CHAPTER 3

RATES OF CONVERGENCE IN LOCAL LIMIT THEOREM: DOMAIN OF NON-NORMAL ATTRACTION OF THE STANDARD NORMAL LAW CASE

3.1 INTRODUCTION:

Let $\{X_n\}$ be a sequence of independent r.v.s each having a common d.f. F. Suppose that F belongs to the domain of non-normal attraction of the standard normal law Φ. Thus, there exist real sequences $\{A_n\}$ and $\{B_n, B_n> 0\}$ such that $Z_n = (S_n-A_n)/B_n$ converges in law to the standard normal r.v. It is well known that A_n may be taken as $nE(X_1)$ and B_n is of the form $B_n = n^{1/2}L(n)$, where L(n) varies slowly at infinity in the sense of Karamata, and $L(n) \to \infty$ as $n \to \infty$ (see: Feller (1971)).

It is known that (e.g. Gnedenko and Kolmogorov (1954, p.227) and Ibragimov and Linnik (1971, p.126)) for large n, Z_n has a p.d.f. v_n for which, as $n \to \infty$,

$$\Delta_n(x) = |v_n(x)-\phi(x)| = o(1) \qquad ...(3.1.1)$$

uniformly in x iff the c.f. f of r.v. X_1 is such that for some integer $r\geq 1$,

$$\int_{-\infty}^{\infty}|f(t)|^r dt < \infty, \qquad ...(3.1.2)$$

$\phi(x)$ being the standard normal p.d.f. In this connection, we may note that the conditions quoted in Gnedenko and Kolmogorov(1954) and Ibragimov and Linnik (1971) in place of (3.1.2) above are equivalent to the later. This is a direct consequence of Theorem 96 in Titchmarsh (1937, p.96).

Smith(1953) and later Smith and Basu(1974) showed that in the case of $EX_1^2 < \infty$ or equivalently if L(n) is constant, actually one can go a step further to show that

$$\sup_{x \in R}(1+|x|)^{\beta}\Delta_n(x) = o(1) \qquad \ldots(3.1.3)$$

for every $\beta \le 2$ iff (3.1.2) holds. In the following theorem, we find that dropping of the assumption of finiteness of EX_1^2 reflects correspondingly in the non-uniform rate of convergence of $\Delta_n(x)$ to zero.

Basu's Theorem: (1984, Metrika)

Let $\{X_n\}$ be a sequence of independent r.v.s each having an absolutely continuous d.f. F *and c.f.* f. *Suppose that* $EX_1^2 = \infty$. *If* F *belongs to the domain of attraction of* Φ, *then for every* $\beta < 2$,

$\sup_{x \in R}(1+|x|)^{\beta}\Delta_n(x) = o(1)$ *as* $n \to \infty$ *iff (3.1.2) holds.*

In this chapter, we obtain the uniform rate of convergence and non-uniform bound for $\Delta_n(x)$ in (3.1.1). We state below the main results of this chapter.

Theorem 3.1.1: *Under the assumptions [A1]-[A5] and notations stated in the Section 3.2 below,*

$$\sup_{x \in R} \Delta_n(x) = O\{nB_n^{-3} \int_{|u| \le B_n} |u|^3 dF(u) + nB_n^{-2-\delta} + nR_1(B_n) + \vartheta_n\},$$

where $\vartheta_n = |nB_n^{-2}H(B_n) - 1|$.

Theorem 3.1.2: *Under the assumptions [A1]-[A5] and notations stated in the Section 3.2 below,*

$$\Delta_n(x) \le C(1+|x|)^{-\beta}$$

$$\{nB_n^{-4} \int_{|u| \le |x^*| B_n} u^4 dF(u) + nB_n^{-3} \int_{|u| \le |x^*| B_n} |u|^3 dF(u)$$

$$+ n^{-1} + |x^*|^{\beta} nB_n^{-2-\delta} + |x^*|^{\beta} nR_1(B_n) + \vartheta_n\}$$

for sufficiently large n *and* $0<\beta<2$ *where* $\vartheta_n = |nB_n^{-2}H(Bn) - 1|$ *and* $x^* = \max(|x|, 1)$.

Remark 3.1.1: The existence of the terms $B_n^{-2-\delta}$ and $R_1(B_n)$ in both the theorems mentioned above is inevitable. This is so because in some situations the first term dominates the second term whereas in the other situations second term dominates the first one.

We prove the Theorems 3.1.1 and 3.1.2 in the Section 3.4. The notations and assumptions are introduced in Section 3.2. In Section 3.3, we prove some Lemmas which will be useful in Section 3.4. Some of these lemmas are also of independent interest.

3.2 NOTATIONS AND ASSUMPTIONS:

Suppose r.v. X_0 has d.f. Φ and r.v. X_1 has d.f. F. Without loss of generality, we assume that $EX_1 = 0$ so that $A_n \equiv 0$. Thus under our assumptions, for all t,

$$\lim_{n \to \infty} \{f(tB_n^{-1})\}^n = \exp\{-t^2/2\} \equiv N(t). \qquad \ldots(3.2.1)$$

$$\text{Let } R_k(z) = P(|X_k| > z) \;,\; k = 0,\ 1. \qquad \ldots(3.2.2)$$

and

$$H(z) = \int_{-z}^{z} u^2 dF(u). \qquad \ldots(3.2.3)$$

Since F belongs to the domain of non-normal attraction of Φ, H(z) varies slowly at infinity. Further,

$$R_1(z) = o(z^{-2}H(z)) \text{ as } z \to \infty \qquad \ldots(3.2.4)$$

and

$$\lim_{n \to \infty} nB_n^{-2}H(B_n) = 1. \qquad \ldots(3.2.5)$$

Now, since H is slowly varying, for large z,

$$\int_z^{\infty} u^{-2}H(u)\,du \sim H(z)\int_z^{\infty} u^{-2}du = z^{-1}H(z). \qquad \ldots(3.2.6)$$

In fact, (3.2.6) is a direct consequence of Proposition 1.5.10 of Bingham et al.(1987, p.27). Using (3.2.4), we observe that

$$\int_{|u|>B_n} |u|\,dF(u) = -\int_{B_n}^{\infty} u\,dR_1(u)$$

$$= B_nR_1(B_n) + \int_{B_n}^{\infty} R_1(u)\,du,$$

by performing integration by parts

$$\leq B_nR_1(B_n) + C\int_{B_n}^{\infty} u^{-2}H(u)\,du,$$

$$\le B_n C_1 B_n^{-2} H(B_n) + C B_n^{-1} H(B_n)$$

$$= C B_n^{-1} H(B_n),$$

for some C and all large n, so that

$$\int_{|u|>B_n} |u| \, dF(u) = O(B_n^{-1} H(B_n)). \qquad \ldots(3.2.7)$$

For each integer n and real x, we define

$$\alpha_n(t,x) = \int_{|u|\le|x|B_n} e^{itu} dF(u), \qquad \ldots(3.2.8)$$

$$\beta_n(t,x) = f(t) - \alpha_n(t,x), \qquad \ldots(3.2.9)$$

$$A_n(t,x) = \{\alpha_n(tB_n^{-1},x)\}^n, \qquad \ldots(3.2.10)$$

$$B_n(t,x)$$

$$= \sum_{j=1}^{n} \binom{n}{j} \{\alpha_n(tB_n^{-1},x)\}^{n-j} \{\beta_n(tB_n^{-1},x)\}^j, \qquad \ldots(3.2.11)$$

so that

$$\{f(tB_n^{-1})\}^n = A_n(t,x) + B_n(t,x). \qquad \ldots(3.2.12)$$

We define a few absolutely convergent inversion integrals.

$$v_n(u) = (2\pi)^{-1} \int_{-\infty}^{\infty} \{f(tB_n^{-1})\}^n e^{-itu} dt, \qquad \ldots(3.2.13)$$

$$a_n(u,x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \{A_n(t,x)\} \, e^{-itu} dt, \qquad \ldots(3.2.14)$$

$$b_n(u,x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \{B_n(t,x)\} \, e^{-itu} dt \qquad \ldots(3.2.15)$$

for each integer n and reals u and x.

For any given $0< \lambda< 2$, we write

$$P_\lambda(t) = \begin{cases} |t|^{-\lambda} & \ldots \text{ if } |t|\geq 1 \\ |t|^{\lambda} & \ldots \text{ if } 0\leq |t|\leq 1 \end{cases} \qquad \ldots(3.2.16)$$

Note that $P_\lambda(t)\leq 1$ for all t.

Let $\Xi = \{(t, n, x): |t|\leq \varepsilon B_n, |x|\geq 1, n\geq n_o\}$, where ε will be as same in Lemma 3.3.2 of Section 3.3, and n_o is a very large positive constant.

We now make the following assumptions:

[A1] The d.f. F is symmetric* and absolutely continuous.

**Remark 3.2.1*: By symmetric d.f. we mean a d.f. F such that $1-F(x) = F(-x)$ for all $x> 0$.)

[A2] There exists an integer $r\geq 1$ such that

$$\int_{-\infty}^{\infty} |f(t)|^r dt< \infty.$$

[A3] $\int_{-\infty}^{\infty} |u|^{2+\delta} |v(u)-\phi(u)| du< \infty$ for some $0< \delta\leq 1$; $v(u)$ being the p.d.f. corresponding to the d.f. F.

[A4] The d.f.F belongs to the domain of non-normal attraction of the standard normal law Φ.

[A5] $nB_n^{-3} \int_{|u|\leq |x|B_n} |u|^3 dF(u) \to 0$ as $n \to \infty$.

