CHAPTER 5

RATES OF CONVERGENCE IN LOCAL LIMIT THEOREM FOR INDEPENDENT SUMMANDS-I

5.1 INTRODUCTION:

Let $\{X_n\}$ be a sequence of mutually independent r.v.s with corresponding sequence of absolutely continuous d.f.s $\{G_n\}$. Suppose, for each n, $G_n \in \{F_1, F_2, \ldots, F_m\}$; m being a fixed positive integer.

For each n, let $\tau_j = \tau_j(n)$ be the number of r.v.s among $X_1, \ldots, X_n$ which have F_j as their d.f., j = 1, 2, ..., m. Note that $\sum_{j=1}^{m} \tau_j = n$.

Suppose that each F_j belongs to the domain of normal attraction of the stable law F_o with index α, $0< \alpha< 2$. By Theorem 3.1 of Sreehari (1970), S_n, properly normalized, converges in distribution to a stable r.v. with d.f. F_o. Kruglov (1968) proved that if the d.f. F_j, j = 1, 2, ..., m are absolutely continuous with p.d.f. v_j, j = 1, 2, ..., m, then

$$\sup_{x \in R}|v_n(x) - v_0(x)| = o(1) \text{ as } n \to \infty,$$

$v_n(x)$ being the p.d.f. of S_n, properly normalized, and, v_o being the p.d.f of F_o. Kruglov did not make any assumptions about parent distribution's membership of any particular stable laws' domains of attraction. Also he assumed that the limit distribution of normalized sums S_n

exists through the necessary and sufficient conditions given by Zinger (1965), and it was not important what the limit distribution is. Basu et al. (1979), under certain regularity conditions, obtained a non-uniform rate of convergence in a local limit theorem concerning i.i.d.r.v.s in the domain of normal attraction of a stable law (they assumed the limit law to be strictly stable in the case $0< \alpha \le 1$). The rate was found to be of the order $n^{1-([\alpha]+1)\gamma}$ $0< \alpha< 2$, $\gamma = 1/\alpha$. In this chapter, we obtain uniform as well as non-uniform rates of convergence of the density v_n to v_o for the above set up. This improves Kruglov's result and generalizes work of Basu et al. (1979) from i.i.d. set up to non-identical set up.

We state our theorems first:

Theorem 5.1.1: *Under the assumptions* **[A1]-[A5]**, *stated below*

$$\sup_{x \in R}|v_n(x)-v_o(x)| = O(n^{1-([\alpha]+1)\gamma}).$$

Theorem 5.1.2: *Under the assumptions* **[A1]-[A5]**, *stated below*

$$\sup_{x \in R}(1+|x|^{\alpha})|v_n(x)-v_o(x)| = O(n^{1-([\alpha]+1)\gamma}).$$

We prove the theorems for special values of m in Section 5.4. We introduce the notations and assumptions in Section 5.2. In Section 5.3, we mention some lemmas which will be needed in Section 5.4. These lemmas can be proved on the lines of corresponding results of Section 2.3 of Chapter 2. We shall mention only necessary changes, wherever required, in the proofs of their counterpart-results of Chapter 2. The outline of the proof for the general case is given in Section 5.5.

5.2 NOTATIONS AND ASSUMPTIONS:

Let Y_o denote a stable or a strictly stable r.v. with index α, according as $1< \alpha\leq 2$ or $0< \alpha< 1$, respectively, having the d.f. F_o with $EY_o = 0$, whenever it exists and let w_o denote its c.f. We assume $EX_n = 0$, whenever it exists.

It is known that $Z_n = S_n/B_n$ converges in distribution to r.v. Y with $B_n^{\alpha} = \sum_{i=1}^{m} d_i^{\alpha} \tau_i$ (see: Sreehari (1970), Theorem 3.1), where d_i is an appropriate constant depending only on d.f. F_i. For the sake of simplicity, we shall take $B_n = n^{\gamma}$, $\gamma = 1/\alpha$, without loss of generality.

We write $\phi_n(t) = E[\exp (itZ_n)] = \prod_{i=1}^{m} \{w_i(tn^{-\gamma})\}^{\tau_i}$, where w_i is the c.f. corresponding to the d.f. F_i.

