alpha)x_2 \end{pmatrix}$$

and

$$\nabla f_\alpha^U(x_1, x_2) = \begin{pmatrix} 2(4-2\alpha)x_1 \\ 2(4-2\alpha)x_2 \end{pmatrix}$$

Therefore,

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = \begin{pmatrix} 9x_1 \\ 9x_2 \end{pmatrix} = 0$$

That is , $x^0 = (x_1, x_2) = (0, 0)$.

Now to verify second order necessary and sufficient conditions, we find fuzzy Hessian matrix of $\tilde{f}(x)$. *The* α*-level matrices of fuzzy Hessian matrix are:*

$$\nabla^2 \tilde{f}_\alpha^L(x) = \begin{pmatrix} 2(1+\alpha) & 0 \\ 0 & 2(1+\alpha) \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}_\alpha^U(x) = \begin{pmatrix} 2(4-2\alpha) & 0 \\ 0 & 2(4-2\alpha) \end{pmatrix}$$

Since both the α*-level matrices* $\nabla^2 \tilde{f}_\alpha^L(x)$ *and* $\nabla^2 \tilde{f}_\alpha^U(x)$ *are positive definite matrices for all* $\alpha \in [0, 1]$. *Therefore,* $x^0 = (0, 0)$ *satisfies the second order necessary and sufficient condition for a locally non-dominated solution. Hence,* $x^0 = (0, 0)$ *is a locally non-dominated solution of given problem.*

Now we consider another example.

Example 4.3.2.

$$Minimize \;\; \tilde{f}(x_1, x_2)$$

$$Subject \;\; to \;\; \bar{x} = (x_1, x_2) \in \mathbb{R}^2,$$

where $\tilde{f} : \mathbb{R}^2 \to F(\mathbb{R})$ *be defined by* $\tilde{f}(x_1, x_2) = (1, 2, 4) \odot x_1^3 \oplus (1, 2, 4) \odot x_2^3 \oplus (1, 3, 5)$, $(1, 2, 4)$ *and* $(1, 3, 5)$ *are triangular fuzzy numbers.*

By the first order necessary condition, we have

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = 0$$

Here, $\tilde{f}_\alpha^L(x_1, x_2) = (1+\alpha)x_1^3 + (1+\alpha)x_2^3 + (1+2\alpha)$ *and*

$\tilde{f}_\alpha^U(x_1, x_2) = (4-2\alpha)x_1^3 + (4-2\alpha)x_2^3 + (5-2\alpha)$. *Therefore,*

$$\nabla f_\alpha^L(x_1, x_2) = \begin{pmatrix} 3(1+\alpha)x_1^2 \\ 3(1+\alpha)x_2^2 \end{pmatrix}$$

and

$$\nabla f_\alpha^U(x_1, x_2) = \begin{pmatrix} 3(4-2\alpha)x_1^2 \\ 3(4-2\alpha)x_2^2 \end{pmatrix}$$

Therefore,

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = \begin{pmatrix} 13.5x_1^2 \\ 13.5x_2^2 \end{pmatrix} = 0$$

That is , $x^0 = (x_1, x_2) = (0, 0)$.
Now to verify second order necessary and sufficient conditions, we find fuzzy Hessian matrix of $\tilde{f}(x)$. *The* α*-level matrices of fuzzy Hessian matrix are:*

$$\nabla^2 \tilde{f}_\alpha^L(x) = \begin{pmatrix} 6(1+\alpha)x_1 & 0 \\ 0 & 6(1+\alpha)x_2 \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}_\alpha^U(x) = \begin{pmatrix} 6(4-2\alpha)x_1 & 0 \\ 0 & 6(4-2\alpha)x_2 \end{pmatrix}$$

Since both the α*-level matrices* $\nabla^2 \tilde{f}_\alpha^L(x)$ *and* $\nabla^2 \tilde{f}_\alpha^U(x)$ *are positive semidefinite matrices for all* $\alpha \in [0, 1]$ *at point* $x^0 = (0, 0)$. *Therefore,* $x^0 = (0, 0)$ *satisfies the second order necessary condition but not the sufficient condition for a locally non-dominated solution, as none of the* α*-level matrices is positive definite at* $x^0 = (0, 0)$. *Hence,* $x^0 = (0, 0)$ *is not a locally non-dominated solution of given problem.*

4.4 Constrained fuzzy optimization problem

Now we prove the necessary and sufficient Kuhn-Tucker like optimality conditions for obtaining a non-dominated solution of a nonlinear constrained fuzzy optimization problem.

4.4.1 Problem definition

Let $X \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f}, \tilde{g}_j$, for $j = 1, ..., m$ be fuzzy-valued functions defined on X. Consider the following nonlinear constrained fuzzy optimization problem (NCFOP)

$$\begin{aligned} &Minimize \;\; \tilde{f}(\bar{x}) = \tilde{f}(x_1, .., x_n) \\ &Subject\;\; to \;\; \tilde{g}_j(\bar{x}) \preceq \tilde{0}, \quad j = 1, .., m, \end{aligned}$$

where $\tilde{0}$ is a fuzzy number defined as $\tilde{0}(r) = 1$ if $r = 0$ and $\tilde{0}(r) = 0$ if $r \neq 0$ and its level set is $\tilde{0}_\alpha = \{0\}$ for $\alpha \in [0, 1]$. For (NCFOP), the solution is defined in terms of a (weak) non-dominated solution in the following sense.

Definition 4.4.1. *Let $\bar{x}_0 \in X_1 = \{\bar{x} \in X : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, .., m\}$. We say that an $\bar{x}_0$ is a non-dominated solution of (NCFOP) if there exists no $\bar{x}_1 (\neq \bar{x}_0) \in X_1$ such that $\tilde{f}(\bar{x}_1) \preceq \tilde{f}(\bar{x}_0)$. It is said to be a weak non-dominated solution if there exists no $\bar{x}_1 \in X_1$ such that $\tilde{f}(\bar{x}_1) \prec \tilde{f}(\bar{x}_0)$. That is, $\bar{x}^0$ is a weak non-dominated solution if there exists no $\bar{x}^1 \in X_1$ such that*

$$\begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) < \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) \leq \tilde{f}^U_\alpha(\bar{x}^0) \end{cases} or \begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) \leq \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) < \tilde{f}^U_\alpha(\bar{x}^0) \end{cases} or \begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) < \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) < \tilde{f}^U_\alpha(\bar{x}^0) \end{cases}$$

for all $\alpha \in [0, 1]$.

4.4.2 Necessary and sufficient optimality conditions

Let f and $g_j, j = 1, .., m$, be real-valued functions defined on $T \subset \mathbb{R}^n$. Then we consider the following optimization problem

$$(P) \qquad Minimize \;\; f(\bar{x}) = f(x_1, .., x_n)$$

$$Subject \;\; to \;\; g_j(\bar{x}) \leq 0, \quad j = 1, .., m.$$

The well-known Kuhn-Tucker optimality conditions for problem (P) by S. Rangarajan in [52] is stated as follows:

Theorem 4.4.1. *Let f be a convex, continuously differentiable function mapping T into $\mathbb{R}$, where $T \subset \mathbb{R}^n$ is open and convex. For $j = 1, ..., m$, the constraint functions $g_j : T \to \mathbb{R}$ are convex, continuously differentiable functions. Suppose there is some $\bar{x} \in T$ such that $g_j(\bar{x}) < 0$, $j = 1, ..., m$.*
Then $\bar{x}^0$ is an optimal solution of problem (P) over the feasible set $\{\bar{x} \in T : g_j(\bar{x}) \leq 0, j = 1, .., m\}$ if and only if there exist multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1, .., m$, such that the Kuhn-Tucker first order conditions hold:

(KT-1) $\nabla f(\bar{x}^0) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}^0) = 0$;

(KT-2) $\mu_j \cdot g_j(\bar{x}^0) = 0$ *for all* $j = 1, .., m$.

Now we present the Kuhn-Tucker like optimality conditions for (NCFOP).

Theorem 4.4.2. *Let a fuzzy-valued objective function $\tilde{f} : X \to F(\mathbb{R})$ be convex and continuously H-differentiable, where $X \subset \mathbb{R}^n$ is open and convex. For $j = 1, .., m$, the fuzzy-valued constraint functions $\tilde{g}_j : X \to F(\mathbb{R})$ are convex and continuously H-differentiable. Let $X_1 = \{\bar{x} \in T \subset \mathbb{R}^n : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, .., m\}$ be a feasible set of problem (NCFOP) and let $\bar{x}^0 \in X_1$. Suppose there is some $\bar{x} \in X$ such that $\tilde{g}^U_{j0}(\bar{x}) < 0$, $j = 1, .., m$.*
Then $\bar{x}^0$ is a weak non-dominated solution of problem (NCFOP) if and only if there exist multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1, .., m$, such that the Kuhn-Tucker first order conditions hold:

(FKT-1) $\displaystyle\int_0^1 \nabla \tilde{f}^L_\alpha(\bar{x}^0)\, d\alpha + \int_0^1 \nabla \tilde{f}^U_\alpha(\bar{x}^0)\, d\alpha + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}^U_{j0}(\bar{x}^0) = 0$;

(FKT-2) $\mu_j \cdot \tilde{g}^U_{j0}(\bar{x}^0) = 0$ *for all* $j = 1, .., m$.

Proof. **Necessary.** Define a new function,

$$F(\bar{x}) = \int_0^1 \tilde{f}^L_\alpha(\bar{x}) d\alpha + \int_0^1 \tilde{f}^U_\alpha(\bar{x}) d\alpha. \tag{4.4.1}$$

Since $\tilde{f}$ is convex and continuously H-differentiable function, by Propositions 2.3.5 and 2.6.1, we say that $F(\bar{x})$ is convex and continuously differentiable real-valued function on

X.
Since $\bar{x}^0$ is a weak non-dominated solution of (NCFOP). Then there exists no $\bar{x}^1 \in X_1$ such that

$$\begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) < \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) \leq \tilde{f}^U_\alpha(\bar{x}^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) \leq \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) < \tilde{f}^U_\alpha(\bar{x}^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}^1) < \tilde{f}^L_\alpha(\bar{x}^0) \\ \tilde{f}^U_\alpha(\bar{x}^1) < \tilde{f}^U_\alpha(\bar{x}^0) \end{cases}$$

for all $\alpha \in [0,1]$. Therefore, there exists no $\bar{x}^1 \in X_1$ such that

$$\tilde{f}^L_\alpha(\bar{x}^1) + \tilde{f}^U_\alpha(\bar{x}^1) < \tilde{f}^L_\alpha(\bar{x}^0) + \tilde{f}^U_\alpha(\bar{x}^0),$$

for all $\alpha \in [0,1]$. That is, there exists no $\bar{x}^1 \in X_1$ such that

$$\int_0^1 \tilde{f}^L_\alpha(\bar{x}^1)d\alpha + \int_0^1 \tilde{f}^U_\alpha(\bar{x}^1)d\alpha < \int_0^1 \tilde{f}^L_\alpha(\bar{x}^0)d\alpha + \int_0^1 \tilde{f}^U_\alpha(\bar{x}^0)d\alpha$$

That is, there exists no $\bar{x}^1 \in X_1$ such that

$$F(\bar{x}^1) < F(\bar{x}^0)$$

Therefore,

$$F(\bar{x}^0) \leq F(\bar{x}^1)$$

for all $\bar{x}^1 \in X_1$. Since $\tilde{g}_j$ are convex and continuously H-differentiable functions for $j = 1,..,m$, implies $\tilde{g}^L_{j\alpha}$ and $\tilde{g}^U_{j\alpha}$ are real-valued convex and continuously differentiable functions for all $\alpha \in [0,1]$ and $j = 1,..,m$.
By definition of partial ordering , we have

$$\begin{aligned} X_1 &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1,...,m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^L_{j\alpha}(\bar{x}) \leq 0 \; and \; \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1,...,m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1,...,m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j0}(\bar{x}) \leq 0, j = 1,...,m\} \end{aligned}$$

Therefore, $\bar{x}^0 \in X_1 = \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j0}(\bar{x}) \leq 0, j = 1,..,m\}$ and there is some $\bar{x} \in X$ such that $\tilde{g}^U_{j0}(\bar{x}) < 0$, $j = 1,..,m$. So our problem becomes an optimization problem with real objective function $F(\bar{x})$ subject to real constraints.
Therefore, by Theorem 4.4.1, there exist multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1,..,m$, such that the following kuhn-Tucker first order conditions hold:

(KT-1) $\nabla F(\bar{x}^0) + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}_{j0}^U(\bar{x}^0) = 0;$

(KT-2) $\mu_j \cdot \tilde{g}_{j0}^U(\bar{x}^0) = 0$ for all $j = 1, .., m$.

