Part II

Research Work

Chapter 3

Controllability of Matrix Second Order Systems: A Trigonometric Matrix Approach

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In this chapter, we study the controllability of finite dimensional matrix second order systems. A necessary and sufficient condition for the controllability of the matrix second order linear (MSOL) system has been proved in (Hughes and Skelton[42]). They converted the second order system into first order system and obtained controllability result. However, no computational scheme for the steering control was proposed. Here, we prove the controllability result, without reducing it to first order state space form and analyse the original form itself. Here we are interested in the steering of the states from given initial state to a desired final state, but velocity vector is not considered. That is, we deal with state controllability only. We take a trigonometric matrix approach to provide a computational algorithm for the actual computation of controlled state and steering control. We use matrix Sine and Cosine operators to find the solution of the matrix second order system. We employ páde approximation for the computation of Sine and Cosine matrices. We also invoke tools of nonlinear functional analysis like fixed point theorem to obtain controllability result for the nonlinear system. We provide numerical example to substantiate our results. Section 3.1 provides introduction to the problem and Section 3.2 deals with the solution of MSOL system and Matrix second order nonlinear(MSON) system. In section 3.3, we prove controllability results for MSOL, and controllability result of MSON is provided in Section 3.4. Section 3.5 deals with the computational algorithm for Sine and Cosine matrices and steering control for linear and nonlinear systems. Examples are provided to illustrate the results. Summary of the work presented in this chapter is given in section 3.6.

3.1 Introduction

Matrix second order systems capture the dynamic behavior of many natural phenomena and have found applications in many fields such as vibration and structural analysis, space craft control and robotics control and hence have attracted much attention(Balas[10], Diwaker and Yedavalli[24], Hughes and Skelton[42], Laub and Arnold [53], Demetriou[23]). Also distributed parameters systems, very often, discretized to second order systems. Generally, second order systems are transformed to first order state-space representation for studying its controllability. But, recent work of (Skelton[70], Diwakar and Yedavalli [24],[25]) showed that there are several problems associated with such transformations as, the second order system losses its physical insight when they are transformed to first order state space form. It become computationally less efficient as the dimension of the system is higher than the second order system. Sparsity and many other special nature of the original matrices are not preserved. So, we do not reduce the system into first order and analyse the original second order form itself. We use matrix Sine and Cosine matrices to find the solution of the MSOL system and MSON system. We employ *Páde* approximation for the computation of matrix Sine and Cosine matrices.

Here we investigate the controllability property of the system governed by a Matrix Second Order Nonlinear (MSON) differential equation:

$$\left.\begin{array}{l}\frac{d^2x(t)}{dt^2}+A^2x(t)=Bu(t)+f(t,x(t))\\ x(0)=x_0,\;\; x'(0)=y_0.\end{array}\right\} \tag{3.1.1}$$

where, the state x(t) is in R^n and the control $u(t)$ is in R^m, A^2 is a constant matrix of order $n\times n$ and B is a constant matrix of order $n\times m$ and $f:[0,T]\times R^n\rightarrow R^n$ is a nonlinear function satisfying Caratheodory conditions, that is, f is measurable with respect to t for all x and continuous with respect to x for almost all $t\in[0,T]$. The initial states x_0 and y_0 are in R^n. The corresponding Matrix Second Order Linear (MSOL) system is :

$$\left.\begin{array}{l}\frac{d^2x(t)}{dt^2}+A^2x(t)=Bu(t)\\ x(0)=x_0,\;\; x'(0)=y_0.\end{array}\right\} \tag{3.1.2}$$

Definition 3.1.1. *The system (3.1.1) is said to be controllable on* $[0,T]$ *if for each pair* $x_0,x_1\in R^n$, *there exists a control* $u(\cdot)\in L^2([0,T];R^m)$ *such that the corresponding solution of (3.1.1) together with* $x(0)=x_0$ *also satisfies* $x(T)=x_1$.

