CHAPTER-7

CONTROLLABILITY OF SINGULAR SYSTEMS USING α-TIMES INTEGRATED SEMIGROUPS AND COSINE FUNCTIONS

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7.1 Introduction

In this chapter, sufficient conditions for the controllability of two abstract singular systems are obtained. First we consider an n-dimensional abstract singular system described by the equation.

$$\left.\begin{array}{l} t\frac{d^2x}{dt^2} - (\alpha - 1 + tA)\frac{dx}{dt} + (\alpha - 1)Ax(t) = 0 \\ x(0) = 0;\ \lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\frac{d}{dt}x(t) = u \end{array}\right\} \tag{7.1.1}$$

where, $0 < \alpha < 1, t > 0, x(t) \in R^n$ for each t and A is a constant $n \times n$ matrix; u is considered as a control applied to the system at initial time.

Also, we consider an n-dimensional singular system described by the equation

$$\left.\begin{array}{l} t\frac{d^2x}{d^2t} + (1-\alpha)\frac{dx}{dt} - tAx(t) - (1-\alpha)A\int_0^t x(\tau)d\tau = 0 \\ x(0) = 0; \qquad \lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\frac{d}{dt}x(t) = u, \end{array}\right\} \tag{7.1.2}$$

where, $0 < \alpha \leq 1, u \in R^n$ for each t and A is a constant $n \times n$ matrix, u is considered as a control applied to the system at the initial time.

Recently, Yang [139] studied the solvability of the above equations and the solutions have been obtained in terms of α- times integrated semigroups $S_\alpha(t)$ and α- times integrated cosine function $C(t)$, respectively. That is, the solution of (7.1.1) is given by

$$x(t) = S_\alpha(t)u$$

and the solution of (7.1.2) is given by

$$x(t) = C(t)u$$

Here we provide the series solution and then sufficient condition for the controllability of (7.1.1) and (7.1.2) by using fixed point theory.

7.2 Solution of Singular differential equation in terms of Semigroup operator

In this section we obtain the solution of the singular differential equation in terms of α -times integrated semigroup $S_\alpha(t)$.

THEOREM 7.2.1 *(Yang [139]) Let $S_\alpha(t)$ be the α -times integrated semigroup generated by A for some $\alpha \in IR^n$ and $t \in R$, we have*

1.
$$AS_\alpha(t)u = S_\alpha(t)Au \tag{7.2.1}$$

2.
$$S_\alpha(t)u = \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t S_\alpha(\tau)Aud\tau \tag{7.2.2}$$

and

3. Solution of (7.1.1) is given by

$$x(t) = S_\alpha(t)u$$

We have the following series representation of the solution of the equation (7.1.1).

THEOREM 7.2.2 *The solution of the system (7.1.1) is given by*

$$x(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}\Big[I + \frac{At}{(\alpha+1)} + \frac{A^2t^2}{(\alpha+1)(\alpha+2)} + \frac{A^3t^3}{(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$+\frac{A^4t^4}{(\alpha+1)....(\alpha+4)} + ...\Big]u \quad (7.2.3)$$

Proof :By Theorem (7.2.1), for $u \in R^n$ we have

$$\begin{aligned}
x(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t S_\alpha(\tau)Aud\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t \Big[\frac{\tau^\alpha}{\Gamma(\alpha+1)}u + S_\alpha(S)Auds\Big]Ad\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + A\int_0^t \frac{\tau^\alpha}{\Gamma(\alpha+1)}ud\tau + A\int_0^t\int_0^\tau S_\alpha(S)Audsd\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + A\int_0^t \frac{\tau^\alpha}{\Gamma(\alpha+1)}ud\tau + A\int_0^t\int_0^\tau \Big[\frac{S^\alpha}{(\alpha+1)}u \\
&\quad + \int_0^s S_\alpha(p)Audp\Big]Adsd\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + A\int_0^t \frac{\tau^\alpha}{\Gamma(\alpha+1)}ud\tau + A^2\int_0^t\int_0^\tau \frac{S^\alpha}{\Gamma(\alpha+1)}udsd\tau \\
&\quad +A^2\int_0^t\int_0^\tau\int_0^s \Big[\frac{p^\alpha}{\Gamma(\alpha+1)}u + \int_0^p S_\alpha(v)Audv\Big]Adpdsd\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \frac{At^{\alpha+1}}{\Gamma(\alpha+1)(\alpha+1)}u + \frac{A^2}{\Gamma(\alpha+1)}\frac{t^{\alpha+2}}{(\alpha+1)(\alpha+2)}u
\end{aligned}$$

