CHAPTER-1

INTRODUCTION

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1.1 Introduction

In the study of system dynamics and design, controllability analysis plays a vital role in most of the practical engineering systems. In controllability analysis, one is concerned about proving the existence of a control which will steer the system from a given initial state to a desired final state in a prescribed time. This basic property of control systems was first introduced by Kalman [80] in 1960's for finite dimensional linear systems and subsequently extended to nonlinear systems (refer Klamka [84], Joshi and George [79]) and to infinite dimensional systems (Triggiani [135], Lasiecka and Triggani [91]) by many researchers. They derived various types of controllability concepts both weak and strong such as approximate controllability (Zhou [144], George [62], McKibben [100]), exact controllability (Zuazua [145]-[147]), partial exact controllability (Nandakumaran and George [107], [108]), trajectory controllability (George [62]), stochastic controllability (Araposthathis, George and Ghosh [5]) etc. Although controllability problem concerning the linear systems has been widely studied with contributions by Kalman [80], Silverman and Meadows [125], there is relatively a modest attempt towards the analysis of nonlinear systems. The most notable ones being Mirza and Womak [102], Klamka [84] and Quinn and Carmichael [117] etc. For linear systems numerous necessary and sufficient conditions for controllability were established for finite and infinite dimensional systems. However, the nonlinear systems need to be analysed for various types of controllability, although there are many controllability results proved by many authors for nonlinear systems with some restricted nonlinearities, refer Dauer and Mahmudov [51].

The main objective of this thesis is to investigate controllability properties of systems described by differential equations, integro-differential equations, singular differential equations and differential inclusions by using the tools of functional analysis. For some time now, functional analysis has been firmly established as one of the basic disciplines of pure as well as applied mathematics. Its ideas and methods have transformed large parts of analysis almost beyond recognition. It has also greatly stimulated the growth of linear control theory. Our attention throughout this thesis is based on this area of mathematical theory of control. Though there has been a considerable development in the theory of nonlinear functional analysis, not much of it has percolated down as an application to control system theory. The main aim of our thesis is to study the basic properties of control systems using the techniques of nonlinear functional analysis.

We begin with the study of exact controllability of a dispersion system described by the third order nonlinear partial differential equation:

$$\frac{\partial w(x,t)}{\partial t} + \frac{\partial^3 w(x,t)}{\partial x^3} = (Gu)(x,t) + f(t, w(x,t)) \qquad (1.1.1)$$

on the domain $t \geq 0,\ 0 \leq x \leq 2\pi$; with periodic boundary condition

$$\frac{\partial^k w(0,t)}{\partial x^k} = \frac{\partial^k w(2\pi,t)}{\partial x^k};\ k = 0,1,2$$

and initial condition

$$w(x,0) = 0$$

where,

$$(Gu)(x,t) = g(x)\{u(t,x) - \int_0^{2\pi} g(s)u(s,t)ds\}$$

where $g(x)$ is a piecewise continuous non-negative function on $[0, 2\pi]$ such that

$$[g] \stackrel{\text{def}}{=} \int_0^{2\pi} g(x)dx = 1$$

and $f : [0,\infty] \times R \longrightarrow R$ is a nonlinear function.

This work extends the work of Russell and Zhang [121], in which authors considered a linear dispersion system. We obtain controllability result using two types of nonlinearities, namely, Lipschitzian and monotone. Also, exact controllability of the same system has been studied through the approach of integral contractors.(refer Altman [3]), see [66]. Further, we study the exact controllability of a more general system having a memory term. Here we replace $f(t, w(x,t))$ in (1) by $f(t, w(x,t), \int_0^t g_1(t,s,w(x,s))ds)$

and obtain the controllability result using fixed point theorem, where g_1 is a nonlinear function.

If a system is exactly controllable, it is possible to steer any initial state x_0 to any arbitrary final state x_1. But it does not give any idea about the path along which the system moves. Practically it is always desirable to steer the system from initial state x_0 to a final state x_1 along a prescribed trajectory. It may minimize certain cost involved in steering the system, depending upon the path chosen. It may safeguard the system also. This motivates to study the concept of trajectory controllability. Trajectory controllability is a stronger controllability notion.