As we know the matrix exponential $y(t)=e^{At}y_0$ provides the solution to the first order differential equation

$$\frac{dy}{dt}=Ay,\;\; y(0)=y_0.$$

Trigonometric matrix functions play a similar role in matrix second order differential equation

$$\frac{d^2y}{dt^2}+Ay=0,\;\; y(0)=x_0,\;\; y'(0)=y_0,$$

That is, the solution of the above second order system, using Sine and Cosine matrices, is given by(refer Hargreaves and Higham [41])

$$y(t)=cos(\sqrt{A}t)x_0+(\sqrt{A})^{-1}sin(\sqrt{A}t)y_0.$$

where $cos(\sqrt{A}t)$ and $sin(\sqrt{A}t)$ are matrix Sine and matrix Cosine as defined in (3.1.3) and (3.1.4).

The complex exponential of a matrix is defined as the series, (refer Chen[17])

$$\begin{aligned} e^{iAt} &= I + iAt + \frac{(iAt)^2}{2!} + \frac{(iAt)^3}{3!} + \frac{(iAt)^4}{4!} + \frac{(iAt)^5}{5!} + \frac{(iAt)^6}{6!} + \frac{(iAt)^7}{7!} + \ldots\ldots.. \\ &= (I - \frac{A^2t^2}{2!} + \frac{A^4t^4}{4!} - \frac{A^6t^6}{6!} + \ldots\ldots.) + i(At - \frac{A^3t^3}{3!} + \frac{A^5t^5}{5!} - \frac{A^7t^7}{7!} + \ldots\ldots.). \end{aligned}$$

Convergence of the above series has been well established, (refer Brockett [14]). We define Cosine and Sine matrix of A as the real and imaginary part of the above series. That is,

$$cos(At) = I - \frac{(At)^2}{2!} + \frac{(At)^4}{4!} - \frac{(At)^6}{6!} + \ldots\ldots.. \tag{3.1.3}$$

$$sin(At) = At - \frac{(At)^3}{3!} + \frac{(At)^5}{5!} - \frac{(At)^7}{7!}\ldots\ldots.. \tag{3.1.4}$$

Since exponential matrix series converges, the sub-series defined in (3.1.3) and (3.1.4) also converge. Further,

$$e^{iAt} = cos(At) + isin(At)$$

and

$$e^{-iAt} = cos(At) - isin(At)$$

Using the above identities, we have the following representation of Cosine and Sine matrices in terms of matrix exponentials:

$$cos(At) = \frac{e^{iAt} + e^{-iAt}}{2} \quad and \quad sin(At) = \frac{e^{iAt} - e^{-iAt}}{2i}$$

Properties: The Sine and Cosine matrices satisfy following properties:

(i) $cos(0) = I$.

(ii) $sin(0) = 0$.

(iii) $\frac{d}{dt}cos(At) = -Asin(At)$.

(iv) $\frac{d}{dt}sin(At) = Acos(At)$.

(v) $cos(At)$ is non-singular matrix, if A is nonsingular.

(vi) $sin(A(t-s)) = sin(At)cos(As) - cos(At)sin(As)$ for all t.

(vii) $A^{-1}cos(At) = cos(At)A^{-1}$.

3.2 Solution Using Cosine and Sine Matrices

We use Sine and Cosine matrices to reduce the system (3.1.1) into an integral equation. It can be shown easily that the matrices $X_1(t) = cos(At)$ and $X_2(t) = A^{-1}sin(At)$ satisfy the homogeneous linear matrix differential equation

$$\frac{d^2X(t)}{dt^2} + A^2X(t) = 0 \tag{3.2.1}$$

Here, if A is a singular matrix, then X_2 is expanded as the power series, (refer Hargreaves and Higham [41])

$$X_2 = It - \frac{A^2t^3}{3!} + \frac{A^4t^5}{5!} - \frac{A^6t^7}{7!}\ldots\ldots \tag{3.2.2}$$

