CHAPTER-5

TRAJECTORY CONTROLLABILITY OF NONLINEAR INTEGRO-DIFFERENTIAL SYSTEM

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5.1 Introduction

The concept of controllability (introduced by Kalman-1960) leads to some very important conclusions regarding the behaviour of linear and nonlinear dynamical systems. Most of the practical systems are nonlinear in nature and hence the study of nonlinear systems is important. There are various notions of controllability such as complete controllability [52], approximate controllability [61], exact controllability ([65],[96]), partial exact controllability [107], null controllability, local controllability, etc. A new notion of controllability, namely, Trajectory controllability (T-controllability) is introduced here for some abstract nonlinear integro-differential systems. In T-controllability problems, we look for a control which steers the system along a prescribed trajectory rather than a control steering a given initial state to a desired final state. Thus, this is a stronger notion of controllability. Under suitable conditions, the

T-controllability of nonlinear system in finite dimensional case has been established in Section 5.2. Then the result is extended to infinite dimensional case in Section 5.3. We use the tools of monotone operator theory and set-valued analysis. We also use Lipschitzian and monotone nonlinearities with coercivity property in Section 5.3. Examples are provided to illustrate our results.

5.2 T-controllability of Finite-dimensional Systems

Consider the nonlinear scalar system

$$\left.\begin{array}{l} x'(t) = a(t)x(t) + b(t, u(t)) + f\Big(t, x(t), \int_0^t g(t, s, x(s))ds\Big) \\ x(0) = x_0, \end{array}\right\} \tag{5.2.1}$$

for all $0 \leq t \leq T < \infty$. Here, $a(t)$ is an L^1 function defined on $J = [0, T]$ and $b : J \times \mathbb{R} \mapsto \mathbb{R}$. For $t \in J$, the state $x(t)$ and the control $u(t)$ belong to $\mathbb{R}$. Further, $f : J \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear function satisfying the caratheodory conditions, i.e. f is measurable with respect to first argument and continuous with respect to second argument. Also, $g : \Delta \times \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear function which also satisfies the caratheodory conditions, where $\Delta = \{(t, s) \in J \times J;\ 0 \leq s \leq t \leq T\}$.

It may be noted that according to the definition of completely controllable system (refer Chapter 2), there is no constraint imposed on the control or on the trajectory.

Let $\mathcal{T}$ be the set of all functions $z(\cdot)$ defined on $J = [0, T]$ such that $z(0) = x_0, z(t) = x_1$ and z is differentiable almost everywhere.

Here we refer the Definition 2.1.4 and Definition 2.1.5.

In the system (5.2.1), both control $u(.)$ and state $x(.)$ appear nonlinearly. First let us look at the following system where the control appears linearly.

$$\left.\begin{array}{l} x'(t) = a(t)x(t) + b(t)u(t) + f\Big(t, x(t), \int_0^t g(t, s, x(s))ds\Big) \\ x(0) = x_0, \end{array}\right\} \tag{5.2.2}$$

Assumptions [A1]

(i) The functions $a(t)$ and $b(t)$ are continuous on J.

(ii) $b(.)$ do not vanish on J.

(iii) f is Lipschitz continuous with respect to second and third argument, i.e. there exist α_1, α_2 such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \alpha_1|x_1 - x_2| + \alpha_2|y_1 - y_2|$$

for all $x_1, x_2, y_1, y_2 \in R, t \in J$.

(iv) g is L^1 - Lipschitz continuous with respect to the third argument in the following sense.

$$\int_0^t |g(t, s, x(s)) - g(t, s, y(s))| \leq \beta|x(t) - y(t)|; \ x, y \in \mathcal{T}, (t, s) \in \Delta.$$

Under the above assumptions, one can easily construct the control explicitly to prove the T- controllability of the nonlinear system (5.2.2). To see this, we proceed as follows:

For each control $u \in L^2(J)$, the existence and uniqueness of the solution for the system (5.2.2) follow from Assumptions [A1] by using the standard arguments.