$$+\frac{A^3}{\Gamma(\alpha+1)}\frac{t^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}u+\frac{A^4}{\Gamma(\alpha+1)}\frac{t^{\alpha+4}}{(\alpha+1).....(\alpha+4)}u$$
$$+.........$$

$$\begin{aligned}
x(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}\Big[I+\frac{At}{(\alpha+1)}+\frac{A^2t^2}{(\alpha+1)(\alpha+2)}+\frac{A^3t^3}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\quad+\frac{A^4t^4}{(\alpha+1)....(\alpha+4)}+.......\Big]u
\end{aligned}$$

$$\begin{aligned}
\dot{x}(t) &= \frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\Big[I+\frac{At}{(\alpha+1)}+\frac{A^2t^2}{(\alpha+1)(\alpha+2)}+\frac{A^3t^3}{(\alpha+1)(\alpha+2)(\alpha+3)}+\\
&\quad......\Big]u+\frac{t^\alpha}{\Gamma(\alpha+1)}\Big[\frac{A}{(\alpha+1)}+\frac{2A^2t}{(\alpha+1)(\alpha+2)}+\\
&\quad\frac{3A^3t^2}{(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{4A^4t^3}{(\alpha+1)....(\alpha+4)}+.......\Big]u
\end{aligned}$$

$$\begin{aligned}
\ddot{x}(t) &= \frac{\alpha(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha+1)}\Big[I+\frac{At}{(\alpha+1)}+\frac{A^2t^2}{(\alpha+1)(\alpha+2)}+\frac{A^3t^3}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\quad+......\Big]u+\frac{2\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\Big[\frac{A}{(\alpha+1)}+\frac{2A^2t}{(\alpha+1)(\alpha+2)}+\frac{3A^3t^2}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\quad+\frac{4A^4t^3}{(\alpha+1)....(\alpha+4)}+.......\Big]u+\frac{t^\alpha}{\Gamma(\alpha+1)}\Big[\frac{2A^2}{(\alpha+1)(\alpha+2)}\\
&\quad+\frac{6A^3t}{(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{12A^4t^2}{(\alpha+1)....(\alpha+4)}+.....\Big]u
\end{aligned}$$

Hence,

$$\begin{aligned}
&t\ddot{x}(t)-(\alpha-1+tA)\dot{x}(t)+(\alpha-1)Ax(t)\\
&= \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\Big[\frac{2A^2}{(\alpha+1)(\alpha+2)}+\frac{6A^3t}{(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{12A^4t^2}{(\alpha+1).....(\alpha+4)}\\
&\quad+......\Big]u+\frac{t^\alpha(\alpha+1-At)}{\Gamma(\alpha+1)}\Big[\frac{A}{\alpha+1}+\frac{2A^2t}{(\alpha+1)(\alpha+2)}
\end{aligned}$$

$$+\frac{3A^3t^2}{(\alpha+1)(\alpha+2)(\alpha+3)}+.....\Big]u-\frac{At^\alpha}{\Gamma(\alpha+1)}\Big[I+\frac{At}{\alpha+1}$$

$$+\frac{A^2t^2}{(\alpha+1)(\alpha+2)}+\frac{A^3t^3}{(\alpha+1)(\alpha+2)(\alpha+3)}+.....\Big]u$$

$$= 0$$

Here, $\lim_{t\to 0} x(t) = 0$

and

$$\lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\frac{d}{dt}x(t)$$

$$= \lim_{t\to 0}\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)}t^{1-\alpha}t^{\alpha-1}\Big[I+\frac{At}{\alpha+1}+\frac{A^2t^2}{(\alpha+1)(\alpha+2)}+.....\Big]u = u$$

∎

7.3 Controllability Result via α-times Integrated Semigroup

We now give the sufficient condition for controllability of the singular system (7.1.1).