3.3 PRELIMINARY RESULTS:

We shall need the following lemmas.

Lemma 3.3.1: Under the assumptions **[A1]** and **[A4]**, there exist positive constants ε, $\lambda < 1$ and c such that for for all t with $|t| \leq \varepsilon B_n$ and all large n,

$$|f(tB_n^{-1})|^n \leq \exp(-ct^2 P_\lambda(t)). \qquad \ldots(3.3.1)$$

Remark 3.3.1: The proof of this lemma follows along the lines of Ibragimov and Linnik (1971, p.123).

Lemma 3.3.2: Under the assumptions **[A1]** and **[A4]**, there exist constants ε, $\lambda < 1$, c and C such that for all $(t, n, x) \in \Xi$, we have

$$|A_n(t,x)| \leq C\exp(-ct^2 P_\lambda(t)). \qquad \ldots(3.3.2)$$

Proof: We recall that in order that a symmetric d.f. F(x) with c.f. f(t) belongs to the domain of attraction of the normal law it is necessary (and sufficient) that in the neighbourhood of origin,

$$\log f(t) = -(1/2)t^2\tilde{h}_1(t), \text{ for } |t| \leq \varepsilon \qquad \ldots(3.3.3)$$

where $\tilde{h}_1(t)$ is a positive, slowly varying function as $t \to 0$ (see: Ibragimov and Linnik (1971, Theorem 2.6.5, p.85)). We, first, obtain an upper bound on $\{f(tB_n^{-1})\}^{-1}$ and use it later in this proof.

Note that using (3.3.3),

$$f(tB_n^{-1}) = \exp\{-(1/2)t^2B_n^{-2}\tilde{h}_1(tB_n^{-1})\}, \quad \ldots(3.3.4)$$

for all t with $|t| \leq \varepsilon B_n$.

Therefore, $0 < f(tB_n^{-1}) \leq 1$ for all t with $|t| \leq \varepsilon B_n$.

Consider

$$\log f(tB_n^{-1})$$

$$= \log [1+f(tB_n^{-1})-1]$$

$$= (f(tB_n^{-1})-1)-(\theta/2)(f(tB_n^{-1})-1)^2, \quad \ldots|\theta|< 2.$$

Hence $-1-\theta/2 \leq \log f(tB_n^{-1}) \leq 0$

or $-2 \leq \log f(tB_n^{-1}) \leq 0$

or $\exp(-2) \leq f(tB_n^{-1}) \leq 1$

or $1 \leq \{f(tB_n^{-1})\}^{-1} \leq \exp(2)$, for $|t| \leq \varepsilon B_n$. ...(3.3.5)

Now, we have from Lemma 3.3.1, for $|t| \leq \varepsilon B_n$ and $|x| \geq 1$,

$$|A_n(t,x)| = |\sum_{j=0}^{n} \binom{n}{j} \{f(tB_n^{-1})\}^{n-j}\{-\beta_n(tB_n^{-1},x)\}^j|$$

$$\leq \sum_{j=0}^{n} \binom{n}{j} \{f(tB_n^{-1})\}^n\{f(tB_n^{-1}\}^{-j} |\{-\beta_n(tB_n^{-1},x)\}|^j$$

$$\leq \sum_{j=0}^{n} (n^j/j!)e^{-ct^{2p}\lambda(t)}e^{2j} \{R_1(B_n)\}^j,$$

using (3.3.5) and the fact that $R_1(|x|B_n) \leq R_1(B_n)$.

$$\leq e^{-ct^{2p}\lambda(t)} \sum_{j=0}^{n} (ne^2\{R_1(B_n)\})^j/j!$$

$$\leq Ce^{-ct^{2p}\lambda(t)}.$$

Note that the last inequality follows from the fact that $nR_1(B_n) \to 0$ as $n \to \infty$ as a consequence of (3.2.4) and (3.2.5).□

Next, for every integer n, define

$$d_n(t,x) = n\{\alpha_n(tB_n^{-1},x) - \exp\{-(t^2/2)B_n^{-2}H(B_n)\}\}, \quad \ldots(3.3.6)$$

$$S_n(t,x) = n^{-1}\{\sum_{k=1}^{n}\{\alpha_n(tB_n^{-1},x)\}^{n-k}\exp\{-(t^2/2)B_n^{-2}H(B_n)(k-1)\}\}. \quad \ldots(3.3.7)$$

Properties of the function $d_n(t,x)$

Lemma 3.3.3: Under the assumptions of Lemma 3.3.2, for all values of t and x with $|x| \geq 1$, we have

(i) $n^{-1}|d_n(t,x)|$

$$\leq P_1(|t|)\{B_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u) + B_n^{-3}\int_{|u|\leq|x|B_n} |u|^3 dF(u) + n^{-2} + R_1(B_n)\}, \quad \ldots(3.3.8)$$

(ii) $n^{-1}|d_n^{(1)}(t,x)|$

$$\leq P_2(|t|)\{B_n^{-3}\int_{|u|\leq|x|B_n} |u|^3 dF(u) + n^{-2}\}, \quad \ldots(3.3.9)$$

(iii) $n^{-1}|d_n^{(2)}(t,x)|$

$$\leq P_3(|t|)\{B_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u) + B_n^{-3}\int_{|u|\leq|x|B_n} |u|^3 dF(u) + n^{-2}\}. \quad \ldots(3.3.10)$$

Proof: (i) Note that in view of assumption **[A1]** and (3.2.8),

$$n^{-1}|d_n(t,x)| = |\alpha_n(tB_n^{-1},x) - \exp\{-(t^2/2)B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\le|x|B_n} \cos\ (tuB_n^{-1})dF(u) - \exp\{-(t^2/2)B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\le|x|B_n} (\cos\ (tuB_n^{-1}) - 1 + (t^2u^2B_n^{-2}/2))dF(u)$$

$$+ \int_{|u|\le|x|B_n} dF(u) - (t^2B_n^{-2}/2) \int_{|u|\le|x|B_n} u^2dF(u)$$

$$- \{\exp\{-(t^2/2)B_n^{-2}H(B_n)\} - 1 + (t^2B_n^{-2}H(B_n)/2)\}$$

$$- 1 + (t^2B_n^{-2}H(B_n)/2)|$$

$$= |\int_{|u|\le|x|B_n} (\cos\ (tuB_n^{-1}) - 1 + (t^2u^2B_n^{-2}/2))dF(u)$$

$$+ 1 - R_1(|x|B_n) - t^2B_n^{-2}H(|x|B_n)/2$$

$$-\{\exp\{-(t^2/2)B_n^{-2}H(B_n)\} - 1 + (t^2B_n^{-2}H(B_n)/2)\}$$

$$- 1 + t^2B_n^{-2}H(B_n)/2|$$

$$= |\int_{|u|\le|x|B_n} (\cos\ (tuB_n^{-1}) - 1 + (t^2u^2B_n^{-2}/2))dF(u)$$

$$- (t^2B_n^{-2}/2)\{H(|x|B_n) - H(B_n)\} - R_1(|x|B_n)$$

$$- \{\exp\{-(t^2/2)B_n^{-2}H(B_n)\} - 1 + (t^2B_n^{-2}H(B_n)/2)\}|$$

$$\le \int_{|u|\le|x|B_n} |\cos\ (tuB_n^{-1}) - 1 + (t^2u^2B_n^{-2}/2)|dF(u)$$

$$+ (t^2B_n^{-2}/2)\{H(|x|B_n) - H(B_n)\} + R_1(B_n) + (t^2B_n^{-2}H(B_n)/2)^2$$

$$\le \int_{|u|\le|x|B_n} (t^4u^4B_n^{-4}/2)\,dF(u) + (t^2B_n^{-3}/2)\int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+ (t^4B_n^{-4}H^2(B_n)/8) + R_1(B_n),$$

since $0\le \cos(x)-1-x^2/2\le x^4/2$ for all x.

$$\le (t^4B_n^{-4}/2)\int_{|u|\le|x|B_n} u^4 dF(u) + (t^2B_n^{-3}/2)\int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+ (t^4B_n^{-4}H^2(B_n)/8) + R_1(B_n)$$

$$\le P_1(|t|)\Big\{B_n^{-4}\int_{|u|\le|x|B_n} u^4 dF(u) + B_n^{-3}\int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+ n^{-2} + R_1(B_n)\Big\}, \text{ using (3.2.5).}$$

This proves (3.3.8). Note that second term in the inequality preceding the last one is derived as follows:

$$(t^2/2)B_n^{-2}\{H(|x|B_n)-H(B_n)\}$$

$$= (t^2/2)B_n^{-2}\int_{B_n<|u|\le|x|B_n} u^2 dF(u)$$

$$\le (t^2/2)B_n^{-3}\int_{B_n<|u|\le|x|B_n} |u|^3 dF(u)$$

$$\le (t^2/2)B_n^{-3}\int_{|u|\le|x|B_n} |u|^3 dF(u).$$

Now to prove (ii), consider

$$n^{-1}|d_n^{(1)}(t,x)|$$

$$= |(d/dt)\{\alpha_n(tB_n^{-1},x) - \exp\{-(t^2/2)B_n^{-2}H(B_n)\}\}|$$

$$= |(d/dt)\{\int_{|u|\le|x|B_n} \cos(tuB_n^{-1})\,dF(u)$$

$$-\exp\{-(t^2/2)B_n^{-2}H(B_n)\}\}|$$

$$= |-B_n^{-1}\int_{|u|\le|x|B_n} u\sin(tuB_n^{-1})\,dF(u)$$

$$+ tB_n^{-2}H(B_n)\exp\{-(t^2/2)B_n^{-2}H(B_n)\}|, \text{ because of DCT}$$

$$= |-B_n^{-1} \int_{|u|\le|x|B_n} u(\sin(tuB_n^{-1}) - tuB_n^{-1})\,dF(u)$$

$$- tB_n^{-2} \int_{|u|\le|x|B_n} u^2 dF(u)$$

$$+ tB_n^{-2}H(B_n)\{\exp\{-(t^2/2)B_n^{-2}H(B_n)\}-1\} + tB_n^{-2}H(B_n)|$$

$$\le B_n^{-1} \int_{|u|\le|x|B_n} |u||\sin(tuB_n^{-1}) - tuB_n^{-1}|\,dF(u)$$

$$+ |t|B_n^{-2}\{H(|x|B_n) - H(B_n)\} + (|t|^3/2)(B_n^{-2}H(B_n))^2$$

$$\le B_n^{-1} \int_{|u|\le|x|B_n} |u|t^2u^2B_n^{-2}dF(u) + |t|B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$+ (|t|^3/2)B_n^{-4}H^2(B_n)$, because $|\sin(x) - x| \le x^2$ for $x \ge 0$.

$$\le t^2B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u) + |t|B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+ (|t|^3/2)(B_n^{-2}H(B_n))^2$$

$\le P_2(|t|)\{B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u) + n^{-2}\}$, using (3.2.5).

