CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION:

The problems investigated in this thesis are all related to the celebrated Central Limit Problem. Major part of the thesis deals with problems concerning Gnedenko's Local Limit Theorem and its extensions. We investigate some related problems connected with two important areas of research proposed by Prof. B.V.Gnedenko, viz., limit distribution of sums of a random number of random variables and limit distributions of sums of random observations from a fixed number of populations.

Suppose $(\Omega, \mathbb{F}, P)$ is a fixed probability space and $X_1, X_2, \ldots$ are random variables (r.v.s) defined on this probability space. Let F_n be the distribution function (d.f.) of r.v. X_n. Suppose $S_k = X_1 + \ldots + X_k$ for $k = 1, 2, \ldots$.

Throughout the study we shall be concerned with r.v.s X_n which are assumed to be mutually independent.

The investigation of the limit distribution of $(S_n - A_n)/B_n$ for appropriate choices of reals A_n and $B_n > 0$, is known as the **Central Limit Problem.**

The class of all possible limit distributions of $(S_n - A_n)/B_n$, is the class of **stable distributions** if the r.v.s X_n are identically distributed. This means, that if d.f. of r.v. $(S_n-A_n)/B_n$ converge to G, where G is a d.f., at every continuity point of G, then G is a stable d.f. In this case, we say that F, d.f. of r.v. X_1, is **attracted** to the stable law G. The totality of all d.f.s F attracted to the stable law G is said to be the **domain of attraction** of the stable law G. It may be noted that only the stable laws have non-empty domains of attractions.

A closely related classical problem of research in probability theory is to find criteria for the existence of the density version of the Central Limit Theorem. This problem is known as a Local Limit Theorem. More specifically, any assertion to the effect that a sequence of probability densities $\{p_n(x)\}$ with the corresponding sequence of d.f.s $\{F_n(x)\}$, converges to a probability density p(x) with the corresponding d.f. G(x), where $F_n \Rightarrow G$, is called the **Local Limit Theorem.**

Suppose X_1, X_2, ... is a sequence of independent and identically distributed r.v.s (i.i.d.r.v.s) having a common d.f. F with mean zero $(E(X_1) = 0)$ and variance unity $(V(X_1) = 1)$. If Φ denotes the d.f. of standard normal law N(O, 1) and if $Z_n = (X_1+...+X_n)/n^{1/2}$ then the Central Limit Theorem asserts that

$$\sup_{x \in R}|P(Z_n \leq x)-\Phi(x)| \to 0 \text{ as } n \to \infty. \qquad ...(1)$$

Here, the d.f. F belongs to the domain of (normal) attraction of the standard normal law. Denote the density of r.v. Z_n by $p_n(x)$, if it exists. Gnedenko (1954) showed: In order that

$$\sup_{x \in R}|p_n(x)-\phi(x)| \to 0 \text{ as } n \to \infty. \qquad \ldots(2)$$

it is necessary and sufficient that there exists an $n_o \in \{1, 2, \ldots\}$ such that $p_{n_o}(x)$ is bounded and d.f. of Z_n converge weakly to Φ. Here, $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$, $x \in R$. Moreover, Ibragimov (see: Ibragimov and Linnik, 1971, p.135) proved:

$$\sup_{x \in R}|p_n(x)-\phi(x)| = O(n^{-\delta/2}),\ 0< \delta< 1, \qquad \ldots(3)$$

as $n \to \infty$ iff there exists an $n_o \in \{1, 2, \ldots\}$ such that $p_{n_o}(x)$ is bounded and

$$\int_{|x|>z} x^2 dF(x) = O(z^{-\delta}) \text{ as } z \to \infty. \qquad \ldots(4)$$

We discuss an important result in this direction: Suppose the r.v.s $\{X_n\}$ are independent with finite third moment. Then the Berry-Esseen inequality for independent r.v.s is well-known (Berry (1941), Esseen (1945)):

$$\sup_{x \in R}|P(Z_n \leq x)-\Phi(x)| \leq c_1 B_n^{-3}\sum_{j=1}^{n} E|X_j|^3 \qquad \ldots(5)$$

where $B_n^2 = V(S_n)$ and c_1 is an absolute constant.

Now observe the following result:

Let $B_n \to \infty$ and $\sum_{j=1}^{n} E|X_j|^3 = O(B_n^2)$ as $n \to \infty$. $\qquad \ldots(6)$

Suppose that

$$\int_{|t|>\varepsilon} \prod_{j=1}^{n} |Ee^{itX_j}|dt = O(B_n^{-2}) \text{ as } n \to \infty, \qquad \ldots(7)$$

for every fixed $\varepsilon > 0$. Then

$$\sup_{x \in R}|p_n(x)-\phi(x)| = O(B_n^{-1}), \text{ as } n \to \infty. \qquad \ldots(8)$$

This result is due to Petrov (1956). By hypothesis (6), we have

$$B_n^{-3} \sum_{j=1}^{n} E|X_j|^3 = O(B_n^{-1}) \text{ as } n \to \infty. \qquad \ldots(9)$$

Observe that in view of equation (9), equation (8) is analogous to the inequality (5) for sufficiently large n.