THEOREM 7.3.1 *Let $S_\alpha(t)$ be the α-times integrated semigroup generated by A. The system (7.1.1) is controllable to state x_1 during $[0,T]$ if*

$$\|S_\alpha(t)\|\|A\|T < \frac{T^\alpha}{\Gamma(\alpha+1)} \tag{7.3.1}$$

Further, the control u can be computed by using the iterative scheme

$$u_{n+1} = \frac{x_1}{k} - \frac{1}{k}\int_0^T S_\alpha(\tau)Ad\tau u_n \tag{7.3.2}$$

where $k = \frac{T^\alpha}{\Gamma(\alpha+1)}$

Proof: By Theorem (7.2.2), the solution of (7.1.1) is given by

$$x(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t S_\alpha(\tau)Au\,d\tau; \quad u \in R^n$$

For some fixed $t = T$, a control u steers the system from $x(0) = 0$ to $x(T) = x_1$ if

$$x_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^T S_\alpha(\tau)Aud\tau$$

$\Rightarrow x_1 - \int_0^T S_\alpha(\tau)Aud\tau = ku;$ where $k = t^\alpha/\Gamma(\alpha+1)$

$\Rightarrow u = \left[\frac{x_1}{k} - \frac{1}{k}\int_0^T S_\alpha(\tau)Aud\tau\right]$

Define an operator $N : R^n \to R^n$ such that

$$\begin{aligned} Nu &= \tfrac{x_1}{k} - \tfrac{1}{k}\textstyle\int_0^T S_\alpha(\tau)Aud\tau \\ \left\|Nu_1 - Nu_2\right\| &= \left\|\left\{\tfrac{x_1}{k} - \tfrac{1}{k}\textstyle\int_0^T S_\alpha(\tau)Au_1d\tau\right\} - \left\{\tfrac{x_1}{k} - \tfrac{1}{k}\textstyle\int_0^T S_\alpha(\tau)Au_2d\tau\right\}\right\| \\ &= \left\|\tfrac{1}{k}\textstyle\int_0^T S_\alpha(\tau)A(u_2 - u_1)d\tau\right\| \\ &\le \tfrac{1}{k}\textstyle\int_0^T \left\|S_\alpha(\tau)\right\| \left\|A\right\| \left\|u_2 - u_1\right\|d\tau \\ &= \tfrac{1}{k}\left\|S_\alpha(\tau)\right\| \left\|A\right\| \left\|u_2 - u_1\right\| \textstyle\int_0^T d\tau \end{aligned}$$

Thus,

$$\left\|Nu_1 - Nu_2\right\| \le \frac{1}{k}\left\|S_\alpha\right\| \left\|A\right\| T \left\|u_2 - u_1\right\|$$

N is contraction if, $\left\|Nu_1 - Nu_2\right\| \le \beta\left\|u_1 - u_2\right\|$; $\beta < 1$

Here,

$$\beta = \frac{1}{k}\|S_\alpha\| \ \|A\|T < 1$$

$$\Rightarrow \|S_\alpha\| \ \|A\|T < k = \frac{T^\alpha}{\Gamma(\alpha+1)}$$

Hence, if $\|S_\alpha\|\|A\|T < k$, then N is a contraction and by the Banach contraction principle (Joshi and Bose [78]), there exists a unique fixed point for N. This fixed point is the steering control for (7.1.1), steering $x(0) = 0$ to $x(T) = x_1$. Moreover by the Banach contraction principle, the control u can be computed by (Joshi and George [79]).

$$\begin{aligned} u_{n+1} &= Nu_n \\ u_{n+1} &= \frac{x_1}{k} - \frac{1}{k}\int_0^T S_\alpha(\tau)A \ d\tau \ u_n \end{aligned}$$

Note: In the system (7.1.1), if A is a real number then (7.1.1) reduces to,

$$t\ddot{x}(t) - (\alpha - 1 + t)\dot{x}(t) - \gamma x(t) = u; \quad \alpha, \gamma \in R$$

which is called confluent hypergeometric equation and its solution is confluent hypergeometric series.