This proves (3.3.9).

(iii) Finally, we consider

$$n^{-1}|d_n^{(2)}(t,x)|$$

$$= |(d/dt)d_n^{(1)}(t,x)|$$

$$= |(d/dt)\{-B_n^{-1} \int_{|u|\le|x|B_n} u\sin(tuB_n^{-1})\,dF(u)$$

$$+ tB_n^{-2}H(B_n)\exp\{-(t^2/2)B_n^{-2}H(B_n)\}|$$

$$= |-B_n^{-2} \int_{|u|\le|x|B_n} u^2 \cos\ (tuB_n^{-1})\, dF(u)$$

$$+ B_n^{-2}H(B_n)\exp\{-(t^2/2)B_n^{-2}H(B_n)\}$$

$$- (tB_n^{-2}H(B_n))^2 \exp\{-(t^2/2)B_n^{-2}H(B_n)\}|, \text{ because of DCT}$$

$$= |-B_n^{-2} \int_{|u|\le|x|B_n} u^2(\cos\ (tuB_n^{-1})-1)\, dF(u)$$

$$-B_n^{-2} \int_{|u|\le|x|B_n} u^2 dF(u)$$

$$+ B_n^{-2}H(B_n)\{\exp\{-(t^2/2)B_n^{-2}H(B_n)\}-1\} + B_n^{-2}H(B_n)$$

$$- (tB_n^{-2}H(B_n))^2 \exp\{-(t^2/2)B_n^{-2}H(B_n)\}|$$

$$\le B_n^{-2} \int_{|u|\le|x|B_n} u^2 |\cos\ (tuB_n^{-1})-1|\, dF(u)$$

$$+ B_n^{-2}\{H(|x|B_n)-H(B_n)\}$$

$$+ B_n^{-2}H(B_n)|\exp\{-(t^2/2)B_n^{-2}H(B_n)\}-1|$$

$$+ (tB_n^{-2}H(B_n))^2 \exp\{-(t^2/2)B_n^{-2}H(B_n)\}$$

$$\le B_n^{-4}t^2 \int_{|u|\le|x|B_n} u^4 dF(u) + B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+(t^2/2)B_n^{-4}H^2(B_n) + t^2B_n^{-4}H^2(B_n), \text{ because } |\cos\ (x)-1| \le x^2.$$

$$\le t^2B_n^{-4} \int_{|u|\le|x|B_n} u^4 dF(u) + B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u)$$

$$+(3t^2/2)B_n^{-4}H^2(B_n)$$

$$\le P_3(|t|)\{B_n^{-4} \int_{|u|\le|x|B_n} u^4 dF(u)$$

$$+ B_n^{-3} \int_{|u|\le|x|B_n} |u|^3 dF(u) + n^{-2}\}, \text{ using } (3.2.5).$$

This proves (3.3.10).□

Before we prove the next lemma, we discuss analogs of Potter-type bounds on the slowly varying function $H(z)$ defined at (3.2.3) using the Karamata indices. This discussion is based on Theorem 2.1.1 and Proposition 2.2.3 of Section 2.1 of Bingham et al.(1987, p.66).

Definition 3.3.1: Let $h(.)$ be positive. Its **Upper Karamata Index** $c(h)$ is the infimum of those p for which, $h(\lambda x)/h(x) \le \{1+o(1)\}\lambda^p$ $(x \to \infty)$, uniformly in $1 \le \lambda \le \Lambda < \infty$. Its **Lower Karamata Index** $d(h)$ is the supremum of those q for which,
$h(\lambda x)/h(x) \ge \{1+o(1)\}\lambda^q$ $(x \to \infty)$, uniformly in $1 \le \lambda \le \Lambda < \infty$. (Here $\inf \varnothing = +\infty$, $\sup \varnothing = -\infty$).

We notice that as a consequence of Karamata Indices Theorem (Theorem 2.1.1 of Bingham et al. (1987)) a positive measurable function h is slowly varying iff $c(h) = d(h) = 0$. In the light of this, Proposition 2.2.3 of Bingham at el.(1987, p.73) becomes: Let h be a slowly varying function. Then for every $\delta_1 > 0$ and $A > 1$ there exists $X = X(A, \delta_1)$ such that

$$h(y)/h(x) \le A(y/x)^{\delta_1} \quad (y \ge x \ge X).$$

Recall that the function $H(z)$ defined at (3.2.3) is a slowly varying function at infinity, and hence applying upper bound type result for $H(z)$ we observe that for every $\delta_1 > 0$,

$$H(|x|B_n) \leq A|x|^{\delta_1} H(B_n) \qquad \ldots(3.3.11)$$

for all x with $|x| \geq 1$ and all large n.

Properties of the function $\alpha_n(t,x)$

Lemma 3.3.4: Under the assumptions of Lemma 3.3.2, for each fixed n and x, $\alpha_n(tB_n^{-1},x)$ is differentiable any number of times under the integral sign. For all values of t and all x with $|x| \geq 1$,

(i) $|\alpha_n^{(1)}(tB_n^{-1},x)| \leq E|X_1|,$ ...(3.3.12)

(ii) $|\alpha_n^{(1)}(tB_n^{-1},x)| \leq |x|^{\delta_1} P(|t|) B_n^{-2} H(B_n),$ ...(3.3.13)

(iii) $|\alpha_n^{(2)}(tB_n^{-1},x)| \leq C|x|^{\delta_1} B_n^{-2} H(B_n).$ ...(3.3.14)

(iv) If, additionally, we assume **[A2]** then, for all $x \neq 0$, sufficiently large but fixed integer s, there exists a constant C such that

$$\int_{-\infty}^{\infty} |\alpha_n(t,x)|^n dt = O(B_n^{-1}), \qquad \ldots(3.3.15)$$

$$\int_{-\infty}^{\infty} |\alpha_n(t,x)|^{2s} dt \leq C, \qquad \ldots(3.3.16)$$

$$\int_{-\infty}^{\infty} |\beta_n(t,x)|^{2s} dt \leq C, \qquad \ldots(3.3.17)$$

(v) $|\alpha_n(t,x)|^m dt = |\alpha_m(t,x(B_n/B_m))|^m,$ ...(3.3.18)

where $x \neq 0$, m is a function of n and $m \leq n$.

Proof: Inequality (3.3.12) is obvious. Using DCT and the fact that $EX_1 = 0$, we find that

$$|\alpha_n^{(1)}(tB_n^{-1},x)| = |i\, B_n^{-1} \int_{|u| \leq |x|B_n} u e^{ituB_n^{-1}} dF(u)|$$

$$= |B_n^{-1} \int_{|u| \leq |x|B_n} u(e^{ituB_n^{-1}}-1) dF(u) + B_n^{-1} \int_{|u| \leq |x|B_n} u dF(u)|$$

$$\leq B_n^{-2}|t| \int_{|u| \leq |x|B_n} u^2 dF(u) + B_n^{-1} \int_{|u| > |x|B_n} |u| dF(u)$$

$= |t|B_n^{-2}H(|x|B_n)+B_n^{-1}O(|x|^{-1}B_n^{-1}H(|x|B_n))$, using (3.2.7)

$= |t|B_n^{-2}C|x|^{\delta_1}H(B_n) + CB_n^{-2}|x|^{\delta_1}H(B_n)$, using (3.3.11)

$\leq P(|t|)|x|^{\delta_1}B_n^{-2}H(B_n)$

$\leq P(|t|)|x|^{\delta_1}n^{-1}$, using (3.2.5).

This proves (3.3.13).

To prove (3.3.14) we proceed as follows:

$|\alpha_n^{(2)}(tB_n^{-1},x)|$

$= |(i\ B_n^{-1})^2 \int_{|u|\leq|x|B_n} u^2 e^{ituB_n^{-1}} dF(u)|$

$\leq B_n^{-2} \int_{|u|\leq|x|B_n} u^2\ dF(u)$

$= B_n^{-2}H(|x|B_n)$

$\leq C|x|^{\delta_1}B_n^{-2}H(B_n)$, using (3.3.11)

$\leq C|x|^{\delta_1}n^{-1}$, for all t, using (3.2.5).

Equations (3.3.15), (3.3.16) and (3.3.17) are proved on lines similar to those in Basu et al. (1980, (3.3)-(3.5)).