If the characteristic function of the r.v. X_1 is absolutely integrable in the r^{th} power for some integer $r \geq 1$, Basu (1978) and Basu and Maejima (1979) obtained non-uniform estimates and L_p-versions of the local limit theorem for a sequence of independent and identically distributed r.v.s. Refinements of these theorems with normal and/or non-normal stable limit laws were obtained by Smith (1953), Banys (1972, 1974), Smith and Basu (1974), Petrov (1975), Maejima (1978, 1980), Basu, Maejima and Patra (1979), Basu and Maejima (1980) and others.

Any limit theorem is only a description of the convergence behaviour of a mathematical quantity for indefinitely large values of its parameter. If the theorem is to have much application outside its own restricted context, we require some knowledge of how large these parameters must be before the quantity can be approximated by its limits. In other words, we need to know the **rate of convergence** of such a mathematical quantity to its limit.

Despite their great utility, very few results exist in the direction of rates of convergence in the local limit theorems for independent summands with stable limit laws. This is mainly due to complexities involved in the dependence of local behaviour of the S_n on the individual summands in any given local limit problem.

At this stage, we introduce the notion of uniform and non-uniform rates of convergence in local limit theorems.

Clearly, from (3), $\sup_{x\in R}|p_n(x)-\phi(x)| = o(n^{-\delta/2}\theta_n)$, where $\theta_n \to \infty$, as $n \to \infty$. We then say that there is **uniform** rate of convergence of $n^{-\delta/2}\theta_n$ in the local limit theorem provided $n^{-\delta/2}\theta_n \to 0$.

If we have a result of the type

$\sup_{x\in R} \Psi(x)|p_n(x)-\phi(x)| = o(\delta_n)$, as $n \to \infty$,

for some $\Psi(x)$, where $\delta_n \to 0$, we shall say that δ_n is the estimate of the rate of the convergence in the **non-uniform** local limit theorem.

We note that Smith and Basu (1974) and Basu (1978) obtained uniform as well as non-uniform rates of convergence in the local limit theorem when the common d.f. F belongs to the domain of **normal** attraction of the (standard) normal law.

Using the assumption of the existence of psuedo-moments up to certain orders and using the techniques of Smith (1953), Hoglund (1970), Banys (1977), Basu, Maejima and Patra (1979) established uniform and non-uniform rates of convergence in the local limit theorem when the common d.f. F belongs to the domain of **normal** attraction of a non-normal stable law.

So far, no rates of convergence results are found to be reported in the local limit theory literature with the assumption that the common d.f. F of i.i.d.r.v.s $\{X_n\}$ belongs to the domain of "**non-normal**" attraction of a normal or non-normal stable law. In this context we recall the 5th unsolved problem proposed by Ibragimov and Linnik in their famous book: Independent And Stationary Sequences Of Random Variables (Chap.20, Some Unsolved Problems, p.391), which we solve in the Chapters 3 and 4.

The uniform rates of convergence and non-uniform bound in the local limit theorem for the densities of normalized sums of i.i.d.r.v.s with common d.f. F belonging to the domain of attraction of the (standard) normal law are obtained in Chapter 3.

In Chapter 4, we obtain uniform rates of convergence in the local limit theorem for densities of normalized sums of i.i.d.r.v.s with common d.f. F belonging to the domain of **non-normal** attraction of the stable law with index α, $0< \alpha< 2$, $\alpha \neq 1$.

If we consider a sequence of independent but non-identically distributed r.v.s, the existence of a bounded density together with a condition ensuring an integral limit theorem is necessary but not sufficient for the local limit relationship (2) to hold. As already mentioned, the most complicated situation concerning the local limit theorem is dependence of local behaviour of the sum S_n on the individual summands. Different sufficient conditions for the local limit theorem, especially for estimates of type (3), were introduced by Smith (1953), Petrov (1956), Survila (1964), Smith and Basu (1974) and others.

As has been stated in beginning of this introduction that the whole work in this thesis is an attempt to solve some of the problems posed by Prof. B.V.Gnedenko in the area of local limit theory; we quote the following from Gnedenko (1962, p.168, The classical Problematic):

"*Among the big problems in the theory of sums of independent random variables, I want to mention the following areas:*

Until now there is no precise evaluation of speed of convergence of distribution functions of sums of the limit distribution and also the speed of approximating them by corresponding infinitely divisible distributions.