7.4 Solution of Singular Differential Equation in Terms of Cosine Function

In this section we obtain the solution of the singular differential equation in terms of the $\alpha-$ times integrated cosine function $C(t)$.

THEOREM 7.4.1 *(Yang [137]) Let $C(t)$ be the $\alpha-$ times integrated cosine function generated by A for some $\alpha > 0$. Then for all $u \in R^n$ and $t \in R$, we have,*

(i)

$$AC(t)u = C(t)Au \tag{7.4.1}$$

(ii)

$$C(t)u = \frac{t^\alpha u}{\Gamma(\alpha+1)} + \int_0^t (t-\tau)C(\tau)Aud\tau \tag{7.4.2}$$

and,

(iii) Solution of (7.1.2) is given by

$$x(t) = C(t)u$$

We have the following series representation of the solution of the equation (7.1.2).

THEOREM 7.4.2 *The solution of the system (7.1.2) is given by*

$$x(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}\Big[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)......(\alpha+4)} + \frac{A^3t^6}{(\alpha+1)......(\alpha+6)} + ...\Big]u \tag{7.4.3}$$

Proof: By Theorem (7.4.1), for $u \in R^n$ we have

$$
\begin{aligned}
x(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t (t-\tau)C(\tau)Aud\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t (t-\tau)C(\tau)uAd\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t (t-\tau)\left[\frac{\tau^\alpha}{\Gamma(\alpha+1)}u + \int_0^\tau (\tau-s)C(s)uAds\right]Ad\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \frac{A}{\Gamma(\alpha+1)}\int_0^t (t-\tau)\tau^\alpha d\tau u \\
&\quad + \frac{A^2}{\Gamma(\alpha+1)}\int_0^t \int_0^\tau (t-\tau)(\tau-s)s^\alpha dsd\tau u \\
&\quad + A^3 \int_0^t \int_0^\tau \int_0^s (t-\tau)(\tau-s)(s-v)C(v)udvdsd\tau \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}u + \frac{A}{\Gamma(\alpha+1)}\int_0^t (t-\tau)\tau^\alpha d\tau u \\
&\quad + \frac{A^2}{\Gamma(\alpha+1)}\int_0^t \int_0^\tau (t-\tau)(\tau-s)s^\alpha ds\ d\tau\ u \\
&\quad + \frac{A^3}{\Gamma(\alpha+1)}\int_0^t \int_0^\tau \int_0^s (t-\tau)(\tau-s)(s-v)v^\alpha dvdsd\tau u \\
&\quad + A^4 \int_0^t \int_0^\tau \int_0^s \int_0^v (t-\tau)(\tau-s)(s-v)(v-p)C(p)udpdvdsd\tau
\end{aligned}
$$

Therefore we have,

$$
\begin{aligned}
x(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}\left[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)....(\alpha+4)}\right. \\
&\quad \left. + \frac{A^3t^6}{(\alpha+1)....(\alpha+6)} + ...\right]u
\end{aligned}
$$

We now verify the solution.

$$\dot{x}(t) = \frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\left[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)....(\alpha+4)} + \frac{A^3t^6}{(\alpha+1)....(\alpha+6)} + ...\right]u$$

$$+\frac{t^\alpha}{\Gamma(\alpha+1)}\left[\frac{2At}{(\alpha+1)(\alpha+2)} + \frac{4A^2t^3}{(\alpha+1).....(\alpha+4)} + \frac{6A^3t^5}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$\ddot{x}(t) = \frac{\alpha(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha+1)}\left[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)...(\alpha+4)} + \frac{A^3t^6}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$+2\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\left[\frac{2At}{(\alpha+1)(\alpha+2)} + \frac{4A^2t^3}{(\alpha+1)....(\alpha+4)} + \frac{6A^3t^5}{(\alpha+1)....(\alpha+6)} + ...\right]u$$

$$+\frac{t^\alpha}{\Gamma(\alpha+1)}\left[\frac{2A}{(\alpha+1)(\alpha+2)} + \frac{4.3A^2t^2}{(\alpha+1)....(\alpha+4)} + \frac{6.5A^3t^4}{(\alpha+1)....(\alpha+6)} + ...\right]u$$