To prove (3.3.18), observe that, from (3.2.8),

$\alpha_n(t,x) = \int_{|u|\leq|x|B_n} e^{itu}dF(u)$.

Hence,

$|\alpha_n(t,x)|^m = |\int_{|u|\leq|x|B_n} e^{itu}dF(u)|^m$

$= |\int_{|u|\leq|x|B_m(B_n/B_m)} e^{itu}dF(u)|^m$

$= |\alpha_m(t,x(B_n/B_m))|^m$.□

Denote, for r = 0, 1, 2,

$$E_n(r) = \sum_{k=1}^{n-r} \{\alpha_n(tB_n^{-1},x)\}^{n-k-r} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)} \qquad ...(3.3.19)$$

and

$$F_n(r) = \sum_{k=1}^{n-r} |\alpha_n(tB_n^{-1},x)|^{n-k-r} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}. \qquad ...(3.3.20)$$

Lemma 3.3.5: Under the assumptions of Lemma 3.3.2, there exists constants C_1 and c such that, for all (t, n, x) $\in$ Ξ, we have

$$F_n(q) \le C_1 n e^{-Ct^2 P_\lambda(t)} \quad \text{for } q = 0, 1, 2. \qquad ...(3.3.21)$$

Proof: Consider $F_n(q)$

$$= \sum_{k=1}^{n-q} |\alpha_n(tB_n^{-1},x)|^{n-k-q} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$= \sum_{k=1}^{[n/2]-1} |\alpha_n(tB_n^{-1},x)|^{n-k-q} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ \sum_{k=[n/2]}^{n-q} |\alpha_n(tB_n^{-1},x)|^{n-k-q} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$\le Ce^{-(ct^2)P_\lambda(t)} \sum_{k=1}^{[n/2]-1} \{Ce^{-(ct^2)P_\lambda(t)}\}^{(-(k+q)/n)}$$

$$+ \sum_{k=[n/2]}^{n-q} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}, \text{ using Lemma 3.3.2}$$

$$\le Ce^{-(ct^2)P_\lambda(t)} \sum_{k=1}^{[n/2]-1} \max[C^{-(1+q)/n}, C^{-(n+q)/n}] \{e^{(ct^2)P_\lambda(t)}\}^{((k+q)/n)}$$

$$+ \sum_{k=[n/2]}^{n-q} e^{-(t^2/2)nB_n^{-2}H(B_n)([n/2]/n)}$$

$$\le n\left[C \max[2,1/C] e^{-(ct^2)P_\lambda(t)((1/2-\varepsilon))} + e^{-(t^2/2)c_1}\right],$$

for some $0< \varepsilon< 1/2$.

$\leq C_1 n e^{-Ct^2 P_\lambda(t)}$, since $P_\lambda(t) \leq 1$. □

Remark 3.3.2: The above result holds true for any fixed positive integer q and n> 2q.

Properties of function $S_n(t,x)$

Lemma 3.3.6: Under the assumptions of Lemma 3.3.2, for all (t, n, x) ∈ Ξ, we have

$$\text{(i)} \quad |S_n(t,x)| \leq C_1 e^{-Ct^2 P_\lambda(t)}, \qquad \ldots (3.3.22)$$

$$\text{(ii)} \quad |S_n^{(1)}(t,x)| \leq |x|^{\delta_1} P_1(|t|) e^{-Ct^2 P_\lambda(t)}, \qquad \ldots (3.3.23)$$

$$\text{(iii)} \quad |S_n^{(2)}(t,x)| \leq |x|^{2\delta_1} P_2(|t|) e^{-Ct^2 P_\lambda(t)}. \qquad \ldots (3.3.24)$$

where δ_1 is as in (3.3.11).

Proof: Note that in view of (3.3.19), we have

$$n\, S_n(t,x) = E_n(0)$$

$$= \sum_{k=1}^{n} \{\alpha_n(tB_n^{-1},x)\}^{n-k} \exp\{-(t^2/2)B_n^{-2}H(B_n)(k-1)\}.$$

Therefore, from the definition of $F_n(q)$ given at (3.3.20) and Lemma 3.3.5,

$$n|S_n(t,x)| \leq F_n(0)$$

$$\leq C_1 n e^{-Ct^2 P_\lambda(t)}.$$

Now $n\, S_n^{(1)}(t,x)$

$$= \sum_{k=1}^{n-1} (n-k)\{\alpha_n(tB_n^{-1},x)\}^{n-k-1}\{\alpha_n^{(1)}(tB_n^{-1},x)\} B_n^{-1} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ \sum_{k=1}^{n} \{\alpha_n(tB_n^{-1},x)\}^{n-k}(-tB_n^{-2}H(B_n))(k-1) e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}.$$

Therefore, using (3.3.20) for q = 0 and q = 1, we get

$n|S_n^{(1)}(t,x)|$

$= n|\alpha_n^{(1)}(tB_n^{-1},x)|B_n^{-1}F_n(1) + |t|B_n^{-2}H(B_n)nF_n(0)$

$\le \{n\{|x|^{\delta_1}P(|t|)n^{-1}\}B_n^{-1} + n|t|n^{-1}\} nC_1 e^{-Ct^{2p}\lambda(t)}$,

using (3.3.13) and (3.3.21)

$\le n|x|^{\delta_1}P_1(|t|)e^{-Ct^{2p}\lambda(t)}$, using (3.2.5).

Finally, $n\ S_n^{(2)}(t,x)$

$$= \sum_{k=1}^{n-2}(n-k)(n-k-1)\{\alpha_n(tB_n^{-1},x)\}^{n-k-2}\{\alpha_n^{(1)}(tB_n^{-1},x)\}^2B_n^{-2} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ \sum_{k=1}^{n-1}(n-k)\{\alpha_n(tB_n^{-1},x)\}^{n-k-1}\{\alpha_n^{(2)}(tB_n^{-1},x)\}B_n^{-2} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ 2\sum_{k=1}^{n-1}(n-k)\{\alpha_n(tB_n^{-1},x)\}^{n-k-1} \{\alpha_n^{(1)}(tB_n^{-1},x)\}B_n^{-1}(-tB_n^{-2}H(B_n))(k-1) e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ \sum_{k=1}^{n}\{\alpha_n(tB_n^{-1},x)\}^{n-k}\{-B_n^{-2}H(B_n)(k-1)\} e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

$$+ \sum_{k=1}^{n}\{\alpha_n(tB_n^{-1},x)\}^{n-k}\{-tB_n^{-2}H(B_n)(k-1)\}^2 e^{-(t^2/2)B_n^{-2}H(B_n)(k-1)}$$

Therefore, using (3.3.20) for q = 0, 1 and 2, we have

$n|S_n^{(2)}(t,x)|$

$\le n^2|\alpha_n^{(1)}(tB_n^{-1},x)|^2B_n^{-2}F_n(2)+n|\alpha_n^{(2)}(tB_n^{-1},x)|B_n^{-2}F_n(1)$

$+ 2n^2|t|B_n^{-3}H(B_n)|\alpha_n^{(1)}(tB_n^{-1},x)|F_n(1)$

$+ nB_n^{-2}H(B_n)F_n(0) + n^2B_n^{-4}H^2(B_n)\ t^2\ F_n(0)$

$\le n^2\{|x|^{\delta_1}P(|t|)n^{-1}\}^2B_n^{-2}F_n(2) + n\{c|x|^{\delta_1}n^{-1}\}B_n^{-2}F_n(1)$

$+ 2n^2|t|B_n^{-3}H(B_n)\{|x|^{\delta_1}P(|t|)n^{-1}\}F_n(1) + C_1F_n(0)$

$+ C_2\ t^2F_n(0)$, using (3.2.5), (3.3.13), (3.3.14), (3.3.21)

$\le n|x|^{2\delta_1}P_2(|t|)e^{-Ct^{2p}\lambda(t)}$, using Lemma 3.3.5.□

Lemma 3.3.7: Let $0< \varepsilon< 1/2$, $0< \lambda< 1$ and c be as in Lemma 3.3.2. Then, under the assumptions of Lemma 3.3.2, there exists a polynomial $P(|t|)$ in $|t|$ such that for all $(t, n, x) \in \Xi$, we have,

$$|A_n^{(2)}(t,x)-N^{(2)}(t)|\le |x|^{2\delta_1}P_2(|t|)e^{-Ct^{2p}\lambda(t)}$$

$$\left\{nB_n^{-4}\int_{|u|\le|x|B_n}u^4dF(u) + nB_n^{-3}\int_{|u|\le|x|B_n}|u|^3dF(u) + n^{-1}+ nR_1(B_n) + \vartheta_n\right\}$$

where $\vartheta_n = |nB_n^{-2}H(B_n)-1|$, $N(t)$ is defined at (3.2.1) and δ_1 is given at (3.3.11).