Note that we have from the canonical representation of c.f. $w_o(t)$ that for all t

$$w_o(t) = \prod_{i=1}^{m} \{w_o(tn^{-\gamma})\}^{\tau_i}. \qquad \ldots(5.2.1)$$

Then, in view of discussion in Section 5.1,

$$\lim_{n \to \infty} \phi_n(t) = w_o(t) \text{ for all } t. \qquad \ldots(5.2.2)$$

For each positive integer n and real number x, we define for k = 0, 1, ..., m,

$$\alpha_{k,\tau_k}(t,x) = \int_{|u|\le|x|\tau_k^{\gamma}} e^{itu} dF_k(u), \qquad \ldots(5.2.3)$$

$$\beta_{k,\tau_k}(t,x) = w_k(t)-\alpha_{k,\tau_k}(t,x), \qquad \ldots(5.2.4)$$

$$A_{k,\tau_k}(t,x) = \{\alpha_{k,\tau_k}(tn^{-\gamma},x)\}^{\tau_k}, \qquad \ldots(5.2.5)$$

$$B_{k,\tau_k}(t,x) = \{w_k(tn^{-\gamma})\}^{\tau_k}-\{\alpha_{k,\tau_k}(tn^{-\gamma},x)\}^{\tau_k}$$

$$=\sum_{h=1}^{\tau_k}\binom{\tau_k}{h}\{\alpha_{k,\tau_k}(tn^{-\gamma},x)\}^{\tau_k-h}\{\beta_{k,\tau_k}(tn^{-\gamma},x)\}^{h}. \qquad \ldots(5.2.6)$$

Note that for

$$v_n(u) = (2\pi)^{-1} \int_{-\infty}^{\infty}\phi_n(t)e^{-itu}dt, \qquad \ldots(5.2.7)$$

the inversion integral on right hand side is absolutely convergent. The absolutely convergent integral provides the continuous p.d.f. that we shall use in our theorems.

Let $\Xi = \{(t, n, x): |t|> \varepsilon n^{\gamma}, n\ge n_o, |x|\ge 1\}$, where ε will be same as in Lemma 5.3.1, and n_o is a large positive integer.

We now make the following assumptions:

[A1] All the d.f.s F_j, $j = 1, 2, \ldots, m$ are absolutely continuous.

[A2] F_j belongs to the domain of normal attraction of the stable law F_o with index α, $\alpha<2$. In case $0< \alpha \leq 1$, F_o is strictly stable. Further let $\{w_j(tn^{-\gamma})\}^n \to w_o(t)$, the c.f. of F_o, as $n \to \infty$.

[A3] $\lim_{n\to\infty} \tau_j/n = t_j>0$, $j = 1, 2, \ldots, m$.

[A4] For some integer $p\geq 1$, $\int_{-\infty}^{\infty}|w_j(t)|^p dt< \infty$, $j = 1, 2, \ldots, m$.

[A5] $\int_{-\infty}^{\infty}|u|^{[\alpha]+1}|v_j(u)-v_o(u)|du< \infty$, $j = 1, 2, \ldots, m$.

Remark 5.2.1: From the proof it will be clear that it is sufficient if we assume that $0< \liminf_{n \to \infty} \tau_j/n$ for all j instead of **[A4]**.

5.3 PRELIMINARY RESULTS:

Now we mention some preliminary lemmas required to prove the theorems of Section 5.1.

Lemma 5.3.1: Under the assumptions **[A1]**, **[A2]** and **[A3]**, there exists positive constants ε, c and C such that for

$k = 0, 1, \ldots, m,$

$$|A_{k,\tau_k}(t,x)| \le C\exp\{-c|t|^{\alpha}\} \qquad \ldots(5.3.1)$$

for all t in the range $|t| \le \varepsilon n^{r}$, all x with $|x| \ge 1$ and all large n.

Throughout the rest of the chapter ε is taken as same as in the Lemma 5.3.1.

Lemma 5.3.2: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**, there exists a polynomial $P_1(.)$ such that for large n, the relation

$$|\alpha_{k,\tau_k}^{\tau_k - j}(tn^{-\gamma},x) - \alpha_{o,\tau_k}^{\tau_k - j}(tn^{-\gamma},x)|$$

$$\le \tau_k^{1-([\alpha]+1)\gamma} P_1(|t|)\exp\{-c|t|^{\alpha}\}, \qquad \ldots(5.3.2)$$

holds for all $(t, n, x) \in \Xi$ and $1 \le j \le \tau_k$, $k = 1, \ldots, m$; where Ξ is defined at p. 106.

Remark 5.3.1: This is a modified version of Lemma 1 in Banys (1977) for i.i.d. case. The proof of Lemma 5.3.2 is along the lines of proof of Lemma 2.3.3.