Hence,

$$t\frac{d^2x(t)}{dt^2} + (1-\alpha)\frac{d}{dt}x(t) - tAx(t) - (1-\alpha)A\int_0^t x(\tau)d\tau$$

$$= \frac{\alpha(\alpha-1)t^{\alpha-1}}{\Gamma(\alpha+1)}\left[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)....(\alpha+4)} + \frac{A^3t^6}{(\alpha+1)....(\alpha+6)} + ...\right]u$$

$$+2\frac{\alpha t^\alpha}{\Gamma(\alpha+1)}\left[\frac{2At}{(\alpha+1)(\alpha+2)} + \frac{4A^2t^3}{(\alpha+1)...(\alpha+4)} + \frac{6A^3t^5}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$+\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\left[\frac{2A}{(\alpha+1)(\alpha+2)} + \frac{4.3A^2t^2}{(\alpha+1)...(\alpha+4)} + \frac{6.5A^3t^4}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$+\frac{(1-\alpha)\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\left[I + \frac{At^2}{(\alpha+1)(\alpha+2)} + \frac{A^2t^4}{(\alpha+1)...(\alpha+4)} + \frac{A^3t^6}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$+\frac{(1-\alpha)t^\alpha}{\Gamma(\alpha+1)}\left[\frac{2At}{(\alpha+1)(\alpha+2)} + \frac{4A^2t^3}{(\alpha+1)...(\alpha+4)} + \frac{6A^3t^5}{(\alpha+1)...(\alpha+6)} + ...\right]u$$

$$-\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}A + \frac{A^2}{\Gamma(\alpha+1)}\frac{t^{\alpha+3}}{(\alpha+1)(\alpha+2)} + \frac{A^3}{\Gamma(\alpha+1)}\frac{t^{\alpha+5}}{(\alpha+1)....(\alpha+4)}\right.$$

$$\left.+\frac{A^4}{\Gamma(\alpha+1)}\frac{t^{\alpha+7}}{(\alpha+1)....(\alpha+6)} + ...\right]u - (1-\alpha)A\int_0^t\left[\frac{\tau^\alpha}{\Gamma(\alpha+1)}u\right.$$

$$+\frac{A}{\Gamma(\alpha+1)}\frac{\tau^{\alpha+2}}{(\alpha+1)(\alpha+2)}u+\frac{A^2}{\Gamma(\alpha+1)}\frac{\tau^{\alpha+4}}{(\alpha+1)....(\alpha+4)}u$$

$$\left.+\frac{A^3}{\Gamma(\alpha+1)}\frac{\tau^{\alpha+6}}{(\alpha+1)....(\alpha+6)}u+....\right]d\tau$$

$$= 2\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\left[\frac{2At}{(\alpha+1)(\alpha+2)}+\frac{4A^2t^3}{(\alpha+1)....(\alpha+4)}+\frac{6A^3t^5}{(\alpha+1)...(\alpha+6)}+....\right]u$$

$$+\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{2}{(\alpha+1)(\alpha+2)}+\frac{4.3At^2}{(\alpha+1)....(\alpha+4)}+\frac{6.5A^2t^4}{(\alpha+1)....(\alpha+6)}+....\right]u$$

$$+\frac{(1-\alpha)t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{2}{(\alpha+1)(\alpha+2)}+\frac{4At^2}{(\alpha+1)....(\alpha+4)}+\frac{6A^2t^4}{(\alpha+1)...(\alpha+6)}+....\right]u$$