Proof: We write

$$A_n(t,x)-\exp(-t^2/2) = D_{1n}(t,x) + D_{2n}(t), \qquad \ldots(3.3.25)$$

where

$$D_{1n}(t,x) = A_n(t,x) - \exp(-(t^2/2)nB_n^{-2}H(B_n)) \qquad \ldots(3.3.26)$$

$$D_{2n}(t) = \exp(-(t^2/2)nB_n^{-2}H(B_n)) - \exp(-t^2/2). \qquad \ldots(3.3.27)$$

We shall first prove that

$$|D_{1n}^{(2)}(t,x)|$$

$$\leq |x|^{2\delta_1}P(|t|)e^{-Ct^{2p}\lambda(t)}$$

$$\Big\{nB_n^{-4}\int_{|u|\leq|x|B_n} u^4dF(u)+nB_n^{-3}\int_{|u|\leq|x|B_n} |u|^3dF(u)$$

$$+n^{-1}+nR_1(B_n)\Big\}. \qquad \ldots(3.3.28)$$

Note that in view of equations (3.3.6) and (3.3.7),

$$D_{1n}(t,x) = d_n(t,x)S_n(t,x). \qquad \ldots(3.3.29)$$

$$D_{1n}^{(2)}(t,x) = d_n^{(2)}(t,x)S_n(t,x) + 2\, d_n^{(1)}(t,x)S_n^{(1)}(t,x)$$

$$+ d_n(t,x)S_n^{(2)}(t,x). \qquad \ldots(3.3.30)$$

From (3.3.8), (3.3.9), (3.3.10), (3.3.22), (3.3.23) and (3.3.24) it follows that

$$|D_{1n}^{(2)}(t,x)|$$

$$\leq |d_n^{(2)}(t,x)||S_n(t,x)|+2|d_n^{(1)}(t,x)||S_n^{(1)}(t,x)|$$

$$+ |d_n(t,x)||S_n^{(2)}(t,x)|$$

$$\leq P_3(|t|)ne^{-Ct^2p\lambda(t)}$$

$$\{B_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u)+B_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)+n^{-2}\}$$

$$+2P_2(|t|)\{|x|^{\delta_1}P_4(|t|)ne^{-Ct^2p\lambda(t)}\}$$

$$\{B_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)+n^{-2}\}$$

$$+\ P_1(|t|)\{|x|^{2\delta_1}P_5(|t|)ne^{-Ct^2p\lambda(t)}\}$$

$$\{B_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u)+B_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)$$

$$+n^{-2}+R_1(B_n)\}$$

$$\leq n|x|^{2\delta_1}P(|t|)e^{-Ct^2p\lambda(t)}$$

$$\{B_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u)+B_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)$$

$$+n^{-2}+R_1(B_n)\}$$

$$\leq |x|^{2\delta_1}P(|t|)e^{-Ct^2p\lambda(t)}$$

$$\{nB_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u)+nB_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)$$

$$+n^{-1}+nR_1(B_n)\}.$$

This proves (3.3.28).

Next we prove that

$$|D_{2n}^{(2)}(t)|\leq e^{-ct^2/2}P(t)\vartheta_n, \qquad \ldots(3.3.31)$$

where $\vartheta_n = |nB_n^{-2}H(B_n)-1|$.

Let $\theta_n = nB_n^{-2}H(B_n)-1$.

Note that $D_{2n}(t)$

$$= \exp(-(t^2/2)nB_n^{-2}H(B_n)) - \exp(-t^2/2)$$

$$= e^{-(t^2/2)\theta_n}e^{-t^2/2} - e^{-t^2/2}$$

$$= e^{-t^2/2}\{e^{-(t^2/2)\theta_n} - 1\}$$

Therefore, $|D_{2n}(t)|$

$$= e^{-t^2/2}|e^{(-(t^2/2)\theta_n} - 1|$$

$$\le ce^{-t^2/2}(t^2/2)|\theta_n|$$

$$= e^{-t^2/2}P(|t|)\vartheta_n. \qquad \ldots(3.3.32).$$

Consider $D_{2n}^{(2)}(t)$

$$= -(\theta_n+1)e^{-(t^2/2)(\theta_n+1)} + t^2(\theta_n+1)^2e^{-(t^2/2)(\theta_n+1)}$$

$$+ e^{-(t^2/2)} - t^2e^{-(t^2/2)}$$

$$= (\theta_n+1)\theta_n t^2e^{-(t^2/2)(\theta_n+1)} - [(\theta_n+1)e^{-(t^2/2)\theta_n} - 1]$$

$$(1-t^2)e^{-(t^2/2)}$$

Therefore, $|D_{2n}^{(2)}(t)|$

$$= (\theta_n+1)\vartheta_n t^2e^{-(t^2/2)(\theta_n+1)}$$

$$+|(\theta_n+1)e^{-(t^2/2)\theta_n} - 1||1-t^2|e^{-(t^2/2)}$$

$$\le (1+\varepsilon)\vartheta_n t^2e^{-(t^2/2)(1-\varepsilon)}$$

$$+ |\{(\theta_n+1)\ \{e^{-(t^2/2)\theta_n} - 1\}\}|\ |1-t^2|\ e^{-(t^2/2)}$$

$$\le (1+\varepsilon)\vartheta_n t^2e^{-(t^2/2)(1-\varepsilon)}$$

$$+ (1+\varepsilon)\ |e^{-(t^2/2)\theta_n} - 1|\ |1-t^2|\ e^{-(t^2/2)}$$

$\leq (1+\varepsilon)\vartheta_n t^2 e^{-(t^2/2)(1-\varepsilon)}$

$$+(1+\varepsilon)\ (t^2/2)\ \vartheta_n\ (1+t^2)\ e^{-(t^2/2)}$$

$\leq e^{-Ct^2} P(|t|)\vartheta_n,$

which proves inequality (3.3.31).

Differentiating equation (3.3.25) on both the sides twice with respect to t, taking absolute value on both the sides and then combining the estimates (3.3.28) and (3.3.31), we get

$|A_n^{(2)}(t,x) - N^{(2)}(t)|$

$\leq |D_{1n}^{(2)}(t,x)| + |D_{2n}^{(2)}(t)|$

$\leq |x|^{2\delta_1} P_2(|t|) e^{-Ct^2} P_\lambda(t)$

$$\Big\{nB_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u) + nB_n^{-3}\int_{|u|\leq|x|B_n} |u|^3 dF(u)$$

$$+n^{-1} + nR_1(B_n) + \vartheta_n\Big\}.\square$$

Remark 3.3.3: The bound on $D_{2n}(t)$ at (3.3.32) holds for all t.

Lemma 3.3.8: Assume **[A1]** and **[A4]** hold. Let $\varepsilon > 0$ be same as in Lemma 3.3.2. Let $\mu = \sup_{\Theta} |\alpha_n(t,x)|$, where $\Theta = \{(t, n, x): |t| > \varepsilon,\ n \geq n_o,\ |x| \geq 1\}$, ε is as same in Lemma 3.3.2, and n_o is a very large positive constant. Then,

$0 \leq \mu < 1.$ ...(3.3.33)

Proof: First of all note that μ can not be greater than unity. If possible let $\mu=1$. Now $|\alpha_n(t,x)| \le \int_{|u|\le|x|B_n} dF(u) < 1$, because d.f.F is in the domain of non-normal attraction of the standard normal law and therefore it can not have bounded support. This implies that the supremum of the positive reals $\{|\alpha_n(t,x)| : |x| \ge 1, |t| \ge \varepsilon, n \ge n_0\}$ can not be assumed at finite point n. Therefore, there exist sequences of reals $\{t_n\}$ such that $t_n > t_o$, and $\{y_n\}$ such that $y_n \to \infty$, with the property that $\int_{-y_n}^{y_n} e^{it_n u} v(u)du \to 1$ as $n \to \infty$. Recall that v is the p.d.f. corresponding to d.f. F. But, since $\int_{|u|>y_n} v(u)du \to 0$, this implies $f(t_n) \to 1$, as $n \to \infty$; here f(u) represents the c.f. corresponding to d.f. F. By the Riemann-Lebesgue Lemma it follows that $\{t_n\}$ is a bounded sequence for otherwise $f(t_n) \to 0$ as $t_n \to \infty$. Thus $\{t_n\}$ must have a finite limit point t^*, say, and continuity of f(t) then requires $f(t^*) = 1$. But we have $t^* \ge \varepsilon$ and are forced to the contradiction that F(x) is lattice.□

Remark 3.3.4: This Lemma has been proved for domain of normal attraction of normal law in Smith and Basu(1974, Lemma 2.4, p.370), and in this case $F \in \mathcal{D}_{NA}(\alpha)$, $\alpha < 2$, in the Lemma 2.3.9.

3.4 PROOFS OF THE THEOREMS:

Proof of Theorem 3.1.1:

By inversion formula for absolutely continuous density, we have

$$2\pi|v_n(x)-\phi(x)|$$

$$= |\int_{-\infty}^{\infty} e^{-itx}\{f^n(tB_n^{-1})-e^{-t^2/2}\}dt|$$

$$\leq \int_{-\infty}^{\infty}|f^n(tB_n^{-1})-e^{-t^2/2}|dt$$

$$\leq \int_{-\infty}^{\infty}|A_n(t,1)-e^{-t^2/2}|dt+\int_{-\infty}^{\infty}|B_n(t,1)|dt$$

$$= I_{1n}+I_{2n}, \text{ say.} \qquad \ldots(3.4.1)$$

In view of (3.3.8), (3.3.22), (3.3.25), (3.3.29), (3.3.32) and assumption **[A5]** we get

$$I_{1n} = \int_{-\infty}^{\infty}|A_n(t,1)-e^{-t^2/2}|dt$$

$$= \int_{|t|\leq\varepsilon B_n}|A_n(t,1)-e^{-t^2/2}|dt + \int_{|t|>\varepsilon B_n}|A_n(t,1)|dt$$

$$+ \int_{|t|>\varepsilon B_n} e^{-t^2/2}dt$$

$$= I_{1n1} + I_{1n2} + I_{1n3}, \text{ say.} \qquad \ldots(3.4.1a)$$

Consider

$$I_{1n1} = \int_{|t|\leq\varepsilon B_n}|A_n(t,1)-e^{-t^2/2}|dt$$

$$= \int_{|t|\leq\varepsilon B_n}|D_{1n}(t,1)|dt + \int_{|t|\leq\varepsilon B_n}|D_{2n}(t)|dt$$

$$\leq \int_{|t|\leq\varepsilon B_n}|d_n(t,1)|\ |S_n(t,1)|dt + \int_{|t|\leq\varepsilon B_n}|D_{2n}(t)|dt$$

$$\le \left\{ nB_n^{-4} \int_{|u|\le B_n} u^4 dF(u) + nB_n^{-3} \int_{|u|\le B_n} |u|^3 dF(u) + n^{-1} + nR_1(B_n) \right\}$$

$$\int_{-\infty}^{\infty} P_1(|t|) e^{-Ct^{2p}\lambda(t)} dt$$

$$+\vartheta_n \int_{-\infty}^{\infty} P_2(|t|) e^{-Ct^{2p}\lambda(t)} dt$$

$$\le C_1\left\{ nB_n^{-3} \int_{|u|\le B_n} |u|^3 dF(u) + n^{-1} + nR_1(B_n) + \vartheta_n \right\}, \qquad \ldots(3.4.2)$$

for all large n, where ϑ_n is defined in Lemma 3.3.7.