Now we define two functions, similar to (2.3.18) and (2.3.19), which will be useful in proving the theorems and some of the lemmas. For $j = 1, 2, \ldots, m$, let

$$d_{\tau_j}(t,x)$$

$$= \tau_j [\{\alpha_{j,\tau_j}(tn^{-\gamma},x)\} - \{\alpha_{o,\tau_j}(tn^{-\gamma},x)\}], \qquad \ldots(5.3.3)$$

$$S_{\tau_j}(t,x)$$

$$= \tau_j^{-1} \sum_{h=0}^{\tau_j - 1} \{\alpha_{j,\tau_j}(tn^{-\gamma},x)\}^{h} \{\alpha_{o,\tau_j}(tn^{-\gamma},x)\}^{\tau_j - h - 1} \qquad \ldots(5.3.4)$$

Properties of the function $d_{\tau_k}(t,x)$

Lemma 5.3.3: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**, for all values of t, all x with $|x|\geq 1$ and all large n, we have for k = 1, 2, ... , m,

(i) whenever $0< \alpha< 1$,

$$|d_{\tau_k}(t,x)|\leq \tau_k^{1-\gamma}P_1(|t|), \qquad ...(5.3.5)$$

$$|d_{\tau_k}^{(1)}(t,x)|\leq c_1\tau_k^{1-\gamma}, \qquad ...(5.3.6)$$

(ii) whenever $1\leq \alpha< 2$,

$$|d_{\tau_k}(t,x)|\leq \tau_k^{1-2\gamma}P_2(|t|), \qquad ...(5.3.7)$$

$$|d_{\tau_k}^{(1)}(t,x)|\leq \tau_k^{1-2\gamma}P_3(|t|), \qquad ...(5.3.8)$$

$$|d_{\tau_k}^{(2)}(t,x)|\leq c_1\tau_k^{1-2\gamma}. \qquad ...(5.3.9)$$

Remark 5.3.2: The proof of this lemma is along the lines of proof of Lemma 2.3.4.

Properties of the function $\alpha_{k,\tau_k}(t,x)$

Lemma 5.3.4: Under the assumptions **[A1]**, **[A2]** and **[A3]**, for each fixed n, x and k = 0, 1, ..., m, $\alpha_{k,\tau_k}(tn^{-\gamma},x)$ is differentiable any number of times under the integral sign. For all values of t, all x with $|x|\geq 1$ and all large n, we have for k = 1, 2, ... , m,

(i) whenever $0< \alpha< 1$,

$$|\alpha_{k,\tau_k}^{(1)}(tn^{-\gamma},x)|\leq C|x|^{1-\alpha}\tau_k^{\gamma-1}; \qquad ...(5.3.10)$$

(ii) whenever $1< \alpha< 2$,

$$|\alpha^{(1)}_{k,\tau_k}(tn^{-\gamma},x)|\le \tau_k^{\gamma-1}P_k(|t|) \qquad \text{...(5.3.11)}$$

$$\le |x|^{2-\alpha}\tau_k^{\gamma-1}P_k(|t|), \qquad \text{...(5.3.12)}$$

$$|\alpha^{(2)}_{k,\tau_k}(tn^{-\gamma},x)|\le C_1|x|^{2-\alpha}\tau_k^{2\gamma-1}. \qquad \text{...(5.3.13)}$$

(iii) Also for all $x\neq 0$, $0< \alpha< 2$ and every sufficiently large but fixed integer s, there exists a constant c such that

$$\int_{-\infty}^{\infty}|\alpha_{k,\tau_k}(t,x)|^{\tau_k}dt = O(\tau_k^{-\gamma}), \qquad \text{...(5.3.14)}$$

$$\int_{-\infty}^{\infty}|\alpha_{k,\tau_k}(t,x)|^{2s}dt\le c, \qquad \text{...(5.3.15)}$$

$$\int_{-\infty}^{\infty}|\beta_{k,\tau_k}(t,x)|^{2s}dt\le c. \qquad \text{...(5.3.16)}$$

Remark 5.3.3: The proof of this lemma is along the lines of proof of Lemma 2.3.5.

Properties of the function $S_{\tau_k}(t,x)$

Lemma 5.3.5: Under the assumptions **[A1]**, **[A2]** and **[A3]**, for all $(t, n, x) \in \Xi$ and large τ_k, $k = 1, 2, \ldots, m$ we have

(i) whenever $0< \alpha< 1$

$$|S_{\tau_k}(t,x)|\le C\exp\{-c|t|^{\alpha}\}, \qquad \text{...(5.3.17)}$$

$$|S^{(1)}_{\tau_k}(t,x)|\le C|x|^{1-\alpha}\exp\{-c|t|^{\alpha}\}; \qquad \text{...(5.3.18)}$$

(ii) whenever $1\le \alpha< 2$,

$$|S_{\tau_k}(t,x)|\le C\exp\{-c|t|^{\alpha}\}, \qquad \text{...(5.3.19)}$$

$$|S^{(1)}_{\tau_k}(t,x)| \le |x|^{2-\alpha}\exp\{-c|t|^{\alpha}\}P_1(|t|), \qquad \ldots(5.3.20)$$

$$|S^{(2)}_{\tau_k}(t,x)| \le |x|^{2-\alpha}\exp\{-c|t|^{\alpha}\}P_2(|t|). \qquad \ldots(5.3.21)$$

Remark 5.3.4: The proof of this lemma is along the lines of proof of Lemma 2.3.6.