$$-\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[I+\frac{At^2}{(\alpha+1)(\alpha+2)}+\frac{A^2t^4}{(\alpha+1)....(\alpha+4)}+\frac{A^3t^6}{(\alpha+1)....(\alpha+6)}+....\right]u$$

$$-\frac{(1-\alpha)t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{1}{(\alpha+1)}+\frac{At^2}{(\alpha+1)....(\alpha+3)}+\frac{A^2t^4}{(\alpha+1)....(\alpha+5)}\right.$$

$$\left.+\frac{A^3t^6}{(\alpha+1)....(\alpha+7)}+....\right]u$$

$$= \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{2}{(\alpha+1)(\alpha+2)}+\frac{4.3At^2}{(\alpha+1)....(\alpha+4)}+\frac{6.5A^2t^4}{(\alpha+1)....(\alpha+6)}+....\right]u$$

$$+\frac{(1+\alpha)t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{2}{(\alpha+1)(\alpha+2)}+\frac{4At^2}{(\alpha+1)....(\alpha+4)}+\frac{6A^2t^4}{(\alpha+1)....(\alpha+6)}+....\right]u$$

$$-\frac{t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[I+\frac{At^2}{(\alpha+1)(\alpha+2)}+\frac{A^2t^4}{(\alpha+1)....(\alpha+4)}+\frac{A^3t^6}{(\alpha+1)....(\alpha+6)}+....\right]u$$

$$-\frac{(1-\alpha)t^{\alpha+1}}{\Gamma(\alpha+1)}A\left[\frac{1}{(\alpha+1)}+\frac{At^2}{(\alpha+1)....(\alpha+3)}+\frac{A^2t^4}{(\alpha+1)....(\alpha+5)}\right.$$

$$\left.+\frac{A^3t^6}{(\alpha+1)....(\alpha+7)}+....\right]u \qquad (7.4.4)$$

By taking the sum of first terms in each expressions in the equation (7.4.4), we get

$$\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)}\frac{2}{(\alpha+1)(\alpha+2)}+(1+\alpha)\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)}\frac{2}{(\alpha+1)(\alpha+2)}-\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)}+\frac{(\alpha-1)At^{\alpha+1}}{\Gamma(\alpha+1)(\alpha+1)}$$

$$=\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)(\alpha+1)(\alpha+2)}\Big[2+2(\alpha+1)-(\alpha+1)(\alpha+2)+(\alpha-1)(\alpha+2)\Big]=0$$

By taking the sum of second terms in each expressions in the equation (7.4.4), we get

$$\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)}\;\frac{At^2}{(\alpha+1)(\alpha+2)}\left[\frac{4.3}{(\alpha+3)(\alpha+4)}+\frac{4(\alpha+1)}{(\alpha+3)(\alpha+4)}-1-\frac{1-\alpha}{(\alpha+3)}\right]\frac{t^{\alpha+1}A}{\Gamma(\alpha+1)}$$

$$\frac{At^2}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\Big[12+4\alpha+4-(\alpha+3)(\alpha+4)+(\alpha-1)(\alpha+4)\Big]=0$$

By taking the sum of third terms in each expressions in the equation (7.4.4), we get,

$$\frac{6t^{\alpha+1}A}{\Gamma(\alpha+1)}\frac{A^2t^4}{(\alpha+1)......(\alpha+6)}\Big[6.5+6(\alpha+1)-(\alpha+5)(\alpha+6)+(\alpha-1)(\alpha+6)\Big]=0$$

and so on.