Consider

$$I_{1n2} = \int_{|t|>\varepsilon B_n} |A_n(t,1)| dt$$

$$= \int_{|t|>\varepsilon B_n} |\alpha_n(tB_n^{-1},1)|^n dt$$

$$= B_n \int_{|t|>\varepsilon} |\alpha_n(t,1)|^n dt$$

$$= B_n\, \mu^{n-2S} \int_{-\infty}^{\infty} |\alpha_n(t,1)|^{2S} dt$$

$$= CB_n\, \mu^{n-2S}$$

$$\le o(n^{-1}).$$

Finally, $I_{1n3} = \int_{|t|>\varepsilon B_n} e^{-t^2/2} dt \le o(n^{-1})$.

Hence,

$$I_{1n} = \le C_1\left\{ nB_n^{-3} \int_{|u|\le B_n} |u|^3 dF(u) + n^{-1} + nR_1(B_n) + \vartheta_n \right\}.$$

Now we consider the estimation of I_2.

From (3.2.11), we find, for fixed s,

$$I_{2n} = \int_{-\infty}^{\infty} |B_n(t,1)| dt$$

$$\le \int_{-\infty}^{\infty} \sum_{j=1}^{n} \binom{n}{j} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta(tB_n^{-1},1)|^j dt$$

$$= \sum_{j=1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta(tB_n^{-1},1)|^{j} dt$$

$$= \left\{ \sum_{j=1}^{n/2} + \sum_{j=n/2+1}^{n-2s} + \sum_{j=n-2s+1}^{n} \right\} \binom{n}{j}$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^{j} dt$$

$$= J_1(n)+J_2(n)+J_3(n), \text{ say.} \qquad \ldots(3.4.3)$$

Firstly let us consider $J_1(n)$.

$$J_1(n) = \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^{j} dt$$

$$\leq \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j}$$

$$|\{\beta_n(tB_n^{-1},1)\}^{j} - \{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}^{j}| dt$$

$$+ \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}^{j}| dt$$

$$= J_{11}(n)+J_{12}(n), \text{ say.} \qquad \ldots(3.4.4)$$

We first consider $J_{11}(n)$.

Note that

$$\{\beta_n(tB_n^{-1},1)\}^{j} - \{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}^{j}|$$

$$= |\beta(tB_n^{-1},1) - \{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}|$$

$$|\sum_{k=1}^{j} \{\beta(tB_n^{-1},1)\}^{j-k} \{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}^{k-1}|$$

$$= |\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\{F(u)-\Phi(u)\}|$$

$$|\sum_{k=1}^{j} \{\beta(tB_n^{-1},1)\}^{j-k} \{\int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u)\}^{k-1}|$$

because F is a symmetric d.f. by **[A1]**

$$\le \sum_{k=1}^{j} |\beta(tB_n^{-1},1)|^{j-k}$$

$$\left| \int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u) \right|^{k-1} \int_{|u|>B_n} |v(u)-\phi(u)|\, du$$

$$\le \sum_{k=1}^{j} \{P(|X_1|> B_n)\}^{j-k}$$

$$\{P(|X_o|> B_n)\}^{k-1} B_n^{-2-\delta} \int_{|u|>B_n} |u|^{2+\delta} |v(u)-\phi(u)|\, du$$

$$\le j\ [\max\{R_1(B_n), R_o(B_n)\}]^{j-1} B_n^{-2-\delta} \int_{-\infty}^{\infty} |u|^{2+\delta} |v(u)-\phi(u)|\, du$$

$$\le j\ [\max\{C_1 B_n^{-2} H(B_n), C_2 B_n^{-1} e^{-B_n^2/2}\}]^{j-1}\ B_n^{-2-\delta}\ C_3,$$

$$\le j\ \{C_1 n^{-1}\}^{j-1}\ B_n^{-2-\delta}\ C_2 \qquad \ldots(3.4.5)$$

for sufficiently large n.

The last but one inequality follows from the fact that $1-\Phi(x) \le (2\pi)^{-1/2} x^{-1} e^{-x^2/2}$, $x > 0$, and assumption **[A3]**.

We use (3.4.5) now to estimate $J_{11}(n)$ as follows:

$$J_{11}(n) \le \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j}$$

$$\left| \{\beta_n(tB_n^{-1},1)\}^j - \left\{ \int_{|u|>B_n} \cos(tuB_n^{-1})\, d\Phi(u) \right\}^j \right| dt$$

$$\le \sum_{j=1}^{[n/2]} (n^j/j!) \{j\ \{C_1 n^{-1}\}^{j-1}\ B_n^{-2-\delta}\ C_2\}$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} dt$$

$$\le CB_n^{-2-\delta} \sum_{j=1}^{[n/2]} (n^j/j!)\ j\ \{C_1 n^{-1}\}^{j-1}$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n/2} dt$$

$$\leq CnB_n^{-2-\delta} \sum_{j=1}^{[n/2]} \{\{nC_1 n^{-1}\}^{j-1}/(j-1)!\}$$

$$B_n \int_{-\infty}^{\infty} |\alpha_{[n/2]}(t, B_n/B_{[n/2]})|^{n/2} dt,$$

using (3.3.18)

$$\leq CnB_n^{-2-\delta} \sum_{j=1}^{[n/2]} \{\{C_1\}^{j-1}/(j-1)!\} \ C, \text{ using (3.3.15)}$$

$$\leq CnB_n^{-2-\delta} \ e^{c_2}$$

$$= C \ nB_n^{-2-\delta}. \qquad \ldots(3.4.6)$$

On the other hand,

$J_{12}(n)$

$$= \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1}, 1)|^{n-j}$$

$$|\{\int_{|u|>B_n} \cos(tuB_n^{-1}) d\Phi(u)\}|^j dt$$

$$\leq \sum_{j=1}^{[n/2]} (n^j/j!) \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1}, 1)|^{n-j} \{P(|X_o| > B_n)\}^j dt,$$

since $X_o \sim \Phi$.

$$\leq (e^{nR_o(B_n)} - 1) \ B_n \int_{-\infty}^{\infty} |\alpha_{[n/2]}(t, B_n/B_{[n/2]})|^{n/2} dt,$$

using (3.3.18)

$$\leq CnR_o(B_n), \text{ using (3.3.15)}$$

$$\leq CnR_o(B_n)$$

$$\leq CnB_n^{-1} e^{-B_n^2/2}, \text{ for sufficiently large n.} \qquad \ldots(3.4.7)$$

And, therefore, (3.4.4), (3.4.6) and (3.4.7) together imply that

$$J_1(n) \leq J_{11}(n) + J_{12}(n)$$

$$\leq C_1 \ nB_n^{-2-\delta} + C_2 \ nB_n^{-1} e^{-B_n^2/2}$$

$$\leq C_1 \ nB_n^{-2-\delta}, \text{ for sufficiently large n.} \qquad \ldots(3.4.8)$$

Next we estimate $J_2(n)$. In view of (3.3.16), we find, for sufficiently large but fixed s, that

$J_2(n)$

$$= \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^j dt$$

$$\le \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} \{P(|X_1|> B_n)\}^j dt$$

$$\le \{P|X_1|> B_n)\}^{n/2+1} \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} dt$$

$$\le \{P|X_1|> B_n)\}^{n/2+1} B_n \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(t,1)|^{2s} dt$$

$$\le \{P|X_1|> B_n)\}^{n/2+1} B_n C\, 2^n, \text{ using (3.3.16)}$$

$$= CB_nR_1(B_n) \ [4R_1(B_n)]^{n/2}$$

$$\le C\, B_nR_1(B_n) \ [4C_1B_n^{-2}H(B_n)]^{n/2}, \text{ using (3.2.4)}$$

$$\cong C\, B_nR_1(B_n) \ [4C_1/n]^{n/2}, \text{ using (3.2.5) for large n}$$

$$\le o(nR_1(B_n)). \qquad \text{...(3.4.9)}$$

Finally, we estimate $J_3(n)$. Using (3.3.17), for sufficiently large but fixed s, we find that

$$J_3(n) = \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^j dt$$

$$\le \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^0 |\beta_n(tB_n^{-1},1)|^{j-2s+2s} dt$$

$$\le \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!) \ \{P(|X_1|> B_n)\}^{j-2s} B_n \int_{-\infty}^{\infty} |\beta_n(t,1)|^{2s} dt$$

$$\le C\, B_n \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!) \ \{P(|X_1|> B_n)\}^{j-2s}$$

$$\le C\, B_n\, n^{2s-1} \ \{ P(|X_1|> B_n)\}^{n-4s+1} \ (n-n+2s-1)$$

$$\le C\, B_n\, n^{2s-1} \ \{C_1B_n^{-2}H(B_n)\}^{n-4s+1}, \text{ using (3.2.4)}$$