By trajectory controllability we mean if any $z \in T$ (T is a set of functions $z(.)$ defined on $J = [0, T]$ such that $z(t_0) = x_0, z(t_1) = x_1$ and z is differentiable.), there exists a control $u \in L^2(J)$ such that the corresponding solution $x(.)$ of the system also satisfies $x(t) = z(t)$ a.e. George [62] studied Trajectory controllability (also known as T- controllability) of nonlinear systems and obtained sufficient conditions for it. In this thesis we obtain T-controllability results for nonlinear systems with memory.

We investigate the trajectory controllability of an integro-differential system (refer [44]) :

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax(t) + Bu(t) + f\Big(t, x(t), \textstyle\int_0^t g(t,s,x(s))ds\Big); \\ x(t_0) &= x_0;\ t \in J = [0, T] \end{aligned}\right\} \tag{1.1.2}$$

where, the state $x(t)$ and the control $u(t)$ lie in some Hilbert space H with the norm $\|.\|$; for each $t \in J$, $A : H \to H$ is a linear operator not necessarily bounded, $B : H \to H$ is a linear operator, $g : \Delta \times H \to H$ and $f : J \times H \times H \to H$ are nonlinear functions; here $\Delta = \{(t, s) \in J^2 : 0 \le s \le t \le T\}$.

There are various sufficient conditions for exact controllability of nonlinear systems governed by integral equations. Balachandran [9], and Balachandran and Sakthivel ([25],[26]) obtained some sufficient conditions for the exact controllability of Volterra type system under the restrictive assumptions on the system components. We study the exact controllability of the generalized Hammerstein integral equation of Volterra type, given by (refer [64])

$$x(t) = \int_0^t h(t,s)u(s)ds + \int_0^t k(t,s,x)f(s,x(s))ds;\ \ 0 \le t \le T < \infty \tag{1.1.3}$$

where the state of the system $x(t)$ lies in a Hilbert space X for each time $t \in J = [0, T]$. The control function $u(.)$ is from $U = L^2([0, T]; V)$, V is assumed to be a Hilbert space. The kernel $h(t, s)$ is an operator mapping V to X, while the kernel $k(t, s, x)$ maps from X to X; both are bounded linear operators, $f : J \times X \to X$ is a nonlinear operator for each $t, s \in J$. From system (1.1.3) it is clear that the initial state of (1.1.3) is zero, that is, $x(0) = 0 \in X$.

Most authors who studied the controllability of nonlinear systems, employed fixed point theory. Here, our approach is via monotone operator theory for the generalized Hammerstein type integral equation.

One can reduce nonlinear evolution systems with internal control in the above frame work to study the exact controllability. It is also possible to use the above system to study the exact controllability problems associated with the partial differential equations with boundary controls. This will be discussed as applications of the system(1.1.3)(refer to Lasiecka and Triggiani [91] and Yanzoo and Max [141]).

In continuation with controllability analysis, we consider the n-dimensional second order singular system of the form (refer [42]):

$$\left.\begin{aligned} t\tfrac{d^2x}{dt^2} - (\alpha - 1 + tA)\tfrac{dx}{dt} + (\alpha - 1)Ax(t) = 0 \\ x(0) = 0;\quad lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\tfrac{d}{dt}x(t) = u \end{aligned}\right\} \qquad (1.1.4)$$

where, $0 < \alpha < 1, t > 0, x(t) \in R^n$ for each t and A is a constant $n \times n$ matrix, u is considered as a control applied to the system at initial time. The controllability of the system (1.1.4) is studied by using α-times integrated cosine function, α is fractional order. The existence and uniqueness of solution of fractional order singular systems were studied by Yang [140] and Kochubei [88]. Such type of systems are very relevant in the applied fields (see Yang [137],[138]). We obtain controllability results for (1.1.4) via operator theory.

We shall also deal with second order singular integro-differential system of the form (refer [63]):

$$\left.\begin{aligned} t\tfrac{d^2x}{dt^2} - (1-\alpha)\tfrac{dx}{dt} - tAx(t) - (1-\alpha)A\textstyle\int_0^t x(\tau)d(\tau) = 0 \\ x(0) = 0;\quad lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}\tfrac{d}{dt}x(t) = u \end{aligned}\right\} \qquad (1.1.5)$$

where, $0 < \alpha \leq 1, u \in R^n, x(t) \in R^n$ for all t, A is a constant $n \times n$ matrix.

In our analysis for systems (1.1.4) and (1.1.5) we make use of α-times integrated semigroup and cosine functions, respectively. To study controllability properties of (1.1.4) and (1.1.5), we first reduce the systems into operator equations by using α -times integrated semi-group and α -times integrated cosine function and results are obtained by analyzing the operator equation.