General solution of the homogeneous system

$$\frac{d^2x(t)}{dt^2} + A^2x(t) = 0$$

is given by

$$x(t) = X_1(t)C_1 + X_2(t)C_2$$

$$x(t) = cos(At)C_1 + A^{-1}sin(At)C_2$$

where, C_1 and C_2 are arbitrary vectors in R^n. Now using the method of variation of parameter, a particular integral(P.I) for the nonhomogeneous system (3.1.2) is given by

$$P.I = -X_1(t)\int_0^t W^{-1}(s)X_2(s)Bu(s)ds + X_2(t)\int_0^t W^{-1}(s)X_1(s)Bu(s)ds$$

where, the Wronskian

$$\begin{aligned} W &= \begin{vmatrix} X_1 & X_2 \\ X_1' & X_2' \end{vmatrix} \\ &= \begin{vmatrix} cos(At) & A^{-1}sin(At) \\ -Asin(At) & A^{-1}Acos(At) \end{vmatrix} = I \end{aligned}$$

$$\begin{aligned}
P.I &= -cos(At)\int_0^t A^{-1}sin(As)Bu(s)ds + A^{-1}sin(At)\int_0^t cos(As)Bu(s)ds \\
&= \int_0^t A^{-1}(-cos(At)sin(As) + sin(At)cos(As))Bu(s)ds \\
&= \int_0^t A^{-1}sin(A(t-s))Bu(s)ds, \text{ (using property (vi))}.
\end{aligned}$$

Hence the solution of (3.1.2) is given by

$$x(t) = cos(At)C_1 + A^{-1}sin(At)C_2 + \int_0^t A^{-1}sin(A(t-s))Bu(s)ds.$$

Applying the initial conditions $x(0) = x_0$ and $x'(0) = y_0$, the solution becomes

$$x(t) = cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(t-s))Bu(s)ds. \tag{3.2.3}$$

Following the same approach the solution of the nonlinear system (3.1.1) can be written as

$$x(t) = cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(t-s))Bu(s)ds + \int_0^t A^{-1}sin(A(t-s))f(s,x(s))ds \tag{3.2.4}$$

We remark that the above form of solution valid even if the matrix A is singular, in that case $A^{-1}sin(At)$ is to be taken as in (3.2.2).

3.3 Controllability: Linear System

In this section we obtain necessary and sufficient conditions for the controllability of the linear system (3.1.2). We make use of the following lemmas to prove the controllability result.

Lemma 3.3.1. (Chen[17]) *Let f_i, for $i = 1, 2, ..., n$, be $1 \times p$ complex vector valued continuous functions defined on $[t_1, t_2]$. Let F be the $n \times p$ matrix with f_i as its i^{th} row. Define*

$$W(t_1, t_2) = \int_{t_1}^{t_2} F(t)F^*(t)dt$$

Then $f_1, f_2, ..., f_n$ are linearly independent on $[t_1, t_2]$ if and only if the $n\times n$ constant matrix $W(t_1, t_2)$ is nonsingular.

Lemma 3.3.2. (Chen *[17]*) *Assume that for each i, f_i is analytic on $[t_1, t_2]$. Let F be the $n \times p$ matrix with f_i as its i^{th} row, and let $F^{(k)}$ be the k^{th} derivative of F. Let t_0 be any fixed point in $[t_1, t_2]$. Then the f_i are linearly independent on $[t_1, t_2]$ if and only if*

$$Rank[F(t_0) : F^{(1)}(t_0) : .... : F^{(n-1)}(t_0) : .....] = n$$

The necessary and sufficient condition for the controllability of the linear system (3.1.2) is given in the following theorem.

Theorem 3.3.1. *The following statements regarding the linear system (3.1.2) are equivalent:*

(a) *The linear system (3.1.2) is controllable on $[0, T]$.*

(b) *The rows of $A^{-1}sin(At)B$ are linearly independent.*

(c) *The controllability Grammian,*

$$W(0,T) = \int_0^T A^{-1}sin(A(T-s))BB^*(A^{-1}sin(A(T-s)))^*ds \qquad (3.3.1)$$

is nonsingular.