But $F(\bar{x}) = \displaystyle\int_0^1 \tilde{f}_\alpha^L(\bar{x}) d\alpha + \int_0^1 \tilde{f}_\alpha^U(\bar{x}) d\alpha$. We obtain the kuhn-Tucker conditions for problem (NCFOP) as follows

(FKT-1) $\displaystyle\int_0^1 \nabla \tilde{f}_\alpha^L(\bar{x}^0)\, d\alpha + \int_0^1 \nabla \tilde{f}_\alpha^U(\bar{x}^0)\, d\alpha + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}_{j0}^U(\bar{x}^0) = 0;$

(FKT-2) $\mu_j \cdot \tilde{g}_{j0}^U(\bar{x}^0) = 0$ for all $j = 1, .., m$.

Sufficient. We are going to prove this part by contradiction. Suppose that $\bar{x}^0$ not a weak non-dominated solution. Then there exists a $\bar{x}^1 \in X_1$ such that $\tilde{f}(\bar{x}^1) \prec \tilde{f}(\bar{x}^0)$. Therefore, we have

$$\tilde{f}_\alpha^L(\bar{x}^1) + \tilde{f}_\alpha^U(\bar{x}^1) < \tilde{f}_\alpha^L(\bar{x}^0) + \tilde{f}_\alpha^U(\bar{x}^0)$$

for all $\alpha \in [0, 1]$. From (4.4.1), we obtain

$$F(\bar{x}^1) < F(\bar{x}^0) \tag{4.4.2}$$

Since F is convex and continuously differentiable function. Furthermore, $\bar{x}^0 \in X_1 = \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}_{j0}^U(\bar{x}) \leq 0, j = 1, .., m\}$, by conditions (FKT-1) and (FKT-2) of this theorem, we obtain the following new conditions:

(KT-1) $\nabla F(\bar{x}^0) + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}_{j0}^U(\bar{x}^0) = 0;$

(KT-2) $\mu_j \cdot \tilde{g}_{j0}^U(\bar{x}^0) = 0$ for all $j = 1, .., m$.

Using Theorem 4.4.1, we say that $\bar{x}^0$ is an optimal solution of real objective function F with real constraints $\tilde{g}_{j0}^U(\bar{x}) \leq 0$, for $j = 1, .., m$. i.e., $F(\bar{x}^0) \leq F(\bar{x}^1)$, which contradicts to (4.4.2). Hence the proof.

$\square$

4.4.3 Illustrations

First we consider a fuzzy optimization problem having a fuzzy-valued objective function and real constraints.

Example 4.4.1.

$$\begin{aligned} &Minimize \quad \tilde{f}(x_1, x_2) = (\tilde{a} \odot x_1^2) \oplus (\tilde{b} \odot x_2^2) \\ &subject\ to\ g(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1, \end{aligned}$$

where $\tilde{a} = (1, 2, 3)$ *and* $\tilde{b} = (0, 1, 2)$ *are triangular fuzzy numbers defined on* $\mathbb{R}$ *as*

$$\tilde{a}(r) = \begin{cases} (r-1), & if\ 1 \leq r \leq 2, \\ (3-r), & if\ 2 < r \leq 3, \\ 0 & otherwise \end{cases} \quad \tilde{b}(r) = \begin{cases} r, & if\ 0 \leq r \leq 1, \\ 2-r, & if\ 1 < r \leq 2, \\ 0 & otherwise \end{cases}$$

Using Definition 2.3.1, we get

$\tilde{f}_\alpha^L(x_1, x_2) = (1+\alpha)x_1^2 + \alpha x_2^2$ *and* $\tilde{f}_\alpha^U(x_1, x_2) = (3-\alpha)x_1^2 + (2-\alpha)x_2^2$, *for* $\alpha \in [0, 1]$.

We obtain

$$\nabla \tilde{f}_\alpha^L(x_1, x_2) = \begin{pmatrix} 2x_1(\alpha + 1) \\ 2x_2\alpha \end{pmatrix},$$

$$\nabla \tilde{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2x_1(3-\alpha) \\ 2x_2(2-\alpha) \end{pmatrix} \text{ and}$$

$$\nabla g(x_1, x_2) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 - 2) \end{pmatrix}$$

Therefore, we have

$$\int_0^1 \nabla \tilde{f}_\alpha^L(x_1, x_2)\, d\alpha = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix},$$

$$\int_0^1 \nabla \tilde{f}_\alpha^U(x_1, x_2)\, d\alpha = \begin{pmatrix} 5x_1 \\ 3x_2 \end{pmatrix}.$$

From Theorem 4.4.2, we have the following Kuhn-Tucker conditions

(FKT-1) $8x_1 + 2\mu(x_1 - 2) = 0,\ 4x_2 + 2\mu(x_2 - 2) = 0,$

(FKT-2) $\mu((x_1 - 2)^2 + (x_2 - 2)^2 - 1) = 0.$

Solving these equations, we get the solution $(x_1, x_2) = (6/5, 3/2)$ *and* $\mu = 6$. *By Theorem 4.4.2, we say that* $(x_1^*, x_2^*) = (6/5, 3/2)$ *is a weak non-dominated solution for given problem . Also the minimum value of objective function is* $\tilde{f}_{min} = (1.44, 5.13, 8.82)$ *and we can find its defuzzified value 5.13 by using center of area method (ref. [29]).*

Now we solve the same fuzzy optimization problem having fuzzy-valued objective function with fuzzy constraints.

Example 4.4.2.

$$\begin{aligned} Minimize \quad & \tilde{f}(x_1, x_2) = (\tilde{a} \odot x_1^2) \oplus (\tilde{b} \odot x_2^2) \\ subject\ to\ & \tilde{g}(x_1, x_2) = (\tilde{b} \odot (x_1 - 2)^2) \oplus (\tilde{b} \odot (x_2 - 2)^2) \preceq \tilde{c}, \end{aligned}$$

where $\tilde{a} = (1, 2, 3)$, $\tilde{b} = (0, 1, 2)$ *and* $\tilde{c} = (0, 2, 4)$ *are triangular fuzzy numbers defined on* $\mathbb{R}$ *as*

$$\tilde{a}(r) = \begin{cases} (r-1), & if\ 1 \le r \le 2, \\ (3-r), & if\ 2 < r \le 3, \\ 0 & otherwise \end{cases} \quad \tilde{b}(r) = \begin{cases} r, & if\ 0 \le r \le 1, \\ 2-r, & if\ 1 < r \le 2, \\ 0 & otherwise \end{cases}$$

$$\tilde{c}(r) = \begin{cases} r/2, & if\ 0 \le r \le 2, \\ (4-r)/2, & if\ 2 < r \le 4, \\ 0 & otherwise \end{cases}$$

Using Definition 2.3.1, we get

$\tilde{f}_\alpha^L(x_1, x_2) = (1+\alpha)x_1^2 + \alpha x_2^2$ *and* $\tilde{f}_\alpha^U(x_1, x_2) = (3-\alpha)x_1^2 + (2-\alpha)x_2^2$ *for* $\alpha \in [0, 1]$.

Moreover, $\tilde{g}_\alpha^U(x_1, x_2) = (2-\alpha)(x_1 - 2)^2 + (2-\alpha)(x_2 - 2)^2 \le (4 - 2\alpha)$

for $\alpha \in [0, 1]$.

Therefore, $\tilde{g}_0^U(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \le 2.$

Now we obtain

$$\nabla \tilde{f}^L_\alpha(x_1, x_2) = \begin{pmatrix} 2x_1(\alpha + 1) \\ 2x_2\alpha \end{pmatrix},$$

$\nabla \tilde{f}^U_\alpha(x_1, x_2) = \begin{pmatrix} 2x_1(3 - \alpha) \\ 2x_2(2 - \alpha) \end{pmatrix}$ *and*

$\nabla g(x_1, x_2) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 - 2) \end{pmatrix}$

Therefore, we have

$$\int_0^1 \nabla \tilde{f}^L_\alpha(x_1, x_2)\, d\alpha = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix},$$

$$\int_0^1 \nabla \tilde{f}^U_\alpha(x_1, x_2)\, d\alpha = \begin{pmatrix} 5x_1 \\ 3x_2 \end{pmatrix}.$$

From Theorem 4.4.2, we have the following Kuhn-Tucker conditions

(FKT-1) $8x_1 + 2\mu(x_1 - 2) = 0,\ 4x_2 + 2\mu(x_2 - 2) = 0,$

(FKT-2) $\mu((x_1 - 2)^2 + (x_2 - 2)^2 - 2) = 0.$

Solving these equations, we get the solution $(x_1, x_2) = ((-6 + 2\sqrt{41})/(1 + \sqrt{41}), (-6 + 2\sqrt{41})/(-1 + \sqrt{41}))$ *and* $\mu = -3 + \sqrt{41}$. *By Theorem 4.4.2, we say that* $(x_1^*, x_2^*) = ((-6 + 2\sqrt{41})/(1 + \sqrt{41}), (-6 + 2\sqrt{41})/(-1 + \sqrt{41}))$ *is a weak non-dominated solution for given problem . Also the minimum value of objective function is* $\tilde{f}_{min} = (0.8453, 3.2773, 5.7094)$ *and we can find its defuzzified value 3.2773 by using center of area method.*

Remark 4.4.1. *By comparing the defuzzified value of minimum objective functions in Example 4.4.1 and 4.4.2, we observe that there is significant effect on minimum value of the fuzzy-valued objective function if consider fuzzy optimization problem with fuzzy constraints. Moreover, if we consider the fuzzy optimization problem having fuzzy constraints then we can not apply Theorem 6.2 from [76] to find the non-dominated solution. In that case, our result is quite useful to get the solution.*

4.4.4 Case study

Here we set out a nonlinear fuzzy optimization problem as case study which describes a possible situation in an exporting company. The problem is the following:

Two products A and B are to be produced using two different processes (p_1 and p_2) for the purpose of export. The production of one unit of product A (B) requires about 10 (about 6) minutes of processing time for p_1 process, and about 5 (about 10) minutes for the p_2 process. The total time available for process p_1 is at least 2000 minutes and can be extended up to 2064 minutes with linearly reducing possibility and time available on process p_2 can be at least 2050 minutes and can be extended up to 2124 minutes with linearly decreasing possibility. When sold abroad, product A (B) yields a profit of around 20 (around 32) per unit. A discount of around 4 percent (around 3 percent) of total quantity purchased of product A (B) is given on unit selling price per unit of product A (B). The managers want to maximize the benefit. Here we take, the parameters in terms of processing times in the different departments as well as profits for the products are approximately only rather than the exact value.