Thus the equation (7.1.2) is satisfied for $\alpha > 0$

As $\lim_{t\to 0} x(t) = 0$ and $\lim_{t\to 0} \frac{d}{dt}x(t) \neq 0$

If $0 < \alpha \leq 1$, we have

$\lim_{t\to 0} \Gamma(\alpha)t^{1-\alpha}\frac{d}{dt}(x(t))$

$$\begin{aligned}
&= \lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\left[\frac{\alpha t^{\alpha-1}}{\Gamma(\alpha+1)}\left\{I+\frac{At^2}{(\alpha+1)(\alpha+2)}+\frac{A^2t^4}{(\alpha+1)....(\alpha+4)}\right.\right.\\
&\quad \left.+\frac{A^3t^6}{(\alpha+1)....(\alpha+6)}+.....\right\}u+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left\{\frac{2At}{(\alpha+1)(\alpha+2)}+\frac{4A^2t^3}{(\alpha+1).....(\alpha+4)}\right.\\
&\quad \left.\left.+\frac{6A^3t^5}{(\alpha+1).....(\alpha+6)}+.....\right\}u\right]\\
&= u
\end{aligned}$$

■

COROLLARY 7.4.3 *The Solution of*

$$t\ddot{x}(t) - tAx(t) = 0 \tag{7.4.5}$$

with

$$x(0) = 0;\ \lim_{t\to 0}\Gamma(1)t^{1-1}\frac{d}{dt}x(t) = u \tag{7.4.6}$$

is given by

$$x(t) = \left[t + A\frac{t^3}{3!} + A^2\frac{t^5}{5!} + A^3\frac{t^7}{7!} + ...\right]u \tag{7.4.7}$$

Proof: Take $\alpha = 1$ in (7.4.2)

$$\begin{aligned}
x(t) &= tu + A\int_0^t (t-\tau)C(\tau)ud\tau \\
&= tu + A\int_0^t (t-\tau)\left[\tau u + A\int_0^\tau (\tau - s)C(s)uds\right]d\tau \\
&= tu + A\int_0^t (t-\tau)\tau ud\tau + A^2\int_0^t\int_0^\tau (t-\tau)(\tau-s)sudsd\tau \\
&\quad + A^3\int_0^t\int_0^\tau\int_0^s (t-\tau)(\tau-s)(s-v)C(v)udvdsd\tau \\
&= tu + A\int_0^t (t-\tau)\tau ud\tau + A^2\int_0^t\int_0^\tau (t-\tau)(\tau-s)sudsd\tau \\
&\quad + A^3\int_0^t\int_0^\tau\int_0^s (t-\tau)(\tau-s)(s-v)vudvdsd\tau \\
&\quad + A^4\int_0^t\int_0^\tau\int_0^s\int_0^v (t-\tau)(\tau-s)(s-v)(v-p)C(p)udpdvdsd\tau \\
&= tu + A\int_0^t (t-\tau)\tau ud\tau + A^2\int_0^t\int_0^\tau (t-\tau)(\tau-s)sudsd\tau \\
&\quad + A^3\int_0^t\int_0^\tau\int_0^s (t-\tau)(\tau-s)(s-v)vudvdsd\tau \\
&\quad + A^4\int_0^t\int_0^\tau\int_0^s\int_0^v (t-\tau)(\tau-s)(s-v)(v-p)pudpdvdsd\tau + ...
\end{aligned}$$

Therefore we have,

$$x(t) = \left[t + A\frac{t^3}{3!} + A^2\frac{t^5}{5!} + A^3\frac{t^7}{7!} + ...\right]u$$

We now verify the solution.

$$\dot{x}(t) = \left[I + A\frac{t^2}{2!} + A^2\frac{t^4}{4!} + A^3\frac{t^6}{6!} + \dots\dots\dots\right] u$$

$$\ddot{x}(t) = \left[At + \frac{A^2t^3}{3!} + \frac{A^3t^5}{5!} \dots\dots\dots\right] u \tag{7.4.8}$$

Hence,

$$t\ddot{x}(t) - tAx(t)$$
$$= t\left[At + \tfrac{A^2t^3}{3!} + \tfrac{A^3t^5}{5!} \dots\dots\dots\right] u - tA\left[t + A\tfrac{t^3}{3!} + \tfrac{t^5}{5!} + A^3\tfrac{t^7}{7!} + \dots\dots\dots\right] u$$
$$= 0$$

As, $\lim_{t\to 0} x(t) = 0$; and $\lim_{t\to 0} \frac{d}{dt}x(t) = u \neq 0$; $\alpha = 1$ ∎