(d)

$$Rank[B : A^2B : (A^2)^2B : .... : (A^2)^{n-1}B] = n. \qquad (3.3.2)$$

□

Proof. First we shall prove the implication $(a) \Rightarrow (b)$ by contradiction. Suppose that the system (3.1.2) is controllable but the rows of $A^{-1}sin(At)B$ are linearly dependent functions on [0,T]. Then there exists a nonzero constant $1 \times n$ row vector α such that

$$\alpha A^{-1}sin(At)B = 0 \ \forall \ t \in [0, T] \qquad (3.3.3)$$

Let us choose $x(0) = x_0 = 0$, $x'(0) = y_0 = 0$. Therefore the solution (3.2.3) becomes

$$x(t) = \int_0^t A^{-1}sin(A(t-s))Bu(s)ds$$

Since the system (3.1.2) is controllable on [0,T], taking $x(T) = \alpha^*$, where α^* is the conjugate transpose of α.

$$x(T) = \alpha^* = \int_0^T A^{-1}sin(A(T-s))Bu(s)ds$$

Now premultiplying both sides by α, we have

$$\alpha\alpha^* = \int_0^T \alpha A^{-1}sin(A(T-s))Bu(s)ds.$$

From equation (3.3.3)

$$\alpha\alpha^* = 0 \text{ and hence } \alpha = 0$$

Hence it contradicts our assumption that α is non-zero. This implies that rows of $A^{-1}sin(At)B$ are linearly independent on [0,T].

Now we prove the implication $(b) \Rightarrow (a)$.

Suppose that the rows of $A^{-1}sin(At)B$ are linearly independent on $[0, T]$. Therefore by Lemma 3.3.1, the $n \times n$ constant matrix

$$W(0,T) = \int_0^T A^{-1}sin(A(T-s))BB^*(A^{-1}sin(A(T-s)))^*ds$$

is nonsingular.

Now we claim that the control

$$u(t) = B^*(A^{-1}sin(A(T-t)))^*W^{-1}(0,T)(x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0) \quad (3.3.4)$$

transfers the initial state x_0 to the final state x_1 during $[0, T]$. Substituting (3.3.4) for u(t) in the solution (3.2.3), we obtain

$$x(t) = cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(t-s))BB^*(A^{-1}sin(A(T-s)))^*$$
$$W^{-1}(0,T)(x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0)ds$$

At t=T, we have

$$\begin{aligned}
x(T) &= cos(AT)x_0 + A^{-1}sin(AT)y_0 + \int_0^T A^{-1}sin(A(T-s))BB^* \\
&\quad (A^{-1}sin(A(T-s)))^* W^{-1}(0,T)(x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0)ds \\
&= cos(AT)x_0 + A^{-1}sin(AT)y_0 + W(0,T)W^{-1}(0,T) \\
&\quad (x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0) \\
&= cos(AT)x_0 + A^{-1}sin(AT)y_0 + (x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0) \\
&= x_1
\end{aligned}$$

Hence the system is controllable.

The implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (b)$ follow directly from Lemma 3.3.1.

Now we shall obtain the implication $(c) \Rightarrow (d)$.

Suppose that the controllability Grammian

$$W(0,T) = \int_0^T A^{-1}sin(A(T-s))BB^*(s)(A^{-1}sin(A(T-s)))^*$$

is nonsingular. Hence by Lemma 3.3.1, the rows of $A^{-1}sin(At)B$ are linearly independent on $[0,T]$. Since the entries of $A^{-1}sin(At)B$ are analytic functions, applying the Lemma 3.3.2, the rows of $A^{-1}sin(At)B$ are linearly independent on [0,T] if and only if

$$Rank[A^{-1}sin(At)B : A^{-1}cos(At)AB : -A^{-1}sin(At)A^2B : -A^{-1}cos(At)A^3B :$$