The fuzzy problem can be formulated as follows:

$$Maximize\ \tilde{20} \odot x_1 \oplus \tilde{32} \odot x_2 \ominus \widetilde{0.04} \odot x_1^2 \ominus \widetilde{0.03} \odot x_2^2$$

subject to the constraints:

$$\begin{aligned} \tilde{10} \odot x_1 \oplus \tilde{6} \odot x_2 &\preceq \widetilde{2000} \\ \tilde{5} \odot x_1 \oplus \tilde{10} \odot x_2 &\preceq \widetilde{2050} \\ x_1, x_2 &\geq 0, \end{aligned}$$

where $\tilde{20} = (18, 20, 21)$, $\tilde{32} = (31, 32, 34)$, $\widetilde{0.04} = (0.03, 0.04, 0.05)$, $\widetilde{0.03} = (0.02, 0.03, 0.05)$, $\tilde{10} = (9, 10, 11)$, $\tilde{5} = (4, 5, 6)$ and $\tilde{6} = (5, 6, 7)$ are triangular fuzzy numbers while $\widetilde{2000}$ and $\widetilde{2050}$ are fuzzy numbers having following membership functions:

$$\mu_{\widetilde{2000}}(r) = \begin{cases} 1, \quad if\ 0 \leq r \leq 2000, \\ \frac{(2064-r)}{64} \quad if\ 2000 < r \leq 2064, \\ 0 \qquad otherwise \end{cases}$$

and

$$\mu_{\widetilde{2050}}(r) = \begin{cases} 1, \quad if\ 0 \leq r \leq 2050, \\ \frac{(2124-r)}{74} \quad if\ 2050 < r \leq 2124, \\ 0 \qquad\qquad otherwise \end{cases}$$

We get the weak non-dominated solution $x_1 = 246.875$ and $x_2 = 496.15$ of given fuzzy optimization problem by applying the above necessary and sufficient Kuhn-Tucker like conditions.

4.5 Conclusions

The main concept pursued in this chapter is the generalization of the optimality conditions for a nonlinear constrained fuzzy optimization problem which are well-established in classical optimization. We have defined a non-dominated solution for a fuzzy optimization problem using a partial order relation- fuzzy max order on fuzzy numbers. Using the concept of Hukuhara differentiability of a fuzzy-valued function, we proved the required results. Here we observed that the solution of fuzzy formulation of an optimization problem differs from its crisp analog. Therefore, we conclude that considering uncertainties inherent in nature while formulating an optimization problem, makes our deduction more realistic.

Chapter 5

Fuzzy optimization problem under generalized convexity

Contents

5.1 Introduction

A. A. K. Majumdar has proved the sufficient optimality conditions for multi objective optimization problems using the concept of convexity and generalized convexity in his paper [47]. Wu has proved the sufficient optimality conditions for an optimization problem with fuzzy-valued objective function and real constraints in [80] using pseudoconvexity of objective function. Using the approach of [47], in this chapter, we prove the sufficient optimality conditions for a non-dominated solution of a fuzzy optimization problem with fuzzy-valued objective function and fuzzy constraints, under the concept of convexity and generalized convexity of fuzzy-valued functions.

5.2 Problem and its solution

We consider the (NCFOP) defined in Chapter 4.

$$\begin{aligned} &Minimize \;\; \tilde{f}(\bar{x}) = \tilde{f}(x_1, .., x_n) \\ &Subject\ to \;\; \tilde{g}_j(\bar{x}) \preceq \tilde{0}, \quad j = 1, .., m, \\ &\qquad\qquad\quad \bar{x} \in X \subseteq \mathbb{R}^n. \end{aligned}$$

where X is an open set and $\tilde{f}$ and $\tilde{g}_j, j = 1, ..., m$, are fuzzy-valued functions defined on X. Here we recall the definition of a (weak) non-dominated solution from Chapter 4.

Definition 5.2.1. *Let $\bar{x}_0 \in X_1 = \{\bar{x} \in X : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, .., m\}$. We say that an $\bar{x}_0$ is a non-dominated solution of (NCFOP) if there exists no $\bar{x}_1(\neq \bar{x}_0) \in X_1$ such that $\tilde{f}(\bar{x}_1) \preceq \tilde{f}(\bar{x}_0)$. It is said to be a weak non-dominated solution if there exists no $\bar{x}_1 \in X_1$ such that $\tilde{f}(\bar{x}_1) \prec \tilde{f}(\bar{x}_0)$.*

To establish the sufficient optimality conditions for (NCFOP), we need the following theorem of alternatives.

Theorem 5.2.1. *[74] (Tucker's theorem of alternatives) Let A and B be matrices of dimension n by m and n by p respectively, and let x, y, u be column vectors of dimensions m, p, n respectively. Then exactly one of the following system has a solution:*

System 1: $A^t u \leq 0$, $A^t u \neq 0$, $B^t u \leq 0$ *for some* u

System 2: $Ax + By = 0$ *for some* $x > 0$, $y \geq 0$.

5.3 Sufficient optimality conditions

Using the concept of convexity and generalized convexity of a fuzzy-valued function defined in Chapter 2, we prove the sufficient optimality conditions for $\bar{x}_0$ to be a (weak) non-dominated solution of (NCFOP).

Theorem 5.3.1. *(Sufficiency condition 1 for a weak non-dominated solution). Assume that an $\bar{x}_0 \in X_1$ satisfies the following conditions (i)-(iii):*

(i) $\tilde{f}(\bar{x})$, $\tilde{g}_j(\bar{x})$, $j = 1, ..., m$, *are H-differentiable at* $\bar{x} = \bar{x}_0 \in X_1$*;*

(ii) *$\tilde{f}(\bar{x})$, $\tilde{g}_j(\bar{x})$, $j = 1, ..., m$, are convex at $\bar{x} = \bar{x}_0 \in X_1$;*

(iii) *there exist $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$, such that*

(a) *$\nabla \tilde{f}^L_\alpha(\bar{x}_0) + \nabla \tilde{f}^U_\alpha(\bar{x}_0) + \sum_{j=1}^m \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j = 0$, for all $\alpha \in [0,1]$;*

(b) *$\mu_j \cdot \tilde{g}^U_{j0}(\bar{x}_0) = 0$, for all $j = 1, .., m$.*

Then, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP).

Proof. Suppose that $\bar{x}_0 \in X_1$ is not weak non-dominated solution. Then there exists $\bar{x}_1 \in X_1$ such that $f(\bar{x}_1) \prec f(\bar{x}_0)$. That is, there exists $\bar{x}_1 \in X_1$ such that

$$\begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) \leq \tilde{f}^U_\alpha(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) \leq \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^U_\alpha(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^U_\alpha(\bar{x}_0) \end{cases}$$

for all $\alpha \in [0,1]$. Therefore

$$\tilde{f}^L_\alpha(\bar{x}_1) + \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) + \tilde{f}^U_\alpha(\bar{x}_0),$$

for all $\alpha \in [0,1]$. That is,

$$F_\alpha(\bar{x}_1) - F_\alpha(\bar{x}_0) < 0, \tag{5.3.1}$$

where $F_\alpha(\bar{x}) = \tilde{f}^L_\alpha(\bar{x}) + \tilde{f}^U_\alpha(\bar{x})$, for all $\alpha \in [0,1]$. By definition of partial ordering, we have

$$\begin{aligned} X_1 &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, ..., m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^L_{j\alpha}(\bar{x}) \leq 0 \textit{ and } \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m \textit{ and } \alpha \in [0,1]\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j0}(\bar{x}) \leq 0, j = 1, ..., m\} \end{aligned}$$

Let $J = \{j : \tilde{g}^U_{j0}(\bar{x}_0) = 0\}$ is an index set of active constraints at $\bar{x} = \bar{x}_0$. Since $\bar{x}_0, \bar{x}_1 \in X_1$, for $j \in J$, we have

$$\tilde{g}^U_{j0}(\bar{x}_1) - \tilde{g}^U_{j0}(\bar{x}_0) \leq 0. \tag{5.3.2}$$

Now using hypothesis **(ii)** of the Theorem and Theorem 2.6.1 from Preliminaries for (5.3.1) and (5.3.2), we have

$\nabla F_\alpha(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) < 0,\ \nabla \tilde{g}_{j0}^U(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) \leq 0,\ j \in J$ and for all $\alpha \in [0, 1]$.

Thus, the following system of inequalities

$\nabla F_\alpha(\bar{x}_0)\bar{z} < 0$, for all $\alpha \in [0, 1]$, $\nabla \tilde{g}_{j0}^U(\bar{x}_0)\bar{z} \leq 0$ possess a solution $\bar{z} = \bar{x}_1 - \bar{x}_0$.

Therefore, by Tucker's theorem of alternatives (refer Theorem 5.2.1), there exist no $\lambda > 0$ and $\mu_j^{'} \geq 0$ such that

$$\nabla F_\alpha(\bar{x}_0)\lambda + \sum_{j \in J} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j^{'} = 0,$$

for all $\alpha \in [0, 1]$. That is,

$$\nabla \tilde{f}_\alpha^L(\bar{x}_0) + \nabla \tilde{f}_\alpha^U(\bar{x}_0) + \sum_{j \in J} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j = 0,$$

where $\mu_j = \mu_j^{'}/\lambda$ and $F_\alpha(\bar{x}_0) = \tilde{f}_\alpha^L(\bar{x}_0) + \tilde{f}_\alpha^U(\bar{x}_0)$, for all $\alpha \in [0, 1]$.

Taking $\mu_j = 0$ for $j = \{1, ..., m\} - J$, we can still say that there exist no $\mu_j \geq 0$ for all $j \in J$ such that

$$\nabla \tilde{f}_\alpha^L(\bar{x}_0) + \nabla \tilde{f}_\alpha^U(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$. Therefore, we can say that there exist no $\mu_j \geq 0, j = 1, ..., m$ such that

$$\nabla \tilde{f}_\alpha^L(\bar{x}_0) + \nabla \tilde{f}_\alpha^U(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$, and $\mu_j \cdot \tilde{g}_{j0}^U(x_0) = 0$ for j =*1,...,m*. This contradicts to hypothesis **(iii)** of the Theorem. Hence, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP). □

Theorem 5.3.2. *(Sufficiency condition for a non-dominated solution). Assume that an* $\bar{x}_0 \in X_1$ *satisfies the following conditions (i)-(iii):*

(i) $\tilde{f}(\bar{x})$, $\tilde{g}_j(\bar{x})$, j =*1,..,m, are strictly pseudoconvex at* $\bar{x} = \bar{x}_0 \in X_1$*;*

(ii) *there exist* $0 \leq \mu_j \in \mathbb{R}$, j = *1,...,m, such that*

(a) $\nabla \tilde{f}^L_\alpha(\bar{x}_0) + \nabla \tilde{f}^U_\alpha(\bar{x}_0) + \sum_{j=1}^m \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j = 0$, *for all* $\alpha \in [0,1]$;

(b) $\mu_j \cdot \tilde{g}^U_{j0}(\bar{x}_0) = 0$, *for all* j *= 1,..,m.*

Then, $\bar{x}_0$ *is a non-dominated solution of (NCFOP).*

Proof. Suppose $\bar{x}_0$ is not non-dominated solution, then there exists an $\bar{x}_1(\neq \bar{x}_0) \in X_1$ such that $\tilde{f}(\bar{x}_1) \preceq \tilde{f}(\bar{x}_0)$.

i.e., $\tilde{f}^L_\alpha(\bar{x}_1) \leq \tilde{f}^L_\alpha(\bar{x}_0)$ and $\tilde{f}^U_\alpha(\bar{x}_1) \leq \tilde{f}^U_\alpha(\bar{x}_0)$, for all $\alpha \in [0,1]$.

By assumption of strictly pseudoconvexity of the function $\tilde{f}(\bar{x})$ at $\bar{x} = \bar{x}_0$, we have $\tilde{f}^L_\alpha(\bar{x})$ and $\tilde{f}^L_\alpha(\bar{x})$ are also strictly pseudoconvex functions. Using the above inequalities , we obtain

$\nabla \tilde{f}^L_\alpha(\bar{x}_0)^t(\bar{x}_1 - \bar{x}_0) < 0$ and $\nabla \tilde{f}^U_\alpha(\bar{x}_0)^t(\bar{x}_1 - \bar{x}_0) < 0$, for all $\alpha \in [0,1]$.