Lemma 5.3.6: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**, there exist polynomials $P_1(.)$ and $P_2(.)$ such that, for all $(t, n, x) \in \Xi$, we have the following:

(i) whenever $0 < \alpha < 1$,

$$|A_{k,\tau_k}(t,x) - A_{0,\tau_k}(t,x)|$$

$$\le P_1(|t|)\exp\{-c|t|^{\alpha}\}\tau_k^{1-\gamma}, \qquad \ldots(5.3.22)$$

$$|A^{(1)}_{k,\tau_k}(t,x) - A^{(1)}_{0,\tau_k}(t,x)|$$

$$\le |x|^{1-\alpha}P_2(|t|)\exp\{-c|t|^{\alpha}\}\tau_k^{1-\gamma}; \qquad \ldots(5.3.23)$$

(ii) whenever $1 \le \alpha < 2$,

$$|A_{k,\tau_k}(t,x) - A_{0,\tau_k}(t,x)|$$

$$\le P_1(|t|)\exp\{-c|t|^{\alpha}\}\tau_k^{1-2\gamma}, \qquad \ldots(5.3.24)$$

$$|A^{(1)}_{k,\tau_k}(t,x) - A^{(1)}_{0,\tau_k}(t,x)|$$

$$\le |x|^{2-\alpha}P_i(|t|)\exp\{-c|t|^{\alpha}\}\tau_k^{1-2\gamma} \qquad \ldots(5.3.25)$$

$i = 1, 2;\ k = 1, 2, \ldots, m.$

Remark 5.3.5: The proof of this lemma is along the lines of proof of Lemma 2.3.7.

Lemma 5.3.7: Under the assumptions **[A1]**, **[A2]** and **[A3]**, there exist polynomials $P_1(.)$ and $P_2(.)$ such that for all $(t, n, x) \in \Xi$, we have,

(i) whenever $0< \alpha< 1$,

$$|A^{(1)}_{k,\tau_k}(t,x)| \le C|x|^{1-\alpha}\exp\{-c|t|^{\alpha}\}; \quad \ldots(5.3.26)$$

(ii) whenever $1\le \alpha< 2$,

$$|A^{(1)}_{k,\tau_k}(t,x)| \le P_1(|t|)\exp\{-c|t|^{\alpha}\} \quad \ldots(5.3.27)$$

$$\le |x|^{2-\alpha}P_1(|t|)\exp\{-c|t|^{\alpha}\}; \quad \ldots(5.3.28)$$

$$|A^{(2)}_{k,\tau_k}(t,x)| \le |x|^{2-\alpha}P_2(|t|)\exp\{-c|t|^{\alpha}\}. \quad \ldots(5.3.29)$$

Remark 5.3.6: The proof of this lemma is along the lines of proof of Lemma 2.3.8.

Lemma 5.3.8: Assume that **[A1] and [A2]** hold. Let $\varepsilon>0$ and integer n_O be fixed and let

for $k = 1, 2, \ldots, m$,

$$\mu_{(O,k)} \equiv \sup_{\Theta}|\alpha_{O,\tau_k}(t,x)|, \quad \ldots(5.3.30)$$

$$\mu_{(k,k)} \equiv \sup_{\Theta}|\alpha_{k,\tau_k}(t,x)|, \quad \ldots(5.3.31)$$

where $\Theta = \{(t,n,x)\,|\,|t|> \varepsilon,\ n\ge n_O,\ |x|\ge 1\}$.

Then, $0\le \mu_{(O,k)}< 1$ and $0\le \mu_{(k,k)}< 1$.

Remark 5.3.7: The proof of this lemma is along the lines of proof of Lemma 2.3.9.

Set $\mu = \max_{1 \le k \le m} (\mu_{(O,k)}, \mu_{(k,k)})$.

Lemma 5.3.9: Assume that **[A1]**, **[A2]**, **[A3]** and **[A5]** hold. Let g(t,x) be a complex-valued continuous function such that $|g(t,x)| \le \max(1, c|x|^{-\alpha})$ for all x with $|x| \ge 1$ and for all t. Then, for k = 1, 2, ... , m, as $n \to \infty$,

$$\left| \int_{-\infty}^{\infty} (B_{k,\tau_k}(t,x) - B_{O,\tau_k}(t,x)) g(t,x) \exp(-itx) dt \right|$$
$$= |x|^{-\alpha} O(\tau_k^{1-([\alpha]+1)\gamma}) \quad \ldots (5.3.32)$$

Remark 5.3.8: The proof of this lemma is along the lines of proof of Lemma 2.3.10.

Lemma 5.3.10: Under the assumptions **[A1]** and **[A2]**, for all the values of t and all x with $|x| \ge 1$ and n large, we have

$$|B_{k,\tau_k}(t,x)| \le c|x|^{-\alpha}, \quad \ldots (5.3.33)$$

k = 0, 1, ... , m.