Let $z(t)$ be a given trajectory in $\mathcal{T}$. We define a control function $u(t)$ by

$$u(t) = \frac{z'(t) - a(t)z(t) - f\left(t, z(t), \int_0^t g(t, s, z(s))ds\right)}{b(t)}$$

With this control, (5.2.2) becomes,

$$\begin{aligned} x'(t) = \ & a(t)x(t) + z'(t) - a(t)z(t) - f\left(t, z(t), \textstyle\int_0^t g(t, s, z(s))ds\right) \\ & + f\left(t, x(t), \textstyle\int_0^t g(t, s, x(s)ds\right) \\ x(0) = \ & x_0. \end{aligned}$$

Setting $w(t) = x(t) - z(t)$, we have

$$\left.\begin{aligned} & w'(t) = a(t)w(t) + f\left(t, x(t), \textstyle\int_0^t g(t, s, x(s))ds\right) - f\left(t, z(t), \textstyle\int_0^t g(t, s, z(s))ds\right) \\ & w(0) = 0. \end{aligned}\right\} \quad (5.2.3)$$

By using the transition function $\phi(t, s) = e^{\int_s^t a(s)ds}$ for the ordinary differential equation $y'(t) = a(t)y(t)$, (5.2.3) can be rewritten as

$$w(t) = \int_0^t \phi(t,s)\Big[f\Big(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau\Big) - f\Big(s, z(s), \int_0^s g(s,\tau,z(\tau))d\tau\Big)\Big]ds.$$

Thus

$$\begin{aligned} |w(t)| &\leq \textstyle\int_0^t |\phi(t,s)|[\alpha_1|x(s) - z(s)| + \alpha_2|\int_0^s g(s,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,z(\tau))d\tau|]ds \\ &\leq \textstyle\int_0^t |\phi(t,s)|[\alpha_1|x(s) - z(s)| + \alpha_2\beta|x(s) - z(s)|]ds. \end{aligned}$$

That is,

$$|x(t) - z(t)| \leq (\alpha_1 + \alpha_2\beta)\int_0^t |\phi(t,s)||x(s) - z(s)|ds.$$

Hence, by Grownwall's inequality, it follows that

$$||x(t) - z(t)|| = 0.$$

This proves T-controllability of the system (5.2.2). ∎

As remarked earlier, in the above nonlinear system (5.2.2), the control $u(t)$ is appearing linearly. Let us now consider the case in which control as well as the state appear nonlinearly as in (5.2.1). We have the following theorem.

THEOREM 5.2.1 *Suppose that*

(i) $b(t,u)$ *is continuous.*

(ii) $b(t,u)$ *is coercive in the second variable, i.e.*

$$b(t,u) \longrightarrow \pm\infty \text{ as } u \longrightarrow \pm\infty$$

(iii) The function f is Lipschitz continuous in the second and third variable, uniformly in t, i.e. there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \alpha_1|x_1 - x_2| + \alpha_2|y_1 - y_2|, \ \ \forall x_1,x_2,y_1,y_2 \in \mathbb{R}, \mathrm{t} \in \mathrm{J}.$$

(iv) The function g is Lipschitz in the third variable uniformly in $(t,s) \in \Delta$, i.e., there exists $\beta > 0$ such that

$$|g(t,s,x) - g(t,s,y)| \leq \beta|x - y| \ \ \forall x,y \in \mathbb{R}, (\mathrm{t},\mathrm{s}) \in \Delta.$$

Then the nonlinear system (5.2.1) is T-controllable.

Proof: For each fixed u, the existence and uniqueness of the solution of the system (5.2.1) follow from the Lipschitz continuity of the functions f and g. Moreover, this solution satisfies the integral equation

$$\begin{aligned} x(t) = \phi(t,0)x_0 &+ \textstyle\int_0^t \phi(t,s)b(s,u(s))ds \\ &+ \textstyle\int_0^t \phi(t,s)f\Big(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau\Big)ds. \end{aligned} \tag{5.2.4}$$

Let $z \in \mathcal{T}$ be the prescribed trajectory with $z(0) = x_0$. We want to find a control u satisfying

$$z(t) = \phi(t,0)x_0 + \int_{t_o}^t \phi(t,s)b(s,u(s))ds + \int_0^t \phi(t,s)f\Big(s,z(s),\int_0^s g(s,\tau,z(\tau))d\tau\Big)ds.$$

The above equation can be written as

$$z(t) - \phi(t,0)x_0 - \int_0^t \phi(t,s)f\Big(s,z(s),\int_0^s g(s,\tau,z(\tau))d\tau\Big)ds = \int_0^t \phi(t,s)b(s,u(s))ds.$$

Differentiating with respect to t, we get

$$\begin{aligned} z'(t) - a(t)\phi(t,0)x_0 - \int_0^t a(t)\phi(t,s)f\Big(s,z(s),\int_0^s g(s,\tau,z(\tau))d\tau\Big)ds \\ -f\Big(t,z(t),\int_0^t g(t,s,z(s))ds\Big) \\ = \int_0^t a(t)\phi(t,s)b(s,u(s))ds + b(t,u(t)). \end{aligned} \tag{5.2.5}$$

The equation (5.2.5) can be written as

$$w(t) = \int_0^t k(t,s)w(s)ds + w_0(t), \tag{5.2.6}$$

where $w(t) = b(t,u(t))$, $k(t,s) = -a(t)\phi(t,s)$ and $w_0(t)$ is the left hand side of (5.2.5).