$\leq C\, B_n\, n^{2s-1-(n-4s+1)}\{C_1 n B_n^{-2} H(B_n)\}^{n-4s+1}$

$\leq C\, n^{-n+6s-2} B_n C_1^{n-4s+1}$, using (3.2.5)

$\leq O(nB_n^{-2-\delta})$. ...(3.4.10)

This fact can be sen by considering the logarithm of the expressions. Combining the results of (3.4.3), (3.4.8), (3.4.9) and (3.4.10), we obtain, for sufficiently large n,

$I_2 \leq J_1(n)+J_2(n)+J_3(n)$

$\leq C_1 n B_n^{-2-\delta} + C_2 n R_1(B_n)+C_3 n B_n^{-2-\delta}$

$\leq C_1 n B_n^{-2-\delta} + C_2 n R_1(B_n)$. ...(3.4.11)

Therefore, (3.4.1), (3.4.2) and (3.4.11) prove the theorem.□

Proof of Theorem 3.1.2:

Observe that

$(1+|x|) \leq 2\max(|x|, 1)$. ...(3.4.12)

In view of Theorem 3.1.1 above,

$|x|^{\beta}\Delta_n(x)\ \ I_{\{x\cdot |x|\leq 1\}}(x)$

$\leq \sup_{|x|\leq 1}\ \Delta_n(x)$

$\leq \sup_{-\infty<x<\infty}\ \Delta_n(x)$

$= O\{nB_n^{-3}\int_{|u|\leq B_n}|u|^3 dF(u)+nB_n^{-2-\delta}+nR_1(B_n)+\vartheta_n\}$. ...(3.4.13)

Thus to prove the theorem, it is enough to show that

$$|x|^{\beta}\Delta_n(x)\ I_{\{x:|x|>1\}}(x)$$

$$\leq C\{nB_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u) + nB_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)$$

$$+n^{-1}+|x|^{\beta}nB_n^{-2-\delta}+|x|^{\beta}nR_1(B_n)+\vartheta_n\} \qquad \ldots(3.4.14)$$

for large n, $0<\beta<2$ and under the assumptions **[A1]-[A5]**. From (3.2.13)-(3.2.15),

$$v_n(x) = a_n(x,x) + b_n(x,x). \qquad \ldots(3.4.15)$$

Thus, the proof of the theorem will be complete once we prove, for sufficiently large n,

$$|x|^{\beta}|a_n(x,x)-\phi(x)|\ I_{\{x:|x|>1\}}(x)$$

$$\leq C\{nB_n^{-4}\int_{|u|\leq|x|B_n} u^4 dF(u) + nB_n^{-3}\int_{|u|\leq|x|B_n}|u|^3 dF(u)$$

$$+ n^{-1} + nR_1(B_n) + nB_n^{-2-\delta} + \vartheta_n\} \qquad \ldots(3.4.16)$$

and

$$|x|^{\beta}|b_n(x,x)|I_{\{x:|x|>1\}}(x)$$

$$\leq C_1|x|^{\beta}nB_n^{-2-\delta} + C_2|x|^{\beta}nR_1(B_n). \qquad \ldots(3.4.17)$$

Now for $|x|>1$,

$$|x|^{\beta}|a_n(x,x)-\phi(x)|$$

$$= |x|^{\beta-2}\ x^2|a_n(x,x)-\phi(x)|$$

$$\leq (2\pi)^{-1}|x|^{\beta-2}\ \{I_{1n}(x) + I_{2n}(x) + I_{3n}(x)\} \qquad \ldots(3.4.18)$$

using integration by parts and the fact that $|A_{1n}(t,x)|$ and $|A_{1n}^{(1)}(t,x)|$ tend to 0 as $|t|\to\infty$. Here

$$I_{1n}(x) = \int_{|t| \le \varepsilon B_n} |A_n^{(2)}(t,x) - N^{(2)}(t)|\,dt \qquad \ldots(3.4.19)$$

$$I_{2n}(x) = \int_{|t| > \varepsilon B_n} |A_n^{(2)}(t,x)|\,dt \qquad \ldots(3.4.20)$$

$$I_{3n}(x) = \int_{|t| > \varepsilon B_n} |N^{(2)}(t)|\,dt \qquad \ldots(3.4.21)$$

$\varepsilon > 0$ being same as in Lemma 3.3.2.

Using the Lemma 3.3.7, for $2\delta_1 < 2-\beta$ and large n (δ_1 is as in Lemma (3.3.11)), we find that,

$$|x|^{\beta-2} I_{1n}(x)$$

$$\le C\Big\{ nB_n^{-4} \int_{|u| \le |x| B_n} u^4 dF(u) + nB_n^{-3} \int_{|u| \le |x| B_n} |u|^3 dF(u)$$

$$+ n^{-1} + nR_1(B_n) + \vartheta_n \Big\}. \qquad \ldots(3.4.22)$$

Using the results of the Lemma 3.3.4 and 3.3.8, we find that

$$I_{2n}(x) = \int_{|t| > \varepsilon B_n} |A_n^{(2)}(t,x)|\,dt$$

$$= \int_{|t| > \varepsilon B_n} |nB_n^{-1}\{(n-1)\{\alpha_n(tB_n^{-1},x)\}^{n-2}$$

$$\{\alpha_n^{(1)}(tB_n^{-1},x)\}^2 B_n^{-1}$$

$$+ \{\alpha_n(tB_n^{-1},x)\}^{n-1}\{\alpha_n^{(2)}(tB_n^{-1},x)\}B_n^{-1}\}|\,dt$$

$$\le n^2 B_n^{-2} \int_{|t| > \varepsilon B_n} |\alpha_n(tB_n^{-1},x)|^{n-2} |\alpha_n^{(1)}(tB_n^{-1},x)|^2 dt$$

$$+ nB_n^{-2} \int_{|t| > \varepsilon B_n} |\alpha_n(tB_n^{-1},x)|^{n-1} |\alpha_n^{(2)}(tB_n^{-1},x)|\,dt$$

$$\le n^2 B_n^{-2} \int_{|t|>\varepsilon B_n} |\alpha_n(tB_n^{-1},x)|^{n-2}\{E|X_1|\}^2 dt$$

$$+ nB_n^{-2} \int_{|t|>\varepsilon B_n} |\alpha_n(tB_n^{-1},x)|^{n-1}\{C|x|^{\delta_1} n^{-1}\} dt,$$

using (3.3.12) and (3.3.14)

$$\le C\, n^2 B_n^{-1} \int_{|t|>\varepsilon} |\alpha_n(t,x)|^{n-2} dt$$

$$+C|x|^{\delta_1} nB_n^{-3} H(B_n) \int_{|t|>\varepsilon} |\alpha_n(t,x)|^{n-1} dt$$

$$\le C\, n^2 B_n^{-1}\, \mu^{n-2} + C|x|^{2\delta_1}\, B_n^{-1}\, \mu^{n-1}$$

$$\le (C_1 n^2 B_n^{-1} \mu^{n-2} + C_2 |x|^{2-\beta} B_n^{-1} \mu^{n-1}),$$ because $2\delta_1 < 2-\beta$

$$\le C|x|^{2-\beta}\{n^2 B_n^{-1}\, \mu^{n-2} + B_n^{-1} \mu^{n-1}\}$$

$$\le C|x|^{2-\beta}\{n^2 B_n^{-1} \mu^{n-2}\},$$

because $nB_n^{-3} H(B_n) \mu^{n-1} = o(n^2 B_n^{-1} \mu^{n-2})$.

Thus,

$$|x|^{\beta-2} I_{2n}(x) \le Cn^2 B_n^{-1}\, \mu^{n-2} \qquad \text{...(3.4.23)}$$

for sufficiently large n.

Finally,

$$I_{3n}(x) = \int_{|t|>\varepsilon B_n} |N^{(2)}(t)| dt$$

$$= \int_{|t|>\varepsilon B_n} e^{-t^2/2} |t^2-1| dt$$

$$= 2 \int_{\varepsilon B_n}^{\infty} e^{-t^2/2} (t^2-1) dt$$

$$= 2 \int_{\varepsilon B_n}^{\infty} d[-te^{-t^2/2}]$$

$$= 2\varepsilon B_n\, e^{-(\varepsilon B_n)^2/2}$$

or equivalently,

$$|x|^{\beta-2} I_{3n}(x) \le CB_n \; e^{-(\varepsilon B_n)^2/2}. \qquad \ldots(3.4.24)$$

Now (3.4.16) follows from (3.4.18), (3.4.22), (3.4.23), and (3.4.24).

Next we establish (3.4.17).