Remark 5.3.9: The proof of this lemma is along the lines of proof of Lemma 2.3.11.

We shall now prove a result which is an extension of Lemma 2.3.12 to non-identically distributed case under consideration.

Lemma 5.3.11: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**, there exists a constant c>o such that, for all (t, n, x) $\in \Xi$, we have

$$|\phi_n(t)-w_o(t)|$$
$$\leq n^{1-([\alpha]+1)\gamma}P_1(|t|)\exp\{-c|t|^{\alpha}\}. \qquad .(5.3.34)$$

Here $\phi_n(t) = E[\exp(itS_n/B_n)]$.

Proof: Using the definition of $\phi_n(t)$, the equation (5.2.1) and by adding and subtracting the terms

$\prod_{i=1}^{j}\{w_i(tn^{-\gamma})\}^{\tau_i}\prod_{k=i+1}^{m}\{w_o(tn^{-\gamma})\}^{\tau_k}$ for j = 1, 2, ... , m-1,

we get on simplification

$$|\phi_n(t)-w_o(t)|$$
$$\leq \sum_{i=1}^{m}|\{w_i(tn^{-\gamma})\}^{\tau_i}-\{w_o(tn^{-\gamma})\}^{\tau_i}|$$
$$\leq n^{1-([\alpha]+1)\gamma}P_1(t)\exp\{-c|t|^{\alpha}\},$$ using Lemma 2.3.12.□

5.4 PROOFS OF THE THEOREMS:

Proof of Theorem 5.1.1:

We shall prove the theorem for m = 2. In case of m>2 but fixed, the proof involves similar steps.

We shall prove the relation

$$\sup_{x\in R}|v_n(x)-v_o(x)| = O(n^{1-([\alpha]+1)\gamma}). \qquad ...(5.4.1)$$

The inversion formula for continuous density gives that

$$2\pi|v_n(x)-v_o(x)|\leq I_{1n}+I_{2n}+I_{3n}, \qquad ...(5.4.2)$$

where

$$I_{1n} = \int_{|t|\le\varepsilon n^{\gamma}} |\phi_n(t) - w_o(t)|dt,$$

$$I_{2n} = \int_{|t|>\varepsilon n^{\gamma}} |\phi_n(t)|dt$$

$$= \int_{|t|>\varepsilon n^{\gamma}} |w_1(tn^{-\gamma})|^{\tau_1}|w_2(tn^{-\gamma})|^{\tau_2}dt$$

and $I_{3n} = \int_{|t|>\varepsilon n^{\gamma}} |w_o(tn^{-\gamma})|^{n}dt,$

$\varepsilon>0$ being as in Lemma 5.3.1.

By Lemma 5.3.11 it now follows that

$$I_{1n} = O(n^{1-([\alpha]+1)\gamma}). \qquad \ldots(5.4.3)$$

By **[A1]**, the d.f.s F_o, F_1 and F_2 are absolutely continuous. We have from the canonical representation of a stable law, $w_o(t) = \{w_o(tn^{-\gamma})\}^{\tau_1}\{w_o(tn^{-\gamma})\}^{\tau_2}$. Also note that there exists, for any $\varepsilon> 0$, a $c(\varepsilon)> 0$ such that $|w_i(t)|\le \exp(-c(\varepsilon))$, $|t|> \varepsilon$, for i = 0, 1, 2. Therefore,

$$I_{2n}\le n^{\gamma} \int_{|t|>\varepsilon} \exp\{-c(\varepsilon)(n-2p)\}|w_1(t)|^{p}|w_2(t)|^{p}dt$$

$$\le n^{\gamma}\exp\{-c(\varepsilon)(n-2p)\} \int_{|t|>\varepsilon} |w_1(t)|^{p}dt$$

$$\le cn^{\gamma}\exp\{-c(\varepsilon)(n-2p)\}$$

$$= O(n^{1-([\alpha])+1)\gamma}) \qquad \ldots(5.4.4)$$

and

$$I_{3n} = O(n^{1-([\alpha]+1)\gamma}). \qquad \ldots(5.4.5)$$

Thus equation (5.4.1) follows from the relations (5.4.2) through (5.4.5).

Proof of Theorem 5.1.2:

We shall prove the theorem for the case $0< \alpha< 1$ and $m = 3$. The case $1\leq \alpha< 2$ can be handled similarly. Also the case $m = 2$ can be worked out exactly on the similar lines. In case $m> 3$ but fixed, the proof will be exactly similar to the case presented here. Modifications necessary for the general case are discussed in Section 5.5.