The equation (5.2.6) is a linear Volterra integral equation of the second kind and it has a unique solution $w(t)$ for each given $w_0(t)$ (refer [95]). Hence, it suffices to extract $u(t)$ from the solution $w(t)$. To extract $u(t)$, we use the technique of Deimling ([53], [54]).

Consider the multi-valued set function $G\colon [0,T] \to 2^{\mathbb{R}}$ defined by $G(t) = \{u \in \mathbb{R} : \mathrm{b(t,u) = w(t)}\}$. Since $b(\cdot,\cdot)$ and $w(\cdot)$ are continuous, by hypothesis (ii) $G(t)$ is nonempty for all t and upper semi-continuous. That is, $t_n \longrightarrow 0$ implies $G(t_n) \subset G(0) + \bar{B}_\varepsilon(0), \quad \forall n \geq n(\varepsilon, 0)$. Further, G has compact values. Hence G is Lebesgue measurable and therefore, has a measurable selection $u(\cdot)$.

This function u is the required control which steers the nonlinear system along the prescribed trajectory $z(\cdot)$. Hence the proof. ∎

REMARK 5.2.2 *(i) The control u obtained in Theorem 5.2.1 is measurable, may not be continuous. But, if we require control u to be continuous, we have to assume more stronger condition on $b(t,u)$.*

(ii) If the nonlinear function $b(t,u)$ is invertible, then $u(t)$ can be computed directly from $w(t) = b(t, u(t))$. For example, if $b(t,u)$ is strongly monotone, i.e., there exists $\beta > 0$ such that

$$|b(t,u) - b(t,v)| \geq \beta |u - v|,$$

then there exists a unique u such that $b(t,u) = w$. Note that the strong monotonicity implies coercivity.

(iii) If $b(t,u)$ is coercive and monotonically increasing with respect to u, then it can be seen that $b(t, \mathbb{R}) = \mathbb{R}$ and $b(t,u) = w(t)$ is solvable.

∎

EXAMPLE 5.2.3 *Consider the nonlinear integro-differential system with the control term $b(t,u) = u|u|$*

$$\left. \begin{array}{l} x'(t) = a(t)x(t) + b(t, u(t)) + sin\Big((x(t) + 3\int_0^t x(s)ds\Big) \\ x(0) = x_0. \end{array} \right\}$$

The control term $b(t,u)$ is continuous and coercive. One can now verify f and g as in Theorem 5.2.1 to get the T-controllability of the above system.

5.3 T-controllability of Infinite-dimensional Systems

In this section we consider a nonlinear integro-differential system defined in infinite dimensional space and generalize the results of Section 5.2. Let H and U be Hilbert spaces and consider the following nonlinear integro-differential system.

$$\left.\begin{array}{l} w'(t) = Aw(t) + B(t,u(t)) + F(t,w(t),\int_0^t G(t,s,w(s)ds), t \in J = [0,T] \\ w(0) = w_0, \end{array}\right\} \quad (5.3.1)$$

where the state $w(t) \in H$ and the control $u(t) \in U$, for each $t \in J$. The operator $A : H \mapsto H$ is a linear operator not necessarily bounded. The maps $B : J \times U \mapsto H$, $G : \Delta \times H \mapsto H$ and $F : J \times H \times H \mapsto H$ are nonlinear operators, where $\Delta = \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$.

We make the following assumptions on (5.3.1).

Assumptions [I]

(i) Let A be an infinitesimal generator of a strongly continuous C_0-semigroup of bounded linear operators $S(t), t \geq 0$. So, there exist constants $M_1 \geq 0$ and $w \in R^+$ such that

$$\| S(t)\| \leq M_1 e^{wt};\ t \geq 0$$

and also let

$$\int_0^T \int_0^t \| S(t-s)\|^2\, ds\, dt < \infty.$$

(ii) B and G satisfy caratheodory conditions, i.e., $B(t,\cdot) : U \mapsto H$ is continuous for $t \in J$ and $B(\cdot,x) : J \mapsto H$ is measurable for $x \in U$ and $G(t,s,\cdot) : H \mapsto H$ is continuous $\forall (t,s) \in \Delta$ and $G(\cdot,\cdot,x) : \Delta \mapsto H$ is measurable $\forall x \in H$.

(iii) F satisfies caratheodory conditions like G.