$$A^{-1}sin(At)A^4B : A^{-1}cos(At)A^5B........] = n.$$

for any $t \in [0,T]$. Let $t = 0$, this reduces to

$$Rank[0 : B : 0 : A^2B : 0 : .... : (A^2)^{n-1}B : ....] = n$$

$$Rank[B : A^2B : (A^2)^2B : ..... : (A^2)^{n-1}B : .....] = n$$

Using Cayley-Hamilton theorem,

$$Rank[B : A^2B : (A^2)^2B : ..... : (A^2)^{n-1}B] = n$$

Now to prove the implication $(d) \Rightarrow (c)$, we assume that

$$Rank[B : A^2B : (A^2)^2B : ..... : (A^2)^{n-1}B] = n$$

Thus by Lemma 3.3.2, the rows of $A^{-1}sin(At)B$ are linearly independent. Hence Lemma 3.3.1 implies

$$W(0,T) = \int_0^T A^{-1}sin(A(T-s))BB^*(s)(A^{-1}sin(A(T-s)))^* ds$$

is nonsingular.

Thus for the linear system(3.1.2) , the control $u(t)$ defined by (3.3.4), steers the state from x_0 to x_1 during $[0, T]$. Since x_0 and x_1 are arbitrary, the system (3.1.2) is controllable. □

Remark 3.3.1. *Hughes and Skelton[42] obtained the condition (3.3.2) by converting the system into first order system. However, our approach is different and the result obtained is directly from the second order system and also it provides a method to compute the steering control.*

3.4 Controllability: Nonlinear System

We now investigate the controllability of the nonlinear system (3.1.1). We assume that the corresponding linear system (3.1.2) is controllable and the control function u belongs to $L^2([0, T], R^m)$. We use the following definition.

Definition 3.4.1. *An $m \times n$ matrix function $P(t)$ with entries in $L^2([0, T])$ is said to be a steering function for (3.1.2) on $[0, T]$ if*

$$\int_0^T A^{-1}sin(A(T-s))BP(s)ds = I,$$

I being the identity matrix on R^n.

The linear system (3.1.2) is controllable if and only if there exists a steering function $P(t)$ for the system (3.1.2) (refer Russel [68]).

Remark 3.4.1. *If the controllability Grammian (3.3.1) is nonsingular then*

$$P(t) = B^*(A^{-1}sin(A(T-t))^*W^{-1}(0,T) \tag{3.4.1}$$

defines a steering function for the linear system (3.1.2).

Now the nonlinear system (3.1.1) is controllable on $[0,T]$ if and only if for every given x_1 and x_0 in R^n there exists a control u, such that

$$x_1 = x(T) = cos(AT)x_0 + A^{-1}sin(AT)y_0 + \int_0^T A^{-1}sin(A(T-s))f(s,x(s))ds$$
$$+ \int_0^T A^{-1}sin(A(T-s))Bu(s)ds$$

Consider the control $u(t)$ defined by

$$u(t) = P(t)\{x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0 - \int_0^T A^{-1}sin(A(T-s))f(s,x(s))ds\} \tag{3.4.2}$$

where, $P(t)$ is the steering function for the linear system (3.1.2). Now substituting this control $u(t)$ into equation (3.2.4), we have

$$\begin{aligned} x(t) &= cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(T-s))f(s,x(s))ds + \\ &\int_0^t A^{-1}sin(A(T-s))BP(s)\{x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0 - \\ &\int_0^T A^{-1}sinA(T-\tau)f(\tau,x(\tau))d\tau\}ds \end{aligned} \tag{3.4.3}$$

If the equation (3.4.3) is solvable then $x(t)$ satisfies $x(0) = x_0$ and $x(T) = x_1$. This implies that the system (3.1.1) is controllable with control $u(t)$ given by (3.4.2). Hence, controllability of the system (3.1.1) is equivalent to the solvability of the equation (3.4.3). Now applying Banach contraction principle, we will prove the solvability of the equation (3.4.3).