Note that

$$|b_n(x,x)|$$

$$= |(2\pi)^{-1} \int_{-\infty}^{\infty} B_n(t,x) e^{-itx} dt|$$

$$\le \int_{-\infty}^{\infty} |B_n(t,x)| dt$$

$$\le \int_{-\infty}^{\infty} \sum_{j=1}^{n} \binom{n}{j} |\alpha_n(tB_n^{-1},x)|^{n-j} |\beta_n(tB_n^{-1},x)|^{j} dt$$

$$= \sum_{j=1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j} |\beta_n(tB_n^{-1},x)|^{j} dt$$

$$= \left\{ \sum_{j=1}^{[n/2]} + \sum_{j=[n/2]+1}^{n-2s} + \sum_{j=n-2s+1}^{n} \right\} \binom{n}{j}$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j} |\beta_n(tB_n^{-1},x)|^{j} dt$$

$$= J_1(n,x) + J_2(n,x) + J_3(n,x), \text{ say.} \qquad \ldots(3.4.25)$$

Consider $J_1(n,x)$.

$$J_1(n,x) = \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j} |\beta_n(tB_n^{-1},x)|^{j} dt$$

$$= \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j}$$

$$\left\{ \left| \{\beta_n(tB_n^{-1},x)\}^{j} - \{\int_{|u|>|x|B_n} \cos(tuB_n^{-1}) d\Phi(u)\}^{j} \right| \right\} dt$$

$$+ \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j} \left| \{ \int_{|u|>|x|B_n} \cos(tuB_n^{-1}) d\Phi(u)\}^{j} \right| dt$$

$$= J_{11}(n,x) + J_{12}(n,x), \text{ say.} \qquad \ldots(3.4.26)$$

We first consider $J_{11}(n,x)$.

Observe that, by assumption **[A1]**

$$|\{\beta_n(tB_n^{-1},x)\}^j - \{\int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\Phi(u)\}^j|$$

$$= |\beta_n(tB_n^{-1},x) - \int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\Phi(u)|$$

$$|\sum_{k=1}^{j}\{\beta_n(tB_n^{-1},x)\}^{j-k}\{\int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\Phi(u)\}^{k-1}|$$

$$= |\int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\{F(u)-\Phi(u)\}|$$

$$|\sum_{k=1}^{j}\{\beta_n(tB_n^{-1},x)\}^{j-k}\{\int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\Phi(u)\}^{k-1}|$$

$$\le \sum_{k=1}^{j}|\beta_n(tB_n^{-1},x)|^{j-k}\,|\int_{|u|>|x|B_n}\cos(tuB_n^{-1})\,d\Phi(u)|^{k-1}$$

$$\int_{|u|>|x|B_n}|v(u)-\phi(u)|\,du$$

$$\le \sum_{k=1}^{j}\{R_1(|X|B_n)\}^{j-k}\{R_o(|X|B_n)\}^{k-1}B_n^{-2-\delta}|x|^{-2-\delta}$$

$$\int_{|u|>|x|B_n}|u|^{2+\delta}|v(u)-\phi(u)|\,du$$

$$\le j[\max\{R_1(|x|B_n),\{R_o(|X|B_n)\}]^{j-1}B_n^{-2-\delta}|x|^{-2-\delta}$$

$$\int_{-\infty}^{\infty}|u|^{2+\delta}\,|v(u)-\phi(u)|\,du$$

$$\le j[\max\{C_1|x|^{-2}B_n^{-2}H(|x|B_n),C_2|x|^{-1}B_n^{-1}e^{-B_n^2/2}\}]^{j-1}$$

$$C_3|x|^{-2-\delta}B_n^{-2-\delta} \qquad \ldots(3.4.27)$$

the last inequality following from assumption **[A3]**.

We use inequality (3.4.27) now to estimate $J_{11}(n,x)$ as follows:

$$J_{11}(n,x) = \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j}$$

$$|\{\beta_n(tB_n^{-1},x)\}^j - \{\int_{|u|>|x|B_n} \cos(tuB_n^{-1})\,d\Phi(u)\}^j|dt$$

$$\le C_3|x|^{-2-\delta}B_n^{-2-\delta} \sum_{j=1}^{[n/2]} (n^j/j!)$$

$$\{j[\max\{C_1|x|^{-2}B_n^{-2}H(|x|B_n),\ C_2|x|^{-1}B_n^{-1}e^{-B_n^2/2}\}]^{j-1}\}$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n/2}dt$$

$$\le C_4|x|^{-2-\delta}nB_n^{-2-\delta}$$

$$\sum_{j=1}^{\infty} \{n[\max\{C_1B_n^{-2}H(B_n), C_2B_n^{-1}e^{-B_n^2/2}\}]\}^{j-1}/((j-1)!)$$

using (3.3.15), and this, in view of (3.2.5), becomes

$$= C\ |x|^{-2-\delta}nB_n^{-2-\delta} \qquad . \quad (3.4.28)$$

for large n.

On the other hand,

$$J_{12}(n,x)$$

$$= \sum_{j=1}^{[n/2]} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j}$$

$$|\{\int_{|u|>|x|B_n} \cos(tuB_n^{-1})\,d\Phi(u)\}|^j dt$$

$$\le \sum_{j=1}^{[n/2]} (n^j/j!) \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j}\{R_o(|x|B_n)\}^j dt$$

$$\le nR_o(|x|B_n)\sum_{j=1}^{n/2}((nR_o(|x|B_n))^{j-1}/(j-1)!)$$

$$\int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n/2}dt$$

$$\le nR_o(|x|B_n)e^{nR_o(B_n)}C,$$

using (3.3.18) and the fact that $R_o(|x|B_n) \le R_o(B_n)$.

$\leq Cn(C_1|x|B_n)^{-1}e^{-x^2B_n^2/2}$,

because $nR_o(B_n) \to 0$ and $1-\Phi(x) \leq (2\pi x^2)^{-1/2}e^{-x^2/2}$, $x \geq 0$

$= C|x|^{-1}nB_n^{-1}e^{-B_n^2/2}$. ...(3.4.29)

Hence, (3.4.26), (3.4.28) and (3.4.29) together imply that

$J_1(n,x) \leq J_{11}(n,x) + J_{12}(n,x)$

$\leq C_1nB_n^{-2-\delta}|x|^{-2-\delta} + C_2|x|^{-1}nB_n^{-1}e^{-B_n^2/2}$

$= |x|^{-1}(C_1nB_n^{-2-\delta} + C_2nB_n^{-1}e^{-B_n^2/2})$

$= C_1|x|^{-1}(nB_n^{-2-\delta} + o(nB_n^{-2-\delta}))$

$= C_1|x|^{-1}nB_n^{-2-\delta}$, for sufficiently large n. ...(3.4.30)

Next we estimate $J_2(n,x)$. In view of (3.3.16), we find that

$J_2(n,x)$

$$= \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j}|\beta_n(tB_n^{-1},x)|^j dt$$

$$\leq \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{2s} dt \{R_1(|x|B_n)\}^j$$

$$\leq \{R_1(|x|B_n)\}^{n/2+1} B_n \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(t,x)|^{2s} dt$$

$$\leq R_1(|x|B_n) \{R_1(|x|B_n)\}^{n/2} B_n \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} C$$

$\leq CB_nR_1(|x|B_n) \{4 R_1(|x|B_n)\}^{n/2}$

$\leq C_1 B_nR_1(|x|B_n) [4C_2x^{-2}B_n^{-2}H(|x|B_n)]^{n/2}$, using (3.2.4)

$\cong CB_nR_1(B_n)\{4C_2/n\}^{n/2}$,

using (3.2.5) and the fact that H is slowly varying

$= o(nR_1(B_n))$. ...(3.4.31)

Finally, we estimate $J_3(n,x)$. Using (3.3.17), we find, for sufficiently large but fixed s, that

$$J_3(n,x) = \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{n-j} |\beta_n(tB_n^{-1},x)|^{j} dt$$

$$\le \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},x)|^{0} |\beta_n(tB_n^{-1},x)|^{j-2s+2s} dt$$

$$\le \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!)\{P(|X_1|> |x|B_n)\}^{j-2s}$$

$$B_n \int_{-\infty}^{\infty} |\beta_n(t,x)|^{2s} dt$$

$$\le B_n \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!)\{P(|X_1|> |x|B_n)\}^{n-4s+1}\ C$$

$$\le CB_n n^{2s-1}\{P(|X_1|> |x|B_n)\}^{n-4s+1}(n-n+2s-1)$$

$$\le CB_n n^{2s-1}\{P(|X_1|> B_n)\}^{n-4s+1}$$

$$\le CB_n\ n^{2s-1}\{C_1 B_n^{-2} H(B_n)\}^{n-4s+1} \quad \text{using (3.2.4)}$$

$$\le CB_n\ n^{2s-1-(n-4s+1)}\{C_1 n B_n^{-2} H(B_n)\}^{n-4s+1}$$

$$= o(nB_n^{-2-\delta}). \qquad \ldots(3.4.32)$$

This fact can be sen by considering the logarithm of the expressions. Combining the results at (3.4.25), (3.4.30), (3.4.31) and (3.4.32), we obtain,

$$|x|^{\beta}|b_n(x,x)|\ I_{\{x:|x|>1\}}(x)$$

$$\le |x|^{\beta}\ \{J_1(n,x) + J_2(n,x) + J_3(n,x)\}$$

$$\le C_1|x|^{\beta-1}nB_n^{-2-\delta} + C_2|x|^{\beta}nR_1(B_n) + C_3|x|^{\beta}nB_n^{-2-\delta}$$

$$\le C_1|x|^{\beta}nB_n^{-2-\delta} + C_2|x|^{\beta}nR_1(B_n)$$

which proves (3.4.17). This completes the proof of the theorem.□

CONCLUDING REMARKS:

In this chapter we have obtained uniform rates of convergence and non-uniform bounds in the local limit theorem when the limit law is normal. In Chapter 4 we shall assume the limit law to be a non-normal and non-Cauchy stable law and obtain uniform rate of convergence type results. It should be noted that we do not assume the d.f. of summands to be in the domain of normal attraction and hence uniform rate type results only are proved.