(iv) B, G, F satisfy the growth conditions:

$$\begin{aligned} \|B(t,u)\|_H &\leq b_0(t) + b_1\|u\|_U\ \forall u \in U,\ t \in J. \\ \|G(t,s,x)\| &\leq q_0(t) + q_1\|x\|_H \quad \forall\, t \in J,\ x \in H. \\ \|F(t,x,y)\|_H &\leq a_0(t) + a_1\|x\|_H + a_2\|y\|_H. \end{aligned}$$

Under Assumptions [I], a mild solution of the system (5.3.1) satisfies the Volterra integral equation

$$w(t) = S(t)w_0 + \int_0^t S(t-s)B(s,u(s))ds + \int_0^t S(t-s)F(s,w(s),\int_0^s G(s,\tau,w(\tau))d\tau)ds. \quad (5.3.2)$$

Let $\mathcal{T}$ be the set of all functions $z \in L^2(J, H)$ which are differentiable and $z(0) = w_0$. We say that the system (5.3.1) is T-controllable if for any $z \in T$, there exists an L^2-function $u : J \mapsto H$ such that the corresponding solution w of (5.3.1) satisfies $w(\cdot) = z(\cdot)$ a.e.

We make the following additional assumptions on F and B.

Assumptions [II]

(i) $F(t, x, y)$ is Lipschitz continuous with respect to x and y; i.e., there exist constants $\alpha_1, \alpha_2 \geq 0$ such that

$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\| \leq \alpha_1 \|x_1 - x_2\| + \alpha_2 \|y_1 - y_2\|$$

for all $x_1, x_2, y_1, y_2 \in H, t \in J$.

(ii) $G(t, s, x)$ is Lipschitz continuous with respect to x; i.e., there exists a constant $\beta > 0$ such that

$$\|G(t, s, x) - G(t, s, y)\| \leq \beta \|x - y\|, \quad x, y \in H, (t, s) \in \Delta.$$

(iii) B satisfies monotonicity and coercivity conditions. i.e.,

$$\langle B(t, u) - B(t, v), u - v \rangle \geq 0, \ \forall u, v \in U, t \in J$$

and

$$\lim_{\|u\| \to \infty} \frac{\langle B(t, u), u \rangle}{\|u\|} = \infty.$$

We now prove the T-controllability result for the system (5.3.1).

THEOREM 5.3.1 *Under Assumptions [I] and [II], the nonlinear system (5.3.1) is T-controllable.*

Proof: Let z be any trajectory in $\mathcal{T}$. Following the proof of the Theorem 5.2.1, we look for a control u satisfying

$$z(t) - S(t)w_0 - \int_0^t S(t - s)F\Big(s, z(s), \int_0^s G(s, \tau, z(\tau))d\tau\Big)ds = \int_0^t S(t - s)B(s, u(s))ds.$$

Differentiating with respect to t, we get

$$\Big[z'(t) - AS(t)w_0 - \int_0^t AS(t-s)F\Big(s, z(s), \int_0^s G(s,\tau,z(\tau))d\tau\Big)ds - F\Big(t, z(t), \int_0^t G(t,s,z(s))ds\Big)\Big] = \int_0^t AS(t-s)B(s,u(s))ds + B(t,u(t)). \tag{5.3.3}$$

Equation (5.3.3) can be rewritten in the form

$$y(t) = \int_0^t k(t,s)y(s)ds + y_0(t), \tag{5.3.4}$$

where $y(t) = B(t,u(t)),\ k(t,s) = -AS(t-s)$ and $y_0(t)$ is the left hand side of (5.3.3)

Define an operator $K: L^2(J,H) \to L^2(J,H)$ by

$$(Ky)(t) = \int_0^t k(t,s)y(s)ds \tag{5.3.5}$$

Assumption [I(i)] assures that K is a bounded linear operator [42]. Also, it can be easily proved that K^n is a contraction for sufficiently large n (refer [54],[107]). Hence by generalized Banach contraction principle, there exists a unique solution y for (5.3.4) for given $y_0 \in L^2(J,H)$. Therefore, T-controllability follows if we can extract $u(t)$ from the relation

$$B(t,u(t)) = y(t). \tag{5.3.6}$$

To see this, define an operator $N: L^2(I,H) \to L^2(I,H)$ by

$$(Nu)(t) = B(t,u(t)). \tag{5.3.7}$$

Assumptions [I(ii),(iii),(iv)] imply that N is well-defined, continuous and bounded operator. Assumption [II(iii)] shows that N is monotone and coercive. A hemi-continuous monotone mapping is of type (M) (see page 78 of [78]). Therefore, by Theorem 3.6.9 of Joshi and Bose [78], the nonlinear map N is onto. Hence there exists a control u satisfying (5.3.6). The measurability of $u(t)$ follows as u is in $L^2(I,H)$. This proves T-controllability of the system (5.3.1). ∎

COROLLARY 5.3.2 *If F and G are Lipschitz continuous and B is strongly monotone; that is, there exists $\beta > 0$ such that*

$$< B(t,u) - B(t,v), u - v > \geq \beta\|u-v\|^2 \quad \forall u,v \in H, \quad t \in J. \tag{5.3.8}$$

Then the system (5.3.1) is T-controllable.