Furthermore, we have

$$\tilde{g}^U_{j0}(\bar{x}_1) - \tilde{g}^U_{j0}(\bar{x}_0) \leq 0$$

where $j \in J = \{j : \tilde{g}^U_{j0}(x_0) = 0\}$ is an index set of active constraints at $\bar{x} = \bar{x}_0$. Therefore, we have

$$\nabla F_\alpha(\bar{x}_0)^t(\bar{x}_1 - \bar{x}_0) < 0$$

and

$$\nabla \tilde{g}^U_{j0}(\bar{x}_0)^t(\bar{x}_1 - \bar{x}_0) < 0$$

where $F_\alpha(\bar{x}_0) = \tilde{f}^L_\alpha(\bar{x}_0) + \tilde{f}^U_\alpha(\bar{x}_0)$, for all $\alpha \in [0,1]$. Thus, the following system of inequalities
$\nabla F_\alpha(\bar{x}_0)^t \bar{z} < 0,\ for\ \alpha \in [0,1]\ \nabla \tilde{g}^U_{j0}(\bar{x}_0)^t \bar{z} < 0$ possess a solution $\bar{z} = \bar{x}_1 - \bar{x}_0$.

Therefore, by the Tucker's theorem of alternatives (refer Theorem 5.2.1), there exist no $\lambda > 0$ and $0 \leq \mu^{'}_j \in \mathbb{R}$, $j \in J$, such that

$$\nabla F_\alpha(\bar{x}_0)\lambda + \sum_{j \in J} \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu^{'}_j = 0,$$

for all $\alpha \in [0,1]$. Using the similar arguments in the proof of Theorem 5.3.1, there exist no $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$ such that

$$\nabla \tilde{f}_\alpha^L(\bar{x}_0) + \nabla \tilde{f}_\alpha^U(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0,1]$ and $\mu_j \cdot \tilde{g}_{j0}^U(x_0) = 0$ for j =*1,...,m*, violating hypothesis **(ii)** of the theorem. Hence, $\bar{x}_0$ is a non-dominated solution of (NCFOP). □

Theorem 5.3.3. *(Sufficiency condition 2 for a weak non-dominated solution). Assume that an $\bar{x}_0 \in X_1$ satisfies the following conditions (i)-(iii):*

(i) *$\tilde{f}(\bar{x})$ is pseudoconvex at $\bar{x} = \bar{x}_0 \in X_1$;*

(ii) *$\tilde{g}_j(\bar{x})$ are quasiconvex and H-differentiable at $\bar{x}_0$, for $j = 1, ..., m$;*

(iii) *there exist $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$, such that*

- **(a)** $\nabla \tilde{f}_\alpha^L(\bar{x}_0) + \nabla \tilde{f}_\alpha^U(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(\bar{x}_0) \cdot \mu_j = 0$, *for all $\alpha \in [0,1]$;*
- **(b)** $\mu_j \cdot \tilde{g}_{0j}^U(\bar{x}_0) = 0$, $j = 1, .., m$.

Then, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP).

Proof. Suppose that $\bar{x}_0 \in X_1$ is not weak non-dominated solution. Then there exists $\bar{x}_1 \in X_1$ such that $f(\bar{x}_1) \prec f(\bar{x}_0)$. That is, there exists $\bar{x}_1 \in X_1$ such that

$$\begin{cases} \tilde{f}_\alpha^L(\bar{x}_1) < \tilde{f}_\alpha^L(\bar{x}_0) \\ \tilde{f}_\alpha^U(\bar{x}_1) \leq \tilde{f}_\alpha^U(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}_\alpha^L(\bar{x}_1) \leq \tilde{f}_\alpha^L(\bar{x}_0) \\ \tilde{f}_\alpha^U(\bar{x}_1) < \tilde{f}_\alpha^U(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}_\alpha^L(\bar{x}_1) < \tilde{f}_\alpha^L(\bar{x}_0) \\ \tilde{f}_\alpha^U(\bar{x}_1) < \tilde{f}_\alpha^U(\bar{x}_0) \end{cases}$$

for all $\alpha \in [0,1]$. Therefore

$$\tilde{f}_\alpha^L(\bar{x}_1) + \tilde{f}_\alpha^U(\bar{x}_1) < \tilde{f}_\alpha^L(\bar{x}_0) + \tilde{f}_\alpha^U(\bar{x}_0),$$

for all $\alpha \in [0,1]$. That is,

$$F_\alpha(\bar{x}_1) - F_\alpha(\bar{x}_0) < 0, \tag{5.3.3}$$

where $F_\alpha(\bar{x}) = \tilde{f}^L_\alpha(\bar{x}) + \tilde{f}^U_\alpha(\bar{x})$, for all $\alpha \in [0,1]$. By definition of partial ordering, we have

$$\begin{aligned}
X_1 &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, ..., m\} \\
&= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^L_{j\alpha}(\bar{x}) \leq 0 \; and \; \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m \; and \; \alpha \in [0,1]\} \\
&= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m\} \\
&= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j0}(\bar{x}) \leq 0, j = 1, ..., m\}
\end{aligned}$$

Let $J = \{j : \tilde{g}^U_{j0}(\bar{x}_0) = 0\}$ is an index set of active constraints at $\bar{x} = \bar{x}_0$. Since $\bar{x}_0, \bar{x}_1 \in X_1$, for $j \in J$, we have

$$\tilde{g}^U_{j0}(\bar{x}_1) - \tilde{g}^U_{j0}(\bar{x}_0) \leq 0. \tag{5.3.4}$$

Now using hypothesis **(i)** and **(ii)** of the Theorem, from (5.3.3) and (5.3.4),

$\nabla F_\alpha(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) < 0$, $\nabla \tilde{g}^U_{j0}(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) \leq 0$, $j \in J$ and for all $\alpha \in [0,1]$.

Thus, the following system of inequalities

$\nabla F_\alpha(\bar{x}_0)\bar{z} < 0$, for all $\alpha \in [0,1]$, $\nabla \tilde{g}^U_{j0}(\bar{x}_0)\bar{z} \leq 0$ possess a solution $\bar{z} = \bar{x}_1 - \bar{x}_0$.

Therefore, by Tucker's theorem of alternatives (refer Theorem 5.2.1), there exist no $\lambda > 0$ and $\mu^{'}_j \geq 0$ such that

$$\nabla F_\alpha(\bar{x}_0)\lambda + \sum_{j \in J} \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu^{'}_j = 0,$$

for all $\alpha \in [0,1]$. Using the similar arguments in the proof of Theorem 5.3.1, there exist no $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$ such that

$$\nabla \tilde{f}^L_\alpha(\bar{x}_0) + \nabla \tilde{f}^U_\alpha(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0,1]$ and $\mu_j \cdot \tilde{g}^U_{j0}(x_0) = 0$ for j *=1,...,m*, violating hypothesis **(iii)** of the theorem. Hence, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP). □

Now we prove sufficient optimality condition for (NCFOP) under quasiconvexity of a fuzzy-valued objective function. We quote the following Theorem of quasiconvex functions from [30].

Theorem 5.3.4. *[30] Let $X^0 \subseteq \mathbb{R}^n$ be open. If a differentiable function $f : X^0 \to \mathbb{R}$ is quasiconvex at a point $\bar{x} \in X^0$, where $\nabla f(\bar{x}) \neq 0$, then it is pseudoconvex at $\bar{x}$.*

Theorem 5.3.5. *(Sufficiency condition 3 for a weak non-dominated solution). Assume that, for $\bar{x}_0 \in X_1 = \{\bar{x} \in X : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, ..., m\}$*

(i) *$\tilde{f}, \tilde{g}_j$, $j = 1, ..., m$ are H-differentiable at $\bar{x}_0$ and $\nabla \tilde{f}^L_\alpha(\bar{x}_0) \neq 0$ and $\nabla \tilde{f}^U_\alpha(\bar{x}_0) \neq 0$, for all $\alpha \in [0, 1]$.*

(ii) *$\tilde{f}$ and $\tilde{g}_j$, $j = 1, ..., m$ are quasiconvex functions at $\bar{x}_0$.*

(iii) *Let $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$ and $\bar{x}_0 \in X_1$ satisfies the following conditions:*

(a) $\nabla \tilde{f}^L_\alpha(\bar{x}_0) + \nabla \tilde{f}^U_\alpha(\bar{x}_0) + \sum_{j=1}^m \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j = 0$, *for all* $\alpha \in [0, 1]$;

(b) $\mu_j \cdot \tilde{g}^U_{0j}(\bar{x}_0) = 0$, $j = 1, .., m$.

Then, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP).

Proof. Suppose that $\bar{x}_0 \in X_1$ is not weak non-dominated solution. Then there exists $\bar{x}_1 \in X_1$ such that $f(\bar{x}_1) \prec f(\bar{x}_0)$. That is, there exists $\bar{x}_1 \in X_1$ such that

$$\begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) \leq \tilde{f}^U_\alpha(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) \leq \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^U_\alpha(\bar{x}_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) \\ \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^U_\alpha(\bar{x}_0) \end{cases}$$

for all $\alpha \in [0, 1]$. Therefore

$$\tilde{f}^L_\alpha(\bar{x}_1) + \tilde{f}^U_\alpha(\bar{x}_1) < \tilde{f}^L_\alpha(\bar{x}_0) + \tilde{f}^U_\alpha(\bar{x}_0),$$

for all $\alpha \in [0, 1]$. That is,

$$F_\alpha(\bar{x}_1) - F_\alpha(\bar{x}_0) < 0, \tag{5.3.5}$$

where $F_\alpha(\bar{x}) = \tilde{f}^L_\alpha(\bar{x}) + \tilde{f}^U_\alpha(\bar{x})$, for all $\alpha \in [0, 1]$. By definition of partial ordering, we have

$$\begin{aligned} X_1 &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}_j(\bar{x}) \preceq \tilde{0}, j = 1, ..., m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^L_{j\alpha}(\bar{x}) \leq 0 \text{ and } \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m \text{ and } \alpha \in [0, 1]\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j\alpha}(\bar{x}) \leq 0, j = 1, ..., m\} \\ &= \{\bar{x} \in X \subset \mathbb{R}^n : \tilde{g}^U_{j0}(\bar{x}) \leq 0, j = 1, ..., m\} \end{aligned}$$

Let $J = \{j : \tilde{g}^U_{j0}(\bar{x}_0) = 0\}$ is an index set of active constraints at $\bar{x} = \bar{x}_0$. Since $\bar{x}_0, \bar{x}_1 \in X_1$, for $j \in J$, we have

$$\tilde{g}^U_{j0}(\bar{x}_1) - \tilde{g}^U_{j0}(\bar{x}_0) \leq 0. \tag{5.3.6}$$

Now using hypothesis **(i)** and **(ii)** of the Theorem, $\tilde{f}^L_\alpha(\bar{x})$ and $\tilde{f}^U_\alpha(\bar{x})$ are pseudoconvex functions at $\bar{x}_0$ (ref. Theorem 5.3.4), for all $\alpha \in [0,1]$. Therefore, from (5.3.5) and (5.3.6),

$\nabla F_\alpha(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) < 0,\ \nabla \tilde{g}^U_{j0}(\bar{x}_0)(\bar{x}_1 - \bar{x}_0) \leq 0,\ j \in J$ and for all $\alpha \in [0,1]$.

Thus, the following system of inequalities

$\nabla F_\alpha(\bar{x}_0)\bar{z} < 0$, for all $\alpha \in [0,1]$, $\nabla \tilde{g}^U_{j0}(\bar{x}_0)\bar{z} \leq 0$ possess a solution $\bar{z} = \bar{x}_1 - \bar{x}_0$.

Therefore, by Tucker's theorem of alternatives (refer Theorem 5.2.1), there exist no $\lambda > 0$ and $\mu_j^{'} \geq 0$ such that

$$\nabla F_\alpha(\bar{x}_0)\lambda + \sum_{j \in J} \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j^{'} = 0,$$

for all $\alpha \in [0,1]$. Using the similar arguments in the proof of Theorem 5.3.1, there exist no $0 \leq \mu_j \in \mathbb{R}$, $j = 1, ..., m$ such that

$$\nabla \tilde{f}^L_\alpha(\bar{x}_0) + \nabla \tilde{f}^U_\alpha(\bar{x}_0) + \sum_{j=1}^{m} \nabla \tilde{g}^U_{j0}(\bar{x}_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0,1]$ and $\mu_j \cdot \tilde{g}^U_{j0}(x_0) = 0$ for j =*1,...,m*, violating assumption **(iii)** of the theorem. Hence, $\bar{x}_0$ is a weak non-dominated solution of (NCFOP). □

5.4 Illustrations

Here we provide two examples to show the effect of fuzzy modeling of the following crisp type optimization problem.