The local limit theorems for variables not identically distributed are still far from complete.

The theory of limit theorems for sums of independent terms should be extended to more general cases. ...".

We discuss first, at this stage, the well-known problem for independent but non-identically distributed r.v.s considered by Gnedenko around 1952.

Let $\{X_n\}$ be a sequence of mutually independent r.v.s. Suppose that for appropriate choices of the normalizing constants $\{A_n\}$ and $\{B_n\}$, $B_n \to \infty$, the normalized sums $Z_n = (S_n - A_n)/B_n$ converge to a proper r.v. with d.f. G. The problem of characterizing the class of distributions $\{G\}$ when among the distributions of the summands X_i's there are only r different ones was posed by Prof.B.V.Gnedenko.

Let this class be denoted by P_r. It is well known that the class P_1 coincides with the class of stable laws. Zolotarev and Korolyuk (1961) proved that P_2 is the class of compositions of pairs of stable laws. Zinger (1965a, 1965b) proved that the class P_r is broader than the class of compositions of stable laws for $r > 2$. He also characterized the class P_r.

Sreehari (1970) considered this problem in greater detail under the restricted hypothesis that the distributions of the summands are in the domains of attraction of p, $1 \leq p \leq r$, stable laws. He proved that:

(i) when $p = 1$, constants A_n and B_n can always be found such that the limit distribution of Z_n exists; and

(ii) the class P_r is the class of compositions of one or more of the above p stable laws.

He also obtained necessary and sufficient conditions for the limit distribution of Z_n to exist. Mason (1970) also studied this problem.

We establish the local version of the Central limit problem considered by Sreehari in (i) above, by working out the uniform rates of convergence in the local limit theorem when $p = 1$. We, also, obtain the non-uniform rates of convergence in this case. These results are discussed in Chapter 5. Results of this chapter are published in Kirkire (1994).

Chapter 6 discusses the local version of the central limit problem considered by Sreehari in (ii) above, under the assumption that the limit distribution of Z_n exists. The uniform as well as non-uniform rates of convergence in local limit theorem are worked out here.

We shall call the problems considered in Chapters 5 and 6 as local limit problems concerning the **fixed sampling scheme.**

Another important area of research in probability theory is to identify the class of limit distributions of sums of random number of random variables. This problem was also discussed by Prof.B.V.Gnedenko around 1968 when he found interesting applications of this in the problems in Physics, Economics and Reliability Theory.

We consider an interesting problem which is a mixture of two different problems proposed by Gnedenko.

Suppose $\{X_n\}$ is a sequence of independent r.v.s with corresponding sequence of d.f. $\{F_n\}$. Suppose for each n, $F_n \in \{G_1, G_2\}$ and G_i is in the domain of normal attraction of a symmetric stable law H_i with index α_i, $0< \alpha_i< 2$, $\alpha_i \neq 1$, $i = 1, 2$.

If, for every fixed n, the numbers of observations coming from G_1 and G_2 are random numbers then what is the limit distribution of the normalized sums Z_n? Here $Z_n = (X_1+\ldots+X_n-A_n)/B_n$, $B_n \to \infty$ as $n \to \infty$.

This type of situations are encountered in practice more naturally, e.g., suppose, for every fixed n, we toss a coin and if head turns up, we take an observation from G_1; otherwise from G_2. The dependence of the sampling scheme on a random mechanism for the purpose of sampling is more natural and application oriented.

The problem mentioned above has been solved in Chapter 7. Result presented here was published in Kirkire (1991).

Though the Central limit problem with random sampling scheme of Chapter 7 produces interesting problems, we do not discuss them further in this study.

It is natural, at this stage, to think of the local analog of the central limit problem with **random sampling scheme** considered in Chapter 7. In fact the local version of the random central limit theorem itself has been reported by Korolev (1992) under the assumption of the existence of the finite variance of the i.i.d.r.v.s $\{X_n\}$. Korolev's results are approximation type results. We give some preliminary results in this direction in Chapter 8.

In Chapter 2, we discuss the main properties of the stable distributions, and introduce basic notations and some preliminary results that are used frequently throughout the thesis. Some of the results are new and may be of independent interest.

Finally in Chapter 9, we discuss applications of some of the results obtained in the thesis. The basic problems considered in Chapters 3, 4, 5 and 6 generate interesting class of problems in the local limit theory. These problems are discussed at the end of Chapters 9. These open problems in the area of local limit theory are presented for future study.

For convenience, " m^{th} Section of Chapter l" and " n^{th} equation in the m^{th} Section of Chapter l" will be denoted by "**l.m**" and "(l.m.n)", respectively.