The controllability of first and second-order differential and integro-differential inclusions with nonlocal conditions are quite important. It is always advantageous to treat the second-order abstract system directly rather than to convert them to first-order system by using the notion of sine and cosine operators. For example, Fitzgibbon [59] used the second-order system for establishing the boundedness of the solutions of the equation governing the transverse motion of an extensible beam. Lastly, in Chapter 8 we study the controllability on infinite time horizon for second-order semi-linear neutral functional differential inclusions of the form:

$$\left.\begin{array}{rcl} \frac{d}{dt}[x'(t) - f(t, x_t)] & \in & Ax(t) + Bu(t) + F(t, x_t, x'(t)), \quad t \in [0, \infty) \\ x_0 & = & \phi, x'(0) = x_0 \end{array}\right\} \quad (1.1.6)$$

Here, the state $x(t)$ takes the values in the real Banach space X with norm $|.|$ and the control $u(.)$ is given in $L^2(J, U)$, a Banach space of all admissible control function with U as a Banach space, B is a bounded linear operator from U to X, A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in X, $F : J \times C \times X \to 2^X, f : J \times C \to X, \phi \in C$, where $C = C([-r, 0], X)$. To study the controllability of the systems (1.1.6) , we invoke the fixed point theorem due to Ma [98], set-valued analysis ([53],[74]) and cosine operator theory ([132],[133]).

The basic methodology in our approach is to reduce the controllability problem into a solvability problem of an operator equation in suitable function space. Subsequently, we prove solvability results for the operator equations, which in turn imply the conditions for controllability of the system. In proving solvability results we invoke the theory of monotone operators, fixed point theory etc.

In this thesis we give an integrated functional analytic approach for investigating the following problems:

- Exact controllability of nonlinear third order dispersion system (1.1.1),
- Controllability of nonlinear integro-differential third order dispersion system of (1.1.1),
- Trajectory controllability of nonlinear integro-differential system (1.1.2),
- Exact controllability of generalized Hammerstein type integral equations (1.1.3) and its applications,
- Controllability of singular systems (1.1.4) and (1.1.5), using α-times integrated semigroups and cosine functions, respectively, and
- Controllability of system (1.1.6) described by second-order neutral functional differential inclusions with nonlocal condition on infinite time horizon.

The thesis will consist of eight chapters. The out-line of the thesis with brief contents is as follows:

After introducing various problems in Chapter 1, we have given necessary concepts of control systems theory and nonlinear functional analysis in Chapter 2.

In Chapter 3, we study the exact controllability of a nonlinear dispersion system. This work extends the work of Russell and Zhang [121], in which the authors considered a linear dispersion system. We obtain controllability result using two types of nonlinearities, namely, Lipschitzian and monotone [78]. In the later part of this chapter we use a weaker condition on the nonlinearity, called the integral contractor condition, to study the exact controllability of system(1.1.1). We enlarge our scope of investigation by considering integro-differential dispersion system and study the controllability by using Schaefer's fixed point theorem in Chapter 4.

In Chapter 5, we investigate the Trajectory controllability (T-controllability) of nonlinear integro-differential system (1.1.2). Here, first we study the $T-$controllability of nonlinear system in finite dimensional set up. For this, we use the tools of monotone operator theory [78] and set valued analysis [53]. We also use Lipschitzian and monotone nonlinearities with coercive property.

In Chapter 6, we focus ourselves in examining the exact controllability of a generalized nonlinear Hammerstein system (1.1.3) with noncompact linear part, see [135]. To study this we use the monotone operator theory. The controllability problem is reduced to solvability problem in appropriate spaces and prove our main result on exact controllability of system (1.1.3). We also give iterative scheme for the computation of control $u(t)$ and trajectory $x(t)$. To substantiate our theoretical study we give various useful applications and some important systems which can be put in the framework of (1.1.3).

Chapter 7 deals with controllability of the singular system (1.1.4) and we obtain results using α-times integrated semigroups. Also, in this chapter we study the controllability of the integro-differential singular system (1.1.5) using α-times integrated cosine function.

Chapter 8 concludes with sufficient conditions for controllability for second-order neutral functional differential inclusion system (1.1.6). We rely on the fixed point theorem due to Ma and multi-valued analysis to study the controllability of the system (1.1.6). Example is given to illustrate the theory.