We define a mapping $F : C([0,T]; R^n) \to C([0,T]; R^n)$ by

$$
\begin{aligned}
(Fx)(t) &= cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(t-s))f(s,x(s))ds \\
&+ \int_0^t A^{-1}sin(A(t-s))BP(s)\{x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0 \\
&- \int_0^T A^{-1}sinA(T-\tau)f(\tau,x(\tau))d\tau\}ds \qquad (3.4.4)
\end{aligned}
$$

The following lemma proves that F is a contraction under some assumptions on the system components.

Lemma 3.4.1. *Under the following assumptions the nonlinear operator F is a contraction:*

(i) $\sup_{t\in[0,T]} ||A^{-1}sin(At)|| = a < \infty$.

(ii) $||B|| = b < \infty$.

(iii) $\sup_{t\in[0,T]} ||P(t)|| = p < \infty$.

(iv) *The nonlinear function $f : [0,T] \times R^n \to R^n$ is Lipschitz continuous. That is, there exists $\alpha > 0$ such that*

$$||f(t,x) - f(t,y)|| \leq \alpha||x - y|| \ \ \forall x, y \in R^n, \ \ t \in [0,T]$$

(v) $\alpha aT(1 + abpT) < 1$.

Proof. From the definition of F, we have

$$
\begin{aligned}
&||Fx - Fy|| \\
&= \sup_{t\in[0,T]} ||(Fx)(t) - (Fy)(t)|| \\
&= \sup_{t\in[0,T]} || \int_0^t A^{-1}sin(A(T-s))(f(s,x(s)) - f(s,y(s))ds + \int_0^t A^{-1} \\
&\quad sin(A(T-s))BP(s) \int_0^T A^{-1}sinA(T-\tau)(f(\tau,x(\tau)) - f(\tau,y(\tau)))d\tau ds|| \\
&\leq \sup_{t\in[0,T]} || \int_0^t A^{-1}sin(A(T-s))(f(s,x(s)) - f(s,y(s))ds|| + \sup_{t\in[0,T]} || \int_0^t A^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& sin(A(T-s))BP(s)\int_0^T A^{-1}sinA(T-\tau)(f(\tau,x(\tau))-f(\tau,y(\tau)))d\tau ds|| \\
\leq\ & \sup_{t\in[0,T]}\int_0^t ||A^{-1}sin(A(T-s))||\ ||(f(s,x(s))-f(s,y(s))||ds+ \\
& \sup_{t\in[0,T]}\int_0^t ||A^{-1}sin(A(T-s))||||B||||P(s)|| \\
& \int_0^T ||A^{-1}sinA(T-\tau)||\ ||(f(\tau,x(\tau))-f(\tau,y(\tau)))||d\tau ds \\
\leq\ & \sup_{t\in[0,T]} a\int_0^t \alpha||x(s)-y(s)||ds+\sup_{t\in[0,T]} a^2bpt\int_0^T \alpha||y(\tau))-x(\tau)||d\tau \\
\leq\ & a\alpha\sup_{t\in[0,T]}\int_0^t ||x(s)-y(s)||ds+a^2bpT\alpha\int_0^T \sup_{t\in[0,T]}||y(\tau))-x(\tau)||d\tau \\
\leq\ & a\alpha T||x-y||+a^2bpT^2\alpha||x-y|| \\
\leq\ & a\alpha T(1+abpT)||x-y||
\end{aligned}
$$

Since $a\alpha T(1+abpT)<1$, we have F is a contraction. □

Now we have the following computational result for the controllability of the nonlinear system (3.1.1).