Example 5.4.1.

$$\mathit{Minimize} \qquad f(x_1, x_2) = 2 \cdot x_1^2 + 2 \cdot x_2^2$$

$$Subject \;\; to \;\; : \;\; g(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \leq 3$$

has the minimum point $(x_1^*, x_2^*) = (2 - \sqrt{3}/\sqrt{2}, 2 - \sqrt{3}/\sqrt{2})$ *and minimum value is* $f(x_1^*, x_2^*) = 6.419$.

Now we consider a fuzzy optimization problem having fuzzy coefficients and find the non-dominated solution using the optimality conditions.

Example 5.4.2. *We consider the following fuzzy optimization problem*

$$Minimize \qquad \tilde{f}(x_1, x_2) = (\tilde{2} \odot x_1^2) \oplus (\tilde{2} \odot x_2^2)$$

$$Subject\ to: \;\; \tilde{g}(x_1, x_2) = (\tilde{1} \odot (x_1 - 2)^2) \oplus (\tilde{1} \odot (x_2 - 2)^2) \preceq \tilde{3}$$

where $\tilde{2} = (0, 2, 3)$, $\tilde{1} = (-1, 1, 2)$ *and* $\tilde{3} = (2, 3, 4)$ *are triangular fuzzy numbers as shown in Figure 5.1.*

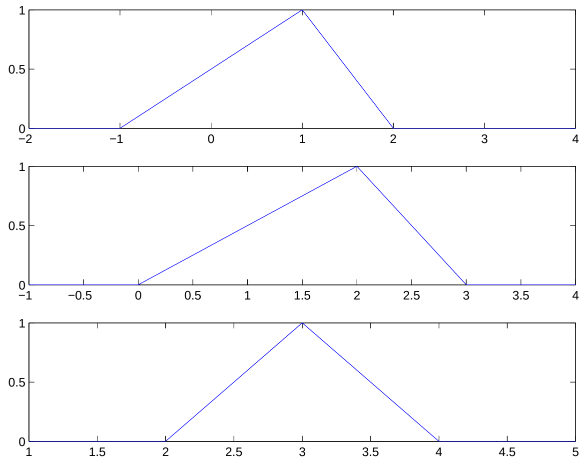

Figure 5.1: Membership functions of triangular fuzzy numbers $\tilde{1} = (-1, 1, 2)$, $\tilde{2} = (0, 2, 3)$ and $\tilde{3} = (2, 3, 4)$

By arithmetics of fuzzy numbers, we obtain

$\tilde{f}_\alpha^L(x_1, x_2) = 2\alpha x_1^2 + 2\alpha x_2^2$,

$\tilde{f}_\alpha^U(x_1, x_2) = (3 - \alpha)x_1^2 + (3 - \alpha)x_2^2$ *and*

$\tilde{g}_\alpha^U(x_1, x_2) = (2 - \alpha)(x_1 - 2)^2 + (2 - \alpha)(x_2 - 2)^2 \leq (4 - \alpha)$.

We also obtain

$$\nabla \tilde{f}_\alpha^L(x_1, x_2) = \begin{pmatrix} 4\alpha x_1 \\ 4\alpha x_2 \end{pmatrix},$$

$$\nabla \tilde{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2(3-\alpha)x_1 \\ 2(3-\alpha)x_2 \end{pmatrix} \text{ and}$$

$$\nabla \tilde{g}_0^U(x_1, x_2) = \begin{pmatrix} 2(x_1-2) \\ 2(x_2-2) \end{pmatrix}.$$

For checking the conditions (a) and (b) in Theorem 5.3.1., we need to solve

the following system of equations:

$$\alpha x_1 + 3x_1 + 2\mu x_1 - 4\mu = 0$$

$$\alpha x_2 + 3x_2 + 2\mu x_2 - 4\mu = 0$$

$$\mu \cdot ((x_1-2)^2 + (x_2-2)^2 - 2) = 0.$$

Then we get $(x_1, x_2) = (1, 1)$ *and* $\mu = (\alpha + 3)/2$.

We see that $(x_1, x_2) = (1, 1)$ *is the feasible solution to the given (NCFOP).*

For any fixed $\alpha \in [0, 1]$, *we see that* $\tilde{f}_\alpha^L(\bar{x})$, $\tilde{f}_\alpha^U(\bar{x})$ *and* $\tilde{g}_\alpha^U(\bar{x})$ *are strictly convex functions at* $\bar{x} = (1, 1)$, *therefore from Theorem 5.3.1, we say that* $(x_1^*, x_2^*) = (1, 1)$ *is a weak non-dominated solution to the given (NCFOP) and minimum value of the fuzzy-valued objective function is* $\tilde{4} = (0, 4, 6)$ *having* $\tilde{4}_\alpha = [4\alpha, 6 - 2\alpha]$. *We defuzzify the minimum value using the center of area method given in [29] as 3.3333. If we compare with this solution with a solution to crisp type of optimization problem in Example 5.4.1 which is 6.419. We observe that by approximating coefficients as fuzzy numbers we get better minimum value.*

Remark 5.4.1. *In the above example, we have solved fuzzy optimization problem having fuzzy coefficients are non symmetric left spread triangular fuzzy numbers. If we spread non symmetric triangular fuzzy numbers on right side as shown in the following Figure*

5.2, then weak non-dominated solution of the same fuzzy optimization problem is given by $(x_1^*, x_2^*) = (2 - 2/\sqrt{6}, 2 - 2/\sqrt{6})$ *and minimum value is* $\tilde{f}(x_1^*, x_2^*) = (2.8, 5.6, 11.2)$. *Its defuzzified value is 6.533.*

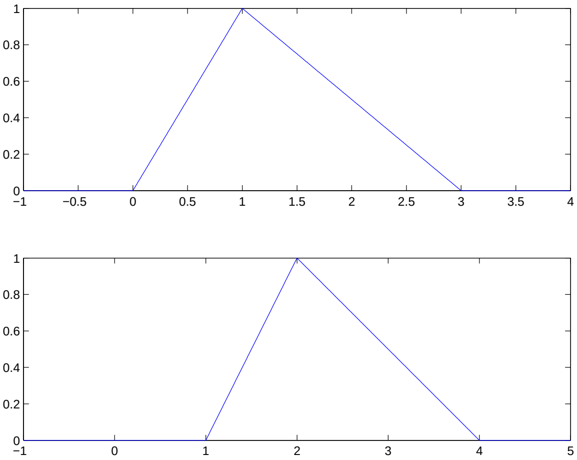

Figure 5.2: Membership functions of triangular fuzzy numbers $\tilde{1} = (0, 1, 3)$ and $\tilde{2} = (1, 2, 4)$

If we consider fuzzy coefficients are symmetric triangular fuzzy numbers as shown in Figure 5.3, then the non-dominated solution will be $(x_1^*, x_2^*) = (1, 1)$ *and* $\mu = 4$. *In this case, minimum value is* $\tilde{4} = (2, 4, 6)$ *and its defuzzified value is 4.*

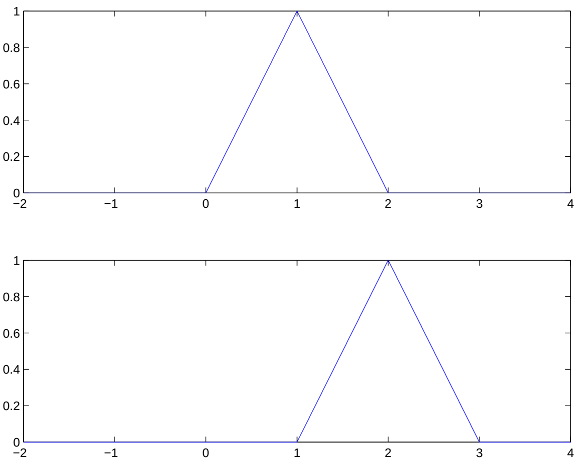

Figure 5.3: Membership functions of triangular fuzzy numbers $\tilde{1} = (0, 1, 2)$ and $\tilde{2} = (1, 2, 3)$

Thus fuzzification of the parameters representing coefficients of the x_1^2 *and* x_2^2 *in* f *and coefficients of* $(x_1-2)^2$, $(x_2-2)^2$ *in* g *has a significant effect on the non-dominated solution and the defuzzified value of the objective function.*

Example 5.4.3. *Consider the fuzzy optimization problem*

$$Minimize \quad \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus (\tilde{2} \odot x_2^2) \oplus (\widetilde{(-3)} \odot x_1) \oplus (\widetilde{(-3)} \odot x_2)$$

$$Subject\ to \quad constraint: \ \tilde{g}(x_1, x_2) = (\tilde{3} \odot x_1) \oplus (\tilde{5} \odot x_2) \oplus (\widetilde{-7}) \preceq \tilde{0}$$

where $\tilde{0} = (0,0,0)$ *and* $\tilde{1} = (0,1,2)$, $\widetilde{(-3)} = (-4,-3,-2)$, $\tilde{3} = (2,3,4)$, $\tilde{5} = (4,5,6)$ *and* $\widetilde{(-7)} = (-8,-7,-6)$ *are triangular fuzzy numbers.*

Using arithmetics of fuzzy numbers, we obtain

$$\tilde{f}_\alpha^L(x_1, x_2) = \alpha x_1^2 + (1+\alpha)x_2^2 + (-4+\alpha)x_1 + (-4+\alpha)x_2 \quad and$$
$$\tilde{f}_\alpha^U(x_1, x_2) = (2-\alpha)x_1^2 + (3-\alpha)x_2^2 + (-2-\alpha)x_1 + (-2-\alpha)x_2.$$

And

$$\tilde{g}_\alpha^L(x_1, x_2) = (2+\alpha)x_1 + (4+\alpha)x_2 + (-8+\alpha) \quad and$$
$$\tilde{g}_\alpha^U(x_1, x_2) = (4-\alpha)x_1 + (6-\alpha)x_2 + (-6-\alpha).$$

Now we have

$$\nabla \tilde{f}_\alpha^L(x_1, x_2) = \begin{pmatrix} 2\alpha x_1 + (-4+\alpha) \\ 2(1+\alpha)x_2 + (-4+\alpha) \end{pmatrix},$$

$$\nabla \tilde{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2(2-\alpha)x_1 - 2 - \alpha) \\ 2(3-\alpha)x_2 - 2 - \alpha) \end{pmatrix},$$

$$\nabla \tilde{g}_0^U(x_1, x_2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$

For checking the conditions (a) and (b) in Theorem 5.3.1, we solve the following system of equations:

$$4x_1 - 6 + 4\mu = 0$$
$$8x_2 - 6 + 6\mu = 0$$
$$\mu \cdot (4x_1 + 6x_2 - 6) = 0.$$

Then, we get $(x_1, x_2) = (\frac{33}{34}, \frac{6}{17})$ *and* $\mu = \frac{9}{17}$ *which is feasible solution to the given (NC-*

FOP).

For any fixed $\alpha \in [0,1]$, *we see that* $\tilde{f}^L_\alpha(\bar{x})$, $\tilde{f}^U_\alpha(\bar{x})$ *are strictly convex, and* $\tilde{g}^U_\alpha(\bar{x})$ *are convex functions at* $(\bar{x}) = (\frac{33}{34}, \frac{6}{17})$.
Therefore, by Theorem 5.3.1 we say that $(x_1, x_2) = (\frac{33}{34}, \frac{6}{17})$ *is a weak non-dominated solution.*

Now we consider one more example which illustrate the Theorem 5.3.3.