Note that in view of Theorem 5.1.1, it is sufficient for us to prove

$$\sup_{|x|\geq 1} |x|^{\alpha}|v_n(x)-v_o(x)| = O(n^{1-\gamma}). \qquad \ldots(5.4.6)$$

Consider, for $|x|\geq 1$,

$$|x|^{\alpha}|\dot{v}_n(x)-v_o(x)|$$

$$\leq |x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)\,[\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_2(tn^{-\gamma})\}^{\tau_2}\{w_3(tn^{-\gamma})\}^{\tau_3}$$

$$-\{w_o(tn^{-\gamma})\}^{\tau_1}\{w_o(tn^{-\gamma})\}^{\tau_2}\{w_o(tn^{-\gamma})\}^{\tau_3}]\,dt|$$

$$\leq \sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)\,\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_2(tn^{-\gamma})\}^{\tau_2}$$

$$[\{w_3(tn^{-\gamma})\}^{\tau_3}-\{w_o(tn^{-\gamma})\}^{\tau_3}]\,dt|$$

$$+\sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)\,\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_o(tn^{-\gamma})\}^{\tau_3}$$

$$[\{w_2(tn^{-\gamma})\}^{\tau_2}-\{w_o(tn^{-\gamma})\}^{\tau_2}]\,dt|$$

$$+\sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)\,\{w_o(tn^{-\gamma})\}^{\tau_2}\{w_o(tn^{-\gamma})\}^{\tau_3}$$

$$[\{w_1(tn^{-\gamma})\}^{\tau_1}-\{w_o(tn^{-\gamma})\}^{\tau_1}]\,dt|$$

$= W_{12} + W_{13} + W_{23}$. . (5.4.7)

In order to prove (5.4.6) it is sufficient to prove that

$W_{12} = O(n^{1-\gamma})$, ...(5.4.8)

$W_{13} = O(n^{1-\gamma})$, ...(5.4.9)

$W_{23} = O(n^{1-\gamma})$. ...(5.4.10)

We shall prove equation (5.4.8) only. Equation (5.4.9) and (5.4.10) can be proved similarly.

Observe that using (5.2.5) and (5.2.6), we can write

$$\int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_2(tn^{-\gamma})\}^{\tau_2}$$

$$[\{w_3(tn^{-\gamma})\}^{\tau_3}-\{w_o(tn^{-\gamma})\}^{\tau_3}]dt$$

$$= \int_{-\infty}^{\infty} \exp(-itx)\{A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)+A_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)$$

$$+B_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)+B_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)\}$$

$$[A_{3,\tau_3}(t,x)-A_{o,\tau_3}(t,x)]dt$$

$$+\int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_2(tn^{-\gamma})\}^{\tau_2}$$

$$[B_{3,\tau_3}(t,x)-B_{o,\tau_3}(t,x)]dt$$

$= I(A_1A_2)+I(A_1B_2)+I(B_1A_2)+I(B_1B_2)+I(B)$, say. ...(5.4.11)

Estimate of $I(A_1A_2)$: We shall prove that

$|I(A_1A_2)| = |x|^{-\alpha}O(n^{1-\gamma})$. ...(5.4.12)

First of all we consider the integral

$$\int_{-\infty}^{\infty}\exp(-itx)A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)A_{3,\tau_3}(t,x)\,dt.$$

Because $A_{k,\tau_k}(t,x)$, $A^{(1)}_{k,\tau_k}(t,x)$ and $A^{(2)}_{o,\tau_k}(t,x)$ are absolutely integrable, simple techniques involving integration by parts give us

$$\int_{-\infty}^{\infty}\exp(-itx)A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)A_{3,\tau_3}(t,x)\,dt$$

$$=ix^{-1}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\exp(-itx)\prod_{\substack{j=1\\ j\neq k}}^{3}A_{j,\tau_j}(t,x)A^{(1)}_{k,\tau_k}(t,x)\,dt. \qquad .(5.4.13)$$

Evaluating

$$\int_{-\infty}^{\infty}\exp(-itx)A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)A_{o,\tau_3}(t,x)\,dt$$

on the lines of (5.4.13), we have then

$$|I(A_1A_2)| = \Big|\int_{-\infty}^{\infty}\exp(-itx)A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)[A_{3,\tau_3}(t,x)-A_{o,\tau_3}(t,x)]\Big|dt$$

$$\leq |x|^{-1}\Big[\int_{|t|\leq\varepsilon n^{\gamma}}+\int_{|t|>\varepsilon n^{\gamma}}\Big]|A^{(1)}_{1,\tau_1}(t,x)||A_{2,\tau_2}(t,x)||A_{3,\tau_3}(t,x)-A_{o,\tau_3}(t,x)|dt$$

$$+|x|^{-1}\Big[\int_{|t|\leq\varepsilon n^{\gamma}}+\int_{|t|>\varepsilon n^{\gamma}}\Big]|A_{1,\tau_1}(t,x)||A^{(1)}_{2,\tau_2}(t,x)||A_{3,\tau_3}(t,x)-A_{o,\tau_3}(t,x)|dt$$