Theorem 3.4.1. *Under the assumptions of Lemma 3.4.1, the system (3.1.1) is controllable and the steering control and the controlled trajectories can be computed by the following iterative scheme:*

$$
u^n(t)=P(t)\{x_1-cos(AT)x_0-A^{-1}sin(AT)y_0-\int_0^T A^{-1}sin(A(T-s))f(s,x^n(s))ds \tag{3.4.5}
$$

$$
\begin{aligned}
x^{n+1}(t)=cos(At)x_0+A^{-1}sin(At)y_0+\int_0^t A^{-1}sin(A(t-s))f(s,x^n(s))ds+ \\
\int_0^t A^{-1}sin(A(t-s))Bu^n(s)ds
\end{aligned} \tag{3.4.6}
$$

$x^0(t)=x_0,\ \ n=1,2,3,4,......$ □

Proof. In Lemma 3.4.1 we have proved that F, defined in the equation (3.4.4), is a contraction. Hence, from the Banach contraction principle, F has a fixed point. Thus the equation (3.4.3) is solvable, subsequently the system (3.1.1) is controllable. Further, Theorem 2.2.1 implies the convergence of the iterative scheme for the computation of control and controlled trajectory. □

3.5 Computational Algorithm

Here we compute Cosine and Sine of a matrix $A \in R^{n \times n}$, using the algorithm proposed by Higham and Hargreaves[41]. The algorithm makes use of Páde approximations of $cos(A)$ and $sin(A)$. We define $C_i = cos(2^{i-m}A)$ and $S_i = sin(2^{i-m}A)$. The value of m is chosen in such a way that $||2^{-m}A||$ is small enough, ensuring a good approximation of $C_0 = cos(2^{-m}A)$ and $S_0 = sin(2^{-m}A)$ by Páde approximation. By applying the Cosine and Sine double angle formulae $cos(2A) = 2cos^2(A) - I$ and $sin(2A) = 2sin(A)cos(A)$, we can compute C_0 and S_0) with the help of recurrence relations $C_{i+1} = 2C_i^2 - I$ and $S_{i+1} = 2C_iS_i$, $i = 0, 1, ...m - 1$. The algorithm for the computation of Sine and Cosine matrices is summarized as follows:

3.5.1 ALGORITHM: Given a matrix $A \in R^{n \times n}$.

Choose m such that $2^{-m}||A||$ is very small.

C_0 = pade approximation to $cos(2^{-m}A)$.

S_0= pade approximation to $sin(2^{-m}A)$.

for $i = 0........m - 1$

$C_{i+1} = 2C_i^2 - I$.

$S_{i+1} = 2C_iS_i$.

end

Steering Control For The Linear System: The control which steers the initial state x_0 of the MSOL system (3.1.2) to a desired state x_1 during $[0, T]$ is given by

$$u(t) = B^*(A^{-1}sin(A(T-t)))^*W^{-1}(0,T)\{x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0\} \quad (3.5.1)$$

where, $sin(At)$ and $cos(At)$ are computed by the *Páde* approximation algorithm given in **Algorithm 3.5.1** (Hargreaves and Higham [41]), and $W^{-1}(0, T)$ is computed by using(3.3.1).

Numerical Experiment For Matrix Second Order Linear System

Example 3.5.1. *Consider the Matrix Second Order Linear(MSOL) System*

$$\frac{d^2x(t)}{dt^2} + A^2x(t) = Bu(t), \quad x(t) \in R^3$$

with initial conditions

$$x(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad x'(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

where,

$$A^2 = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 7 & -2 \\ 4 & -4 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and hence,

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

The controllability matrix is given by

$$Q = [B \;\; A^2B \;\; (A^2)^2B] = \begin{pmatrix} 0 & 2 & 24 \\ 0 & -2 & -28 \\ 1 & 3 & 25 \end{pmatrix}$$

and the rank(Q) = 3. Hence the system is controllable. The matrices $Sin(At)$ and $Cosine(At)$ for $t = 1$ are given by