Example 5.4.4. *Consider the fuzzy optimization problem*

$$Minimize \quad \tilde{f}(x) = \tilde{2} \odot x^3 \oplus \widetilde{(-2)} \odot x$$

$$subject\ to \quad constraint: \ \tilde{g}(x) = \tilde{2} \odot x^3 \oplus \widetilde{(-2)} \preceq \tilde{0}$$

where $\tilde{2} = (1,2,3)$, $\widetilde{(-2)} = (-3,-2,-1)$ *and* $\tilde{0} = (0,0,0)$ *are triangular fuzzy numbers.*

Using arithmetics of fuzzy numbers, we obtain

$$\tilde{f}^L_\alpha(x) = (1+\alpha)x^3 + (-3+\alpha)x,$$

$$\tilde{f}^U_\alpha(x) = (3-\alpha)x^3 + (-1-\alpha)x.$$

And

$$\tilde{g}^U_\alpha(x) = (3-\alpha)x^3 + (-1-\alpha).$$

Now we have

$D\tilde{f}^L_\alpha(x) = 3(1+\alpha)x^2 + (-3+\alpha)$,

$D\tilde{f}^U_\alpha(x) = 3(3-\alpha)x^2 - (-1-\alpha)$,

$D\tilde{g}^U_0(x) = 9x^2.$

By the conditions (a) and (b) in Theorem 5.3.3, we have the following system of equations:

$$3x^2 - 3 + 9x^2 - 1 + \mu 9x^2 = 0$$

$$\mu \cdot (3x^3 - 1) = 0.$$

Solving the system, we get $x_0 = -1/\sqrt{3}$ *and* $\mu = 0$ *feasible solution to the given (NCFOP).*

For any fixed $\alpha \in [0, 1]$, *we see that* $\tilde{f}^L_\alpha(x) = (1+\alpha)x^3 + (-3+\alpha)x$, $\tilde{f}^U_\alpha(x) = (3-\alpha)x^3 + (-1-\alpha)x$ *are pseudoconvex functions at* x_0 *(refer Example 2.6.8 in Preliminaries), and* $\tilde{g}^U_0 = 3x^3 - 1$ *is quasiconvex at* x_0 *(refer Property 2.6.3 in Preliminaries).*
Therefore, by Theorem 5.3.3 we say that $-1/\sqrt{3}$ *is a weak non-dominated solution.*

5.5 Conclusions

In the current chapter, we have proposed the sufficient optimality conditions for obtaining a non-dominated solution of a constrained fuzzy optimization problem. The optimality conditions have been proved based on the assumptions of convexity and generalized convexity- pseudoconvexity and quasiconvexity of fuzzy-valued objective function and fuzzy constraints. We have worked out some examples of fuzzy optimization problems using these optimality conditions which shows the effect of fuzzy modeling also.

Chapter 6

Nonlinear fuzzy optimization methods

Contents

6.1 Introduction

Numerical optimization techniques play a significant role in solving nonlinear crisp optimization problems. In this Chapter, we attempt to develop a method analogous to Newton's method for solving fuzzy optimization problems defined in the context of fuzzy-max order. Using the first order necessary condition for obtaining non-dominated solution of the unconstrained fuzzy optimization problem, we propose the Newton's method. We also give convergence criteria and algorithms for proposed method for both single variable and multi variable problems. We illustrate the methods by giving concrete examples.

6.2 Newton's method

In this section, we propose the Newton's method for single and multi variable unconstrained fuzzy optimization problems.

6.2.1 Single variable fuzzy optimization problem

We consider an unconstrained single variable fuzzy optimization problem (USFOP):

$$Minimize \quad \tilde{f}(x), \qquad x \in X$$

where $X \subseteq \mathbb{R}$ is an open set and $\tilde{f} : X \to F(\mathbb{R})$ is a fuzzy-valued function.

Using partial order relation defined in Chapter 2, we recall here a locally non-dominated solution of the (USFOP) from Chapter 4.

Definition 6.2.1. *Let $X \subseteq \mathbb{R}$ be an open set and Let $\tilde{f} : X \to F(\mathbb{R})$ be a fuzzy-valued function. A point $x^0 \in X$ is said to be a locally non-dominated solution, if there exists no $x^1 \in N_\epsilon(x^0) \cap X$ such that $\tilde{f}(x^1) \preceq \tilde{f}(x^0)$, where $N_\epsilon(x^0)$ is ϵ-neighborhood of x^0.*

To propose the Newton's method to find the non-dominated solution of unconstrained single variable fuzzy optimization, we need the first order necessary condition for a locally non-dominated solution of (USFOP). This optimality condition, we have already established in Chapter 4 for unconstrained multi variable fuzzy optimization problem (refer Theorem 4.3.3). For one dimensional minimization, the first order necessary condition is as follows:

Theorem 6.2.1. *Suppose $\tilde{f} : X \to F(\mathbb{R})$ is continuously H-differentiable fuzzy-valued function, X is an open subset of $\mathbb{R}$. If $x^0 \in X$ is a locally non-dominated solution of (USFOP) and for any direction d and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{f}(x^0 + \lambda \cdot d)$ and $\tilde{f}(x^0)$ are comparable, then $D\tilde{f}(x^0) = \tilde{0}$ ($\tilde{f}'(x^0) = \tilde{0}$).*

Now we propose the Newton's method to find the non-dominated solution of unconstrained single variable fuzzy-valued function. We assume that at each measurement point $x^{(k)}$ we can calculate $\tilde{f}(x^{(k)})$, $\tilde{f}'(x^{(k)})$ and $\tilde{f}''(x^{(k)})$. We can approximate $\tilde{f}$ by a quadratic fuzzy-valued function $\tilde{q}$ so that its value, first and second derivatives matches at $x = x^{(k)}$ of the function $\tilde{f}$. Theorem 2.4.1 of [29] suggests that

$$\tilde{q}(x) = \tilde{f}(x^{(k)}) \oplus \tilde{f}'(x^{(k)}) \odot (x - x^{(k)}) \oplus \tilde{f}''(x^{(k)}) \odot \frac{(x - x^{(k)})^2}{2!}$$

Note that $\tilde{q}(x^{(k)}) = \tilde{f}(x^{(k)})$, $\tilde{q}'(x^{(k)}) = \tilde{f}'(x^{(k)})$ and $\tilde{q}''(x^{(k)}) = \tilde{f}''(x^{(k)})$. Given $x^{(k)}$ we try to approximate minimizer of $\tilde{f}$ by finding minimizer of $\tilde{q}$. The first order necessary condition for $\tilde{q}(x)$ to have a locally non-dominated solution at x is

$$\tilde{q}'(x) = \tilde{0}.$$

That is

$$(\tilde{q}^L_\alpha(x))' = 0 = (\tilde{f}^L_\alpha(x^{(k)}))' + (\tilde{f}^L_\alpha(x^{(k)}))'' \cdot (x - x^{(k)})$$

and

$$(\tilde{q}^U_\alpha(x))' = 0 = (\tilde{f}^U_\alpha(x^{(k)}))' + (\tilde{f}^U_\alpha(x^{(k)}))'' \cdot (x - x^{(k)}),$$

for all $\alpha \in [0,1]$. We can write

$$\int_0^1 (\tilde{q}^L_\alpha(x))' d\alpha = 0 = \int_0^1 (\tilde{q}^U_\alpha(x))' d\alpha$$

implies

$$\int_0^1 (\tilde{f}^L_\alpha(x^{(k)}))' d\alpha + \int_0^1 (\tilde{f}^L_\alpha(x^{(k)}))'' d\alpha \cdot (x - x^{(k)}) = 0 \tag{6.2.1}$$

and

$$\int_0^1 (\tilde{f}^U_\alpha(x^{(k)}))' d\alpha + \int_0^1 (\tilde{f}^U_\alpha(x^{(k)}))'' d\alpha \cdot (x - x^{(k)}) = 0 \tag{6.2.2}$$

By adding (6.2.1) and (6.2.2), we have

$$\int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))' + (\tilde{f}^U_\alpha(x^{(k)}))'\} d\alpha + \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))'' + (\tilde{f}^U_\alpha(x^{(k)}))''\} d\alpha \cdot (x - x^{(k)}) = 0.$$

Now we define a real-valued function F in following way.

$$F(x) = \int_0^1 \{\tilde{f}^L_\alpha(x) + \tilde{f}^L_\alpha(x)\} d\alpha$$

Therefore, we can write the above equation as

$$F'(x^{(k)}) + F''(x^{(k)})(x - x^{(k)}) = 0 \tag{6.2.3}$$

where

$$F'(x^{(k)}) = \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))' + (\tilde{f}^U_\alpha(x^{(k)}))'\} d\alpha$$

and

$$F''(x^{(k)}) = \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))'' + (\tilde{f}^U_\alpha(x^{(k)}))''\} d\alpha,$$

this we can write using Proposition 4.2.3 in Chapter 4. By putting $x = x^{(k+1)}$ in (6.2.3), we get

$$x^{(k+1)} = x^{(k)} - \frac{F'(x^{(k)})}{F''(x^{(k)})} \tag{6.2.4}$$

Thus starting with an initial approximation to minimizer of $\tilde{f}$, we can generate a sequence of approximations to the minimizer of $\tilde{f}$ using the formula (6.2.4). The procedure is terminated when $|x^{(k+1)} - x^{(k)}| < \epsilon$, ϵ is prespecified positive real number.

Remark 6.2.1. *The method is well-defined only when $F''(x^{(k)}) \neq 0$ for each k.*

Convergence

To prove the convergence of Newton's method, we need following Theorem from Numerical analysis.

Theorem 6.2.2. *[61] Let $x = x^*$ be a root of $f(x) = 0$ and let I be an interval containing the point $x = x^*$. Let $\phi(x)$ and $\phi'(x)$ be continuous on I, where $\phi(x)$ is defined by the equation $x = \phi(x)$ which is equivalent to $f(x) = 0$. Then if $|\phi'(x)| < 1$ for all $x \in I$, the sequence of approximations $x_0, x_1, ..., x_n$ defined by*

$$x_{n+1} = \phi(x_n)$$

converges to the root x^*, provided that the initial approximation x_0 is chosen in I.

Now we show the convergence of Newton's method.

Theorem 6.2.3. *Suppose that $\tilde{f}$ is three times continuously H-differentiable fuzzy-valued function defined on $\mathbb{R}$ and $x^* \in \mathbb{R}$ a point such that*

(1) $F'(x^*) = 0$

(2) $F''(x^*) \neq 0$.

Then for all x^0 sufficiently close to x^*, Newton's method is well-defined for all x and converges to x^* with order of convergence at least 2.

Proof. Since $\tilde{f}$ is three times continuously H-differentiable, $F''(x)$ is continuous function. We have,

$$|F''(x)| \geq \epsilon, \tag{6.2.5}$$

for some $\epsilon > 0$ in a suitable neighborhood of x^*. Within this neighborhood we can select an interval I such that, for all $x \in I$

$$|F'(x)F'''(x)| < \epsilon^2, \tag{6.2.6}$$

this is possible since $F'(x^*) = 0$ and $F(x)$ is also three times continuously differentiable function. Now taking

$$\phi(x) = x - \frac{F'^{(x)}}{F''(x)}$$

We observe that

$$\phi'(x) = \frac{F'(x)F'''(x)}{(F''(x))^2},$$

for all $x \in I$. Hence, from (6.2.5) and (6.2.6), we get

$$|\phi'(x)| < 1.$$

Thus, $\phi(x)$ satisfying all the hypothesis of Theorem 6.2.2, by taking an initial approximation $x^{(0)} \in I$, we get the sequence of approximations $x^{(0)}, x^{(1)}, ..., x^{(n)}$ satisfying (6.2.4) converge to non-dominated solution $x = x^*$.