7.5 Controllability Result via $\alpha-$times Integrated Cosine Function

We now give the sufficient condition for controllability of the singular system

$$\left.\begin{array}{l} t\ddot{x} + (1-\alpha)\dot{x} - tAx - (1-\alpha)A\int_0^t x(\tau)d\tau = 0 \\ \text{with } x(0) = 0;\ \lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\frac{d}{dt}x(t) = u;\ 0 < \alpha \leq 1, u \in R^n \end{array}\right\} \tag{7.5.1}$$

where u is considered as a control applied to the system at the initial time.

THEOREM 7.5.1 *Let $C(t)$ be the α- times integrated cosine function generated by A. The system (7.5.1) is controllable to state x_1 during [0,T] if*

$$\|C(t)\| \ \ \|A\|T^2 < 2\frac{T^\alpha}{\Gamma(\alpha+1)} \tag{7.5.2}$$

Further, the control u can be computed by using the iterative scheme

$$u_{n+1} = \frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)Ad\tau u_n \tag{7.5.3}$$

where $k = \frac{T^\alpha}{\Gamma(\alpha+1)}$

Proof: By Theorem 7.4.2, the solution of (7.5.1)is given by

$$x(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}u + \int_0^t (t-\tau)C(\tau)Au d\tau$$

For some fixed $t = T$, a control u steers the system from $x(0) = 0$ to $x(T) = x_1$ if

$x_1 = \frac{T^\alpha}{\Gamma(\alpha+1)}u + \int_0^T (T-\tau)C(\tau)Au d\tau$

$\Rightarrow x_1 - \int_0^T (T-\tau)C(\tau)Au d\tau = ku$; where $k = \frac{T^\alpha}{\Gamma(\alpha+1)} = \text{constant}$

$\Rightarrow u = \left[\frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)Au d\tau\right]$

Define an operator $N : R^n \to R^n$ such that

$$Nu = \frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)Au d\tau$$

$$\begin{aligned}
&\|Nu_1 - Nu_2\| \\
&= \left\| \left\{\frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)Au_1 d\tau\right\} - \left\{\frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)Au_2 d\tau\right\} \right\| \\
&= \left\| \frac{1}{k}\int_0^T (T-\tau)C(\tau)A(u_2 - u_1) d\tau \right\| \\
&\leq \frac{1}{k}\int_0^T |(T-\tau)| \, \|C(\tau)\| \|A\| \|(u_2 - u_1)\| d\tau \\
&= \frac{1}{k}\|C(\tau)\| \, \|A\| \|(u_2 - u_1)\| \int_0^T (T-\tau) d\tau \\
&= \frac{1}{k}\|C(\tau)\| \, \|A\| \, \|(u_2 - u_1)\| \frac{T^2}{2}
\end{aligned}$$

Thus, $\|Nu_1 - Nu_2\| \leq \frac{1}{k}\|C\| \, \|A\| \frac{T^2}{2} \|(u_2 - u_1)\|$
N is a contraction if

$$\|Nu_1 - Nu_2\| \leq \beta \|(u_2 - u_1)\|; \; \beta < 1$$

where,

$$\beta = \frac{1}{k}\|C\|\|A|\frac{T^2}{2} < 1$$

$$\Rightarrow \|C\|\|A\|T^2 < 2k = 2\frac{T^\alpha}{\Gamma(\alpha+1)}$$

Hence, if $\|C\|\|A\|T^2 < 2k$, then N is a contraction and by the Banach contraction principle (Joshi and Bose [78]) there exits a unique fixed point for N. This fixed point is the steering control for (7.5.1), steering $x(0) = 0$ to $x(T) = x_1$. Moreover, by the Banach contraction principle, the control u can be computed by (Joshi and George [79])

$$u_{n+1} = Nu_n$$

i.e.,

$$u_{n+1} = \frac{x_1}{k} - \frac{1}{k}\int_0^T (T-\tau)C(\tau)A \, d\tau \; u_n$$

∎