Proof: The proof follows from the fact that the condition (5.3.8) implies Assumption [II(iii)]. ∎

REMARK 5.3.3 *We have not directly used the Assumptions [II(i)] and [II(ii)] of the Lipschitz continuity of f in the proof of the Theorem 5.3.1. Actually it is needed for the existence and uniqueness of the solution $w(\cdot)$ satisfying (5.3.2) for each control $u(\cdot)$. There are also other verifiable conditions for the uniqueness of the solution, in the literature, (see [65]).* ∎

EXAMPLE 5.3.4 *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Consider the system*

$$\left.\begin{array}{l}\dfrac{\partial y}{\partial t} = \Delta y + u(x,t) + \dfrac{1}{2}[sin^2 x(t) + siny(t)] \;\; in\; \Omega \times (0,T) \\ y(x,0) = 0 \;\; in\; \Omega \\ y(x,t) = 0 \;\; in\; \partial\Omega \times (0,T).\end{array}\right\}$$

The above system can be put into the form of (5.3.1) by defining $Aw(t) = \Delta w(t)$ for all $w(t) \in \mathcal{D}(A)$, where $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is the domain of A and $H = U = L^2(\Omega)$. Here the control term $B(t, u(t)) = u(t)$ is linear. The above system is T-controllable under the assumptions on F and G as in the theorem.

In the one dimensional case, that is say, $\Omega = (0,1)$, one can explicitly write $A : L^2(0,1) \to L^2(0,1)$ by $Aw = w''$, where $\mathcal{D}(A) = \{w \in H : w, w'$ are absolutely continuous, $w(0) = w(1) = 0\}$ and

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n.$$

Here $w_n(s) = \sqrt{2}\; sinns; n = 1, 2, 3 \ldots$ is the orthogonal set of eigenfunctions of A and (w, w_n) is the L^2 inner product. Further, A generates an analytic compact semigroup $S(t), t \geq 0$ in H given by

$$S(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in H.$$

Here $F(t, x(t), y(t)) = \frac{1}{2}[sin^2 x(t) + siny(t)]$ and $G(t, s, y(s)) = \frac{1}{2}[cosy(s)]$, both are Lipschitz continuous. ∎

We now specialize Theorem 5.3.1 for the case $H = \mathbb{R}^n$. So we consider the following finite dimensional nonlinear system in $\mathbb{R}^n$.

$$\left.\begin{array}{l} w'(t) = A(t)w(t) + B(t, u(t)) + F\Big(t, w(t), \int_0^t G(t,s,w(s))ds\Big) \\ w(0) = (w_0). \end{array}\right\} \qquad (5.3.9)$$

where A, B, F and G are as in (5.3.1) with H replaced by $\mathbb{R}^n$. Therefore Theorem 5.3.1 can be specialized for the system. The following theorem can be proved as in Theorem 5.2.1.

THEOREM 5.3.5 *Suppose that*

(i) F is Lipschitz continuous with respect to x and y and G is Lipschitz continuous in x

(ii) $B(t, u)$ satisfies

$$\lim_{\|u\|\to\infty} \frac{\langle B(t,u), u\rangle}{\|u\|} = \infty.$$

Then the nonlinear system (5.3.9) is T-controllable by a measurable control $u : J \mapsto \mathbb{R}^n$. ∎

EXAMPLE 5.3.6 *Consider the nonlinear 2-dimensional system,*

$$\begin{aligned} x_1'(t) &= a_{11}x_1 + a_{12}x_2 + sin\Big(y_1(t) + 3\int_0^t y_1(s)ds\Big) + cos\Big(y_2(t) + 3\int_0^t y_2(s)ds\Big) + u_1^2, \\ x_1(0) &= x_{01}. \\ x_2'(t) &= a_{21}x_1 + a_{22}x_2 + cos\Big(y_1(t) + 3\int_0^t y_1(s)ds\Big) + sin\Big(y_2(t) + 3\int_0^t y_2(s)ds\Big) + u_2^2, \\ x_2(0) &= x_{02} \end{aligned}$$

It can be easily verified that the above system satisfies the hypotheses of Theorem 5.3.5, and hence it is T-controllable.