Now to obtain the rate of convergence of Newton's method, we note that $F'(x^*) = 0$ so that Taylor's expansion gives

$$F'(x^{(k)}) + F''(x^{(k)})(x^* - x^{(k)}) + F'''(x^{(k)})\frac{(x^* - x^{(k)})^2}{2!} + ... = 0,$$

where

$$F'(x^{(k)}) = \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))' + (\tilde{f}^U_\alpha(x^{(k)}))'\}d\alpha,$$

$$F''(x^{(k)}) = \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))'' + (\tilde{f}^U_\alpha(x^{(k)}))''\}d\alpha$$

and

$$F'''(x^{(k)}) = \int_0^1 \{(\tilde{f}^L_\alpha(x^{(k)}))''' + (\tilde{f}^U_\alpha(x^{(k)}))'''\}d\alpha.$$

From which we obtain

$$-\frac{F'^{(x^{(k)})}}{F''(x^{(k)})} = (x^* - x^{(k)}) + \frac{1}{2}(x^* - x^{(k)})^2 \frac{F'''^{(x^{(k)})}}{F''(x^{(k)})} \tag{6.2.7}$$

From (6.2.4) and (6.2.7), we have

$$x^{(k+1)} - x^* = \frac{1}{2}(x^{(k)} - x^*)^2 \frac{F'''(x^{(k)})}{F''(x^{(k)})} \tag{6.2.8}$$

Setting

$$\epsilon_k = x^{(k)} - x^*,$$

Equation (6.2.8) gives

$$\epsilon_{k+1} \propto \frac{-F'''(x^*)}{2F''(x^*)} \cdot (\epsilon_{(k)})^2,$$

so that the Newton's method has quadratic convergence. □

Algorithm and illustration

Now we present the algorithm of proposed Newton's method.

Algorithm 1 Newton's method

Input x^0, ϵ
Calculate $F'(x)$ and $F''(x)$
$k \leftarrow 0$
repeat
    $x^{(k+1)} = x^{(k)} - \frac{F'(x^{(k)})}{F''(x^{(k)})}$
    $k \leftarrow k + 1$
until
$|x^{(k+1)} - x^{(k)}| < \epsilon$
Optimal solution $x^* \leftarrow x^{(k)}$

Example 6.2.1.

$$Maximize \quad \tilde{f}(x) = (\tilde{1} \odot x^3) \oplus (\widetilde{-12} \odot x^2),\ x \in \mathbb{R}$$

where $\tilde{1} = (-1, 1, 3)$ *and* $\widetilde{-12} = (-13, -12, -11)$ *are triangular fuzzy numbers and initial approximation for minimizer is* $x^0 = 1$.

Here $\tilde{f}^L_\alpha(x) = (-1 + 2\alpha)x^3 + (-13 + \alpha)x^2$ *and* $\tilde{f}^U_\alpha(x) = (3 - 2\alpha)x^3 + (-11 - \alpha)x^2$.

We obtain

$$F'(x)) = \int_0^1 \{(\tilde{f}^L_\alpha(x))' + (\tilde{f}^U_\alpha(x))'\} d\alpha$$

and

$$F''(x) = \int_0^1 \{(\tilde{f}_\alpha^L(x))'' + (\tilde{f}_\alpha^U(x))''\} d\alpha.$$

Therefore

$$F'(x) = 6x^2 - 48x$$

and

$$F''(x) = 12x - 48$$

Using (6.2.4),

$$x^{(k+1)} = x^{(k)} - \frac{F'(x^{(k)})}{F''(x^{(k)})}$$

we find sequence of non-dominated solutions given in following table:

k	$x^{(k)}$	$x^{(k+1)}$	$\tilde{f}(x^{(k)})$
0	1	-0.1667	$(-14, -11, -8)$
1	-0.1667	-0.0033	$(-0.3566232, -0.3380991, -0.319570)$
2	-0.0033	$-1.3877e-006$	$(-0.000145, -0.0001307, -0.0001199)$
3	$-1.3877e-006$	0	$(0, 0, 0)$

Therefore, the non-dominated solution of given problem is $x^{(3)} = x^* = 0$.

6.2.2 Multi variable fuzzy optimization problem

We consider following unconstrained multi variable fuzzy optimization problem (UMFOP).

$$Minimize \quad \tilde{f}(\bar{x}), \qquad \bar{x} \in X$$

where $X \subseteq \mathbb{R}^n$ is an open set and $\tilde{f} : X \to F(\mathbb{R})$ is a fuzzy-valued function.

Using partial order relation defined in Chapter 2, we recall here a locally non-dominated solution of the (UMFOP) from Chapter 4.

Definition 6.2.2. *Let $X \subseteq \mathbb{R}^n$ be an open set and Let $\tilde{f} : X \to F(\mathbb{R})$ be a fuzzy-valued function. A point $\bar{x}^0 \in X$ is said to be a locally non-dominated solution, if there exists no $\bar{x}^1 \in N_\epsilon(\bar{x}^0) \cap X$ such that $\tilde{f}(\bar{x}^1) \preceq \tilde{f}(\bar{x}^0)$, where $N_\epsilon(\bar{x}^0)$ is ϵ-neighborhood of $\bar{x}^0$.*

We propose the Newton's method to find the non-dominated solution of (UMFOP). We assume that at each measurement point $\bar{x}^{(k)}$, we can calculate $\tilde{f}(\bar{x}^{(k)})$, $\nabla\tilde{f}(\bar{x}^{(k)})$ and

$\nabla^2 \tilde{f}(\bar{x}^{(k)})$. We can approximate $\tilde{f}$ by a quadratic fuzzy-valued function $\tilde{h}$ so that its value, first and second derivatives matches at $\bar{x} = \bar{x}^{(k)}$ of the function $\tilde{f}$. Theorem 2.4.2 of [28] suggests that

$$\tilde{h}(\bar{x}) = \tilde{f}(\bar{x}^{(k)}) \oplus \left\{ (\bar{x} - \bar{x}^{(k)})^T \odot \nabla \tilde{f}(\bar{x}^{(k)}) \right\} \oplus \left\{ \frac{1}{2} (\bar{x} - \bar{x}^{(k)})^T \odot \nabla^2 \tilde{f}(\bar{x}^{(k)}) \odot (\bar{x} - \bar{x}^{(k)}) \right\}$$

Note that $\tilde{h}(\bar{x}^{(k)}) = \tilde{f}(\bar{x}^{(k)})$, $\nabla \tilde{h}(\bar{x}^{(k)}) = \nabla \tilde{f}(\bar{x}^{(k)})$ and $\nabla^2 \tilde{h}(\bar{x}^{(k)}) = \nabla^2 \tilde{f}(\bar{x}^{(k)})$. Given $\bar{x}^{(k)}$ we try to approximate minimizer of $\tilde{f}$ by finding minimizer of $\tilde{h}$. The first order necessary condition for $\tilde{h}(\bar{x})$ (refer Theorem 4.3.3) to have a locally non-dominated solution at $\bar{x}$ is

$$\nabla \tilde{h}(\bar{x}) = \tilde{0}.$$

Therefore, we have

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^{(k)}) + \nabla \tilde{f}_\alpha^U(\bar{x}^{(k)})\} d\alpha + \int_0^1 \{\nabla^2 \tilde{f}_\alpha^L(\bar{x}^{(k)}) + \nabla^2 \tilde{f}_\alpha^U(\bar{x}^{(k)})\} d\alpha \cdot$$
$$(\bar{x} - \bar{x}^{(k)}) = 0.$$

That is

$$\nabla F(\bar{x}^{(k)}) + \nabla^2 F(\bar{x}^{(k)})(\bar{x} - \bar{x}^{(k)}) = 0 \tag{6.2.9}$$

where

$$\nabla F(\bar{x}^{(k)}) = \int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^{(k)}) + \nabla \tilde{f}_\alpha^U(\bar{x}^{(k)})\} d\alpha$$

and

$$\nabla^2 F(\bar{x}^{(k)}) = \int_0^1 \{\nabla^2 \tilde{f}_\alpha^L(\bar{x}^{(k)}) + \nabla^2 \tilde{f}_\alpha^U(\bar{x}^{(k)})\} d\alpha.$$

By putting $\bar{x} = \bar{x}^{(k+1)}$ in (6.2.9), we have

$$\bar{x}^{(k+1)} = \bar{x}^{(k)} - \nabla F(\bar{x}^{(k)}) \cdot [\nabla^2 F(\bar{x}^{(k)})]^{-1} \tag{6.2.10}$$

where $[\nabla^2 F(\bar{x}^{(k)})]^{-1}$ is inverse of matrix $[\nabla^2 F(\bar{x}^{(k)})]$. Thus starting with an initial approximation to minimizer of $\tilde{f}$, we can generate a sequence of approximations to the minimizer of $\tilde{f}$ using the formula (6.2.10). The procedure is terminated when $\|\bar{x}^{(k+1)} - \bar{x}^{(k)}\| < \epsilon$, ϵ is prespecified positive real number.

Remark 6.2.2. *The method is well-defined only when $\nabla^2 F(\bar{x}^{(k)})$ is nonsingular for each k.*

Convergence

Now we show the convergence of Newton's method.

Theorem 6.2.4. *Suppose that $\tilde{f}$ is three times continuously H-differentiable fuzzy-valued function defined on $\mathbb{R}^n$ and $\bar{x}^* \in \mathbb{R}^n$ is a point such that*

(1) $\nabla F(\bar{x}^*) = 0$

(2) $\nabla^2 F(\bar{x}^*)$ *is invertible*

Then for all $\bar{x}^{(0)}$ is sufficiently to close to $\bar{x}^*$, Newton's method is well-defined for all k and converges to $\bar{x}^*$ with order of converges at least 2.

Here

$$F(\bar{x}) = \int_0^1 \tilde{f}_\alpha^L(\bar{x})d\alpha + \int_0^1 \tilde{f}_\alpha^U(\bar{x})d\alpha$$

Proof. By Fuzzy Taylor's formula of $\nabla\tilde{f}(\bar{x})$ around $\bar{x}^{(0)}$, we get

$$\nabla\tilde{f}(\bar{x}) = \nabla\tilde{f}(\bar{x}^{(0)}) \oplus \nabla^2\tilde{f}(\bar{x}^{(0)}) \odot (\bar{x} - \bar{x}^{(0)}) \oplus o(\|\bar{x} - \bar{x}^{(0)}\|^2).$$

That is

$$\int_0^1 \nabla\tilde{f}_\alpha^L(\bar{x})d\alpha = \int_0^1 \nabla\tilde{f}_\alpha^L(\bar{x}^{(0)})d\alpha + \int_0^1 \nabla^2\tilde{f}_\alpha^L(\bar{x}^{(0)})d\alpha \cdot (\bar{x} - \bar{x}^{(0)}) + \int_0^1 o_\alpha^L(\|\bar{x} - \bar{x}^{(0)}\|^2)d\alpha$$

and

$$\int_0^1 \nabla\tilde{f}_\alpha^U(\bar{x})d\alpha = \int_0^1 \nabla\tilde{f}_\alpha^U(\bar{x}^{(0)})d\alpha + \int_0^1 \nabla^2\tilde{f}_\alpha^U(\bar{x}^{(0)})d\alpha \cdot (\bar{x} - \bar{x}^{(0)}) + \int_0^1 o_\alpha^U(\|\bar{x} - \bar{x}^{(0)}\|^2)d\alpha$$

Therefore,

$$\nabla F(\bar{x}) = \nabla F(\bar{x}^{(0)}) + \nabla^2 F(\bar{x}^{(0)})(\bar{x} - \bar{x}^{(0)}) + O(\|\bar{x} - \bar{x}^{(0)}\|^2)$$

where

$$\nabla F(\bar{x}) = \int_0^1 \nabla\tilde{f}_\alpha^L(\bar{x})d\alpha + \int_0^1 \nabla\tilde{f}_\alpha^U(\bar{x})d\alpha,$$

$$\nabla F(\bar{x}^{(0)}) = \int_0^1 \nabla\tilde{f}_\alpha^L(\bar{x}^{(0)})d\alpha + \int_0^1 \nabla\tilde{f}_\alpha^U(\bar{x}^{(0)})d\alpha,$$

$$\nabla^2 F(\bar{x}^{(0)}) = \int_0^1 \nabla^2\tilde{f}_\alpha^L(\bar{x}^{(0)})d\alpha + \int_0^1 \nabla^2\tilde{f}_\alpha^U(\bar{x}^{(0)})d\alpha$$

and

$$O(\|\bar{x}-\bar{x}^{(0)}\|^2) = \int_0^1 o_\alpha^L(\|\bar{x}-\bar{x}^{(0)}\|^2)d\alpha + \int_0^1 o_\alpha^U(\|\bar{x}-\bar{x}^{(0)}\|^2)d\alpha.$$