$$+|x|^{-1}\Big[\int_{|t|\leq\varepsilon n^{\gamma}}+\int_{|t|>\varepsilon n^{\gamma}}\Big]|A_{1,\tau_1}(t,x)||A_{2,\tau_2}(t,x)||A^{(1)}_{3,\tau_3}(t,x)-A^{(1)}_{o,\tau_3}(t,x)|dt$$

$$= M_1(x)+\ldots+M_6(x), \text{ say.} \qquad (5\ 4.14)$$

Observe that (5.3.22) and (5.3.26) together with Lemma 5.3.1 imply that,

$$|M_i(x)| = |x|^{-\alpha} O(n^{1-\gamma}), \text{ for } i = 1, 3; \qquad \ldots(5.4.15)$$

whereas Lemma 5.3.1 and (5.3.23) imply that,

$$|M_5(x)| = |x|^{-\alpha} O(n^{1-\gamma}). \qquad \ldots(5.4.16)$$

Finally, as a consequence of equation (5.3.10), the assumption **[A2]** and Lemma 5.3.8, we get for i = 2, 4, 6,

$$|M_i(x)| = |x|^{-\alpha} o(n^{1-\gamma}). \qquad \ldots(5.4.17)$$

Thus from equation (5.4.14) to (5.4.17) it follows that,

$$|I(A_1A_2)| = |x|^{-\alpha} O(n^{1-\gamma}) \qquad \ldots(5.4.18)$$

which is same as (5.4.12).

Estimate of $I(A_1B_2)$: Write $I(A_1B_2)$ as

$$I(A_1B_2) = \Big(\int_{|t|\le\varepsilon n^{\gamma}} + \int_{|t|>\varepsilon n^{\gamma}} \Big) \exp(-itx) A_{1,\tau_1}(t,x) B_{2,\tau_2}(t,x)$$

$$\{A_{3,\tau_3}(t,x) - A_{0,\tau_3}(t,x)\} dt$$

$$= I_1(A_1B_2) + I_2(A_1B_2), \text{ say.} \qquad \ldots(5.4.19)$$

Now, $|I_1(A_1B_2)| = |x|^{-\alpha} O(n^{1-\gamma})$, $\qquad \ldots(5.4.20)$

is evident from Lemmas 5.3.1 and 5.3.10 and (5.3.22); whereas, using Lemmas 5.3.8 and 5.3.10, we get .

$$|I_2(A_1B_2)| \le c|x|^{-\alpha} \int_{|t|>\varepsilon n^{\gamma}} |A_{1,\tau_1}(t,x)| dt$$

$$\le c|x|^{-\alpha} n^{\gamma} \mu^{\tau_1 - p} \int_{|t|>\varepsilon} |\alpha_{1,\tau_1}(t,x)|^p dt$$

$\leq c|x|^{-\alpha} n^{\gamma} \mu^{\tau_1 - p}$.

Therefore, it follows that,

$$|I_2(A_1B_2)| = |x|^{-\alpha} O(n^{1-\gamma}). \qquad ...(5.4.21)$$

Thus, $|I(A_1B_2)| = |x|^{-\alpha} O(n^{1-\gamma})$, ..(5.4.22)

follows from (5.4.19), (5.4.20) and (5.4.21). On similar lines we can prove

$$|I(B_1A_2)| = |x|^{-\alpha} O(n^{1-\gamma}). \qquad ...(5.4.23)$$

Observing the fact that $|B_{k,\tau_k}(t,x)| \leq \max(1, c|x|^{-\alpha})$ for k = 1, 2, 3, and once again using the techniques used to get estimate of $I(A_1B_2)$ we get

$$|I(B_1B_2)| = |x|^{-\alpha} O(n^{1-\gamma}). \qquad ...(5.4.24)$$

Estimate of I(B): we have

$$I(B) = {}_{-\infty}\int^{\infty} \exp(-itx)\{w_1(tn^{-\gamma})\}^{\tau_1}\{w_2(tn^{-\gamma})\}^{\tau_2}$$

$$[B_{3,\tau_3}(t,x) - B_{o,\tau_3}(t,x)]dt.$$

Note that

$\{A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x) + A_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)$

$+B_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x) + B_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)\}$

is a complex valued function with absolute value of each summand (component) being less than or equal to $\max(1, c|x|^{-\alpha})$. Each component satisfies all the properties of the function $g(t,x)$ introduced in Lemma 5.3.9. We therefore take each component $g_j(t,x)$, say, j = 1, 2, 3, 4 as $g(t,x)$ of Lemma 5.3.9 and apply Lemma 5.3.9.