This implies that given a constant c_1, there exists $\epsilon > 0$ such that

$$\|\nabla F(\bar{x}) - \nabla F(\bar{x}^{(0)}) - \nabla^2 F(\bar{x}^{(0)})\cdot(\bar{x}-\bar{x}^{(0)})\| \leq c_1\|\bar{x}-\bar{x}^{(0)}\|^2 \tag{6.2.11}$$

for all $\bar{x} \in B_\epsilon(\bar{x}^*) = \{\bar{x} : \|\bar{x}-\bar{x}^*\| < \epsilon\}$. Also $\tilde{f}$ is three times continuously H-differentiable fuzzy-valued function implies F is also. Since

$$\nabla^2 F(\bar{x}^{(*)}) = \int_0^1 \nabla^2 \tilde{f}_\alpha^L(\bar{x}^{(*)})d\alpha + \int_0^1 \nabla^2 \tilde{f}_\alpha^U(\bar{x}^{(*)})d\alpha$$

is invertible, $[\nabla^2 F(\bar{x}^*)]^{-1}$ is continuous at $\bar{x}^*$ and hence there exists $c_2 > 0$

$$\|\nabla^2 F(\bar{x}^*)\| \leq c_2 \tag{6.2.12}$$

for all $\bar{x} \in B_\epsilon(\bar{x}^*)$. As $\nabla F(\bar{x}^*) = 0$, equation (6.2.11) gives

$$\|\nabla^2 F(\bar{x}^{(0)})\cdot(\bar{x}^{(0)}-\bar{x}^*) - \nabla F(\bar{x}^{(0)})\| \leq c_1\|\bar{x}^{(0)}-\bar{x}^*\|^2 \tag{6.2.13}$$

Also by Newton's algorithm

$$\bar{x}^{(1)} = \bar{x}^{(0)} - [\nabla^2 F(\bar{x}^{(0)})]^{-1}\cdot\nabla F(\bar{x}^{(0)})$$

This gives

$$\begin{aligned}
\|\bar{x}^{(1)}-\bar{x}^*\| &= \|\bar{x}^{(0)}-\bar{x}^* - [\nabla^2 F(\bar{x}^{(0)})]^{-1}\cdot\nabla F(\bar{x}^{(0)})\| \\
&\leq \|[\nabla^2 F(\bar{x}^{(0)})]^{-1}\|\cdot\|\nabla^2 F(\bar{x}^{(0)})(\bar{x}^{(0)}-\bar{x}^*) - \nabla F(\bar{x}^{(0)})\| \\
&\leq c_1c_2\|\bar{x}^{(0)}-\bar{x}^*\|^2
\end{aligned}$$

by inequalities (6.2.12) and (6.2.13). Choose $\bar{x}^{(0)}$ sufficiently close to $\bar{x}^*$ in such a way that

$$\|\bar{x}^{(0)}-\bar{x}^*\| \leq \frac{\beta}{c_1c_2}$$

$\beta \in (0,1)$. Therefore

$$\|\bar{x}^{(1)}-\bar{x}^*\| \leq \beta\|\bar{x}^{(0)}-\bar{x}^*\|$$

Proceeding inductively we obtain

$$\|\bar{x}^{(k+1)}-\bar{x}^*\| \leq c_1c_2\|\bar{x}^{(k)}-\bar{x}^*\|^2 \tag{6.2.14}$$

and also

$$\|\bar{x}^{(k+1)} - \bar{x}^*\| \leq \beta \|\bar{x}^{(k)} - \bar{x}^*\|$$

This implies that $\bar{x}^{(k)} \to \bar{x}^*$ and equation (6.2.14) gives the quadratic convergence of $\bar{x}^{(k)}$ to $\bar{x}^*$. $\square$

Algorithm and illustration

Now we present the algorithm of proposed Newton's method.

Algorithm 2 Newton's method

```
Input x̄^0, ϵ
Calculate ∇F(x) and ∇²F(x)
k ← 0
repeat
    x̄^(k+1) = x̄^(k) − (∇F(x̄^(k) · ∇²F(x̄^(k))^(−1))
    k ← k + 1
until
‖x̄^(k+1) − x̄^(k)‖ < ϵ
Optimal solution x̄* ← x̄^(k)
```

Example 6.2.2.

$$Minimize \quad \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^3) \oplus (\tilde{2} \odot x_2^3) \oplus (\tilde{1} \odot x_1 \cdot x_2), \;\; x_1, x_2 \in \mathbb{R}$$

where $\tilde{1} = (-1, 1, 3)$ *and* $\tilde{2} = (1, 2, 3)$ *are triangular fuzzy numbers and initial approximation for minimizer is* $\bar{x}^0 = (1, 1)$.

Here $\tilde{f}^L_\alpha(x_1, x_2) = (-1 + 2\alpha)x_1^3 + (1 + \alpha)x_2^3 + (-1 + 2\alpha)(x_1 \cdot x_2)$ *and*

$\tilde{f}^U_\alpha(x_1, x_2) = (3 - 2\alpha)x_1^3 + (3 - \alpha)x_2^3 + (3 - 2\alpha)(x_1 \cdot x_2)$

Now we have

$$\nabla F(\bar{x})) = \int_0^1 \{\nabla \tilde{f}^L_\alpha(\bar{x}) + \nabla \tilde{f}^U_\alpha(\bar{x})\} d\alpha$$

and

$$\nabla^2 F(\bar{x}) = \int_0^1 \{\nabla^2 \tilde{f}^L_\alpha(x) + \nabla^2 \tilde{f}^U_\alpha(x)\} d\alpha.$$

Therefore

$$\nabla F(\bar{x}) = \begin{pmatrix} 6x_1^2 + 2x_2 \\ 12x_2^2 + 2x_1 \end{pmatrix}$$

and

$$\nabla^2 F(\bar{x}) = \begin{pmatrix} 12x_1 & 2 \\ 2 & 24x_2 \end{pmatrix}$$

Using following equation (6.2.10)

$$\bar{x}^{(k)} = \bar{x}^{(k+1)} - \nabla F(\bar{x}^{(k)}) \cdot \nabla^2 F(\bar{x}^{(k)})$$

We get the nondominated solution of given problem as $\bar{x}^* = (0,0)^T$. *The iterations of* $x^{(k)}$ *are given in the following table:*

k	$x_1^{(k)}$	$x_2^{(k)}$	$\tilde{f}(x_1^{(k)}, x_2^{(k)})$
0	1	1	$(-1, 4, 9)$
1	0.422535	0.464789	$(-0.1714194, 0.4726429, 1.1167052)$
2	0.128703	0.209319	$(-0.0199007, 0.0474143, 0.1147292)$
3	-0.146928	0.163154	$(0.0314868, -0.0184577, -0.0684021)$
4	0.012076	0.075409	$(-0.0004836, -0.00177, 0.0040236)$
5	0.036090	-0.002177	$(0.0000316, -0.0000316, -0.0000997)$
6	0.000130	0.003879	$(-0.0000004, 0.0000006, 0.0000017)$
7	0	0	0

6.3 Conclusions

In this chapter, we have proposed a generalization of Newton's method for single-variable and multi-variable unconstrained fuzzy optimization problems. Using the concept of a H-differentiability of a fuzzy-valued function, we showed the convergence of the Newton's methods for both single and multi variable problems. Moreover, numerical results presented indicate that the methods work well.

Summary

In the current work, we have dealt with nonlinear fuzzy optimization problems with and without constraints. We have studied the nonlinear optimization problems using various approaches and concepts. To study the fuzzy optimization problems, two concepts play very important role- one is the ranking method of fuzzy numbers or order relation. Ranking of fuzzy numbers have been extensively studied by researchers over the years. A lot of articles have been published regarding this. In general fuzzy numbers are not comparable in a natural way like the real numbers. There are many different partial order relations of fuzzy numbers are defined in the literature. One of them called fuzzy-max order, which is the most used partial order relation in many applications of fuzzy number theory. It is very convenient extension of order relation from real numbers to fuzzy numbers. In this work, we have mainly used this partial order relation of fuzzy numbers to establish the necessary and sufficient optimality conditions for fuzzy optimization problems.

A parametric total order relation is another order relation by which we can compare two fuzzy numbers in natural way. This order relation depends upon a fixed parameter value and can be defined on a particular class of fuzzy numbers called L-fuzzy numbers. In spite of these limitations, it is significant to develop some systematic theory in terms of optimality conditions for fuzzy optimization problems. In some problems, we have used this order relation and proved the results to find optimal solution of the fuzzy optimization problems.

Another very important concept which is used in fuzzy optimization theory is of differential calculus of fuzzy-valued functions. Differentiability of fuzzy-valued functions have also been studied and applied by many authors in a variety of ways. This we have already discussed in the second chapter. We have used Hukuhara differentiability of fuzzy-valued functions in our research work.

We have used the concepts of L-fuzzy numbers and a parametric total order relation "$\preceq_\lambda$" on space of L-fuzzy numbers as introduced in [58]. We have proved a necessary and sufficient condition for optimality of a fuzzy-valued function defined on $\mathbb{R}$ using Hukuhara differentiability of fuzzy-valued functions. It is interesting to note that if instead of fuzzy-valued functions we restrict our attention to real-valued functions, then the total order relation "$\preceq_\lambda$" for all $\lambda \in [0, 1]$, reduces to the usual total order relation "$\preceq$" on $\mathbb{R}$ and the optimality conditions reduce to usual optimality conditions for real-valued functions defined on $\mathbb{R}$. Under these settings, we have derived the first and second order necessary conditions as well as second order sufficient conditions for optimality of a fuzzy-valued function defined on $\mathbb{R}^n$. At the end, We have given illustrations to verify the proposed results.

Using partial order relation on fuzzy number space, the necessary and sufficient Kuhn-Tucker like optimality conditions for nonlinear fuzzy optimization problem have been derived in this problem. We have used Hukuhara differentiability and convexity of fuzzy-valued functions for proving the same.

The concept of generalized convexity of fuzzy-valued functions has been defined in this problem. We have applied two weaker forms of convexity of fuzzy-valued functions called quasiconvexity and pseudoconvexity to establish the necessary results. We have proved the sufficient optimality conditions for x to be the non-dominated solution of a fuzzy optimization problem. We have also provided some examples to illustrate the application of the theorems and have shown then the fuzzification of a crisp optimization problem does have significant effect on the solution.

In the last two problems, we have established generalization of a well-known Newton's method for single-variable as well as multi variable fuzzy optimization problems. In this process, we have used fuzzy-max order relation on set of fuzzy numbers and the concept of Hukuhara differentiability of a fuzzy-valued function. We have given convergence criteria and algorithm for both the methods.

Publications

1. **V.D.Pathak and U.M. Pirzada,** *The Optimality Conditions for Single variable Fuzzy-valued functions*, International Conference on Computational Intelligence and Multimedia Applications 2007, IEEE Computer Society, Volume 2(2007), pp. 357–363.

2. **V.D.Pathak and U.M. Pirzada,** *The Optimality Conditions for Fuzzy Optimization Problem under the concept of Generalized Convexity*, Advances in Applied Mathematical Analysis, Volume 5, Number 1, (2010), pp. 23–38.

3. **V.D.Pathak and U.M. Pirzada,** *First and Second-order Optimality Conditions for L-Fuzzy Optimization Problems*, Advanced Modeling and Optimization, Volume 13, Number 1, (2011), pp. 77–88.

4. **V.D.Pathak and U.M. Pirzada,** *Necessary and Sufficient Optimality Conditions for Nonlinear Fuzzy Optimization Problem*, International Journal of Mathematical Science Education (to be published in November 2011).

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