Therefore,

$|I(B)|$

$$=|\int_{-\infty}^{\infty}\exp(-itx)\{A_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)+A_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)$$

$$+B_{1,\tau_1}(t,x)A_{2,\tau_2}(t,x)+B_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x)\}$$

$$[B_{3,\tau_3}(t,x)-B_{o,\tau_3}(t,x)]dt|$$

$$\leq \sum_{j=1}^{4}|\int_{-\infty}^{\infty}\exp(-itx)g_j(t,x)[B_{3,\tau_3}(t,x)-B_{o,\tau_3}(t,x)]dt|$$

$$\leq \sum_{j=1}^{4}|x|^{-\alpha}O(n^{1-\gamma})$$

$$\leq |x|^{-\alpha}O(n^{1-\gamma}),\ \text{using Lemma 5.3.9.} \qquad \ldots(5.4.25)$$

Equation(5.4.8) now follows from (5.4.11), (5.4.18), (5.4.22), (5.4.23), (5.4.24) and (5.4.25). In view of the remarks following equations (5.4.10) the proof of Theorem 5.1.2 is complete.□

5.5 GENERAL CASE:

Remark 5.5.1: In the case m>3 (but fixed) in place of (5.4.7) we will have

$$\sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)$$

$$[\prod_{k=1}^{m}\{w_k(tn^{-\gamma})\}^{\tau_k}-\prod_{k=1}^{m}\{w_o(tn^{-\gamma})\}^{\tau_k}]dt|$$

$$\le \sup_{|x|\ge 1} |x|^{\alpha} \left| \int_{-\infty}^{\infty} \exp(-itx) \prod_{k=1}^{m-1} \{w_k(tn^{-\gamma})\}^{\tau_k} \right.$$

$$\left. [\{w_m(tn^{-\gamma})\}^{\tau_m} - \{w_o(tn^{-\gamma})\}^{\tau_m} dt \right|$$

$$+ \sum_{s=2}^{m-1} \sup_{|x|\ge 1} |x|^{\alpha} \left| \int_{-\infty}^{\infty} \exp(-itx) \right.$$

$$\prod_{k=1}^{m-s} \{w_k(tn^{-\gamma})\}^{\tau_k} \prod_{j=m-s+2}^{m} \{w_o(tn^{-\gamma})\}^{\tau_j}$$

$$\left. [\{w_{m-s+1}(tn^{-\gamma})\}^{\tau_{m-s+1}} - \{w_o(tn^{-\gamma})\}^{\tau_{m-s+1}}] dt \right|$$

$$+ \sup_{|x|\ge 1} |x|^{\alpha} \left| \int_{-\infty}^{\infty} \exp(-itx) \prod_{k=2}^{m} \{w_o(tn^{-\gamma})\}^{\tau_k} \right.$$

$$\left. [\{w_1(tn^{-\gamma})\}^{\tau_1} - \{w_o(tn^{-\gamma})\}^{\tau_1}] dt \right|$$

As in Theorem 5.1.2, we shall consider 1^{st} term only. Proceeding as before, this can be expressed as sum of five terms say I_1, I_2, I_3, I_4 and I_5, similar to (5.4.11) where I_1 has in the integrand the product term involving A_1, ..., A_{m-1} with (A_m-A_o), I_2 is the sum of (m-1) integrals with each integral containing the product of one B_i, $i \le m-1$ with (m-2) A_i's and (A_m-A_o), I_3 is the sum of $2^{m-1}-(m+1)$ integrals with each integrand being the product of (m-1) terms with (A_m-A_o) of which at least two are B_i's and at least one is A_i; I_4 is an integral whose integrand is the product of (m-1) B_i with (A_m-A_o) and I_5 is an integral whose integrand is the product

$$\prod_{k=1}^{m} \{w_k(tn^{-\gamma})\}^{\tau_k}(B_m-B_o).$$

Proceeding as in equations (5.4.12) to (5.4.25) we get $O(n^{1-\gamma})$.

Remark 5.5.2: It is well-known that the limit distribution of normalized sums of independent r.v.s exists irrespective of the sampling scheme under consideration (see: Sreehari (1970)). We are unable to prove the rate in the local theorem of this result in case $\tau_i/n \to 0$ as $n \to \infty$ for some i, mainly because of the failure of some of our estimates to hold in this case.

Remark 5.5.3: In the case m = 2, terms containing either one A factor or one B factor will appear. For m> 2, factors involving A's and B's simultaneously occur and it is more complex to handle. For this reason we have presented the case with m = 3.

CONCLUDING REMARKS:

In this chapter we have considered a situation wherein observation come from m population F_1, F_2, ..., F_m, where each F_j belongs to the domain of normal attraction of a (single) stable law with $\alpha \neq 1$, $\alpha \neq 2$. In chapter 6, we relax the assumption that all F_j are in the domain of attraction of the same stable law and obtain uniform and non-uniform rates of convergence type results.