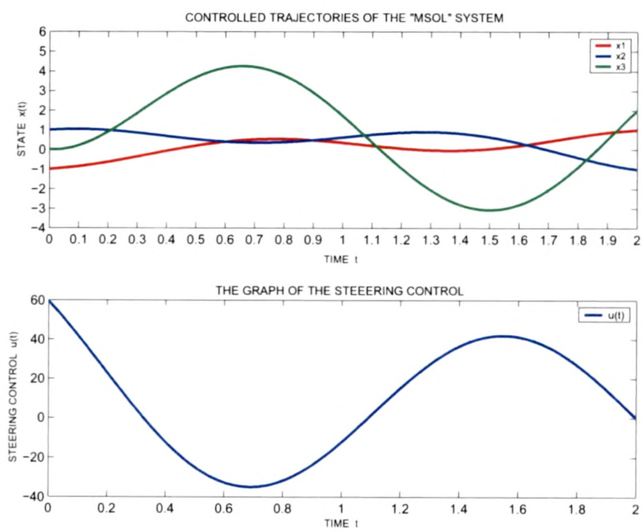
$$sin(A) = \begin{pmatrix} -0.1512 & -0.2810 & -0.4965 \\ -0.2810 & -0.6478 & -0.1405 \\ -0.9931 & -0.2810 & 0.3453 \end{pmatrix}$$

$$cos(A) = \begin{pmatrix} -0.0972 & 0.4385 & -0.3188 \\ 0.4385 & -0.4160 & 0.2192 \\ -0.6375 & 0.4385 & 0.2215 \end{pmatrix}$$

The controllability Grammian matrix, $W(0,T)$ is given by:

$$W = \begin{pmatrix} 0.0733 & -0.0406 & -0.2130 \\ -0.0406 & 0.0272 & 0.1255 \\ -0.2130 & 0.1255 & 0.6915 \end{pmatrix}$$

taking $T = 2$. Now using the equation(3.5.1) along with $P\acute{a}de$ approximation for computing Sine and Cosine matrices, we compute the steering control $u(t)$, steering the state from $x_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ to $x_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ during the time interval $[0,2]$. Furthermore, the controlled trajectory and steering control u are computed and are depicted in the following Figure.



Steering Control For The Nonlinear System: The steering control and controlled trajectories of the MSON system steering from x_0 to x_1 during $[0,T]$ can be approximated from the following algorithm:

$$u^n(t) = P(t)\{x_1 - cos(AT)x_0 - A^{-1}sin(AT)y_0 - \int_0^T A^{-1}sin(A(T-s))f(s,x^n(s))ds$$

$$x^{n+1}(t) = cos(At)x_0 + A^{-1}sin(At)y_0 + \int_0^t A^{-1}sin(A(t-s))f(s,x^n(s))ds+$$

$$\int_0^t A^{-1} sin(A(t-s))Bu^n(s)ds \tag{3.5.2}$$

with $x^0(t) = x_0$, $n = 1, 2, 3, 4, \ldots\ldots$, and $P(t)$ being the steering function given in equation (3.4.1).

Numerical Experiment For Matrix Second Order Nonlinear System

Example 3.5.2. *Consider the Matrix Second Order Nonlinear(MSON) system described by:*

$$\frac{d^2x(t)}{dt^2} + A^2x(t) = Bu(t) + f(t, x(t))$$

where,

$$x(t) \in R^3 \text{ and } f(t, x(t)) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}$$

with the initial conditions

$$x(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad x'(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

and

$$A^2 = \begin{pmatrix} 14 & -2 & 12 \\ 10 & 14 & 30 \\ 0 & -12 & 16 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and hence

$$A = \begin{pmatrix} -2 & 2 & 3 \\ 2 & 4 & 3 \\ 2 & -2 & 4 \end{pmatrix}$$

The controllability matrix is given by:

$$Q = [B \;\; A^2B \;\; (A^2)^2B] = \begin{pmatrix} 0 & 10 & 100 \\ 1 & 44 & 836 \\ 1 & 4 & -464 \end{pmatrix}$$

and the rank(Q) $= 3$. Hence the corresponding linear system is controllable. We have the following numerical estimate, for the parameters given in Lemma 3.4.1, taking $T = 1$,

$$a = \sup_{t \in [0,T]} ||A^{-1} sin(At)|| = 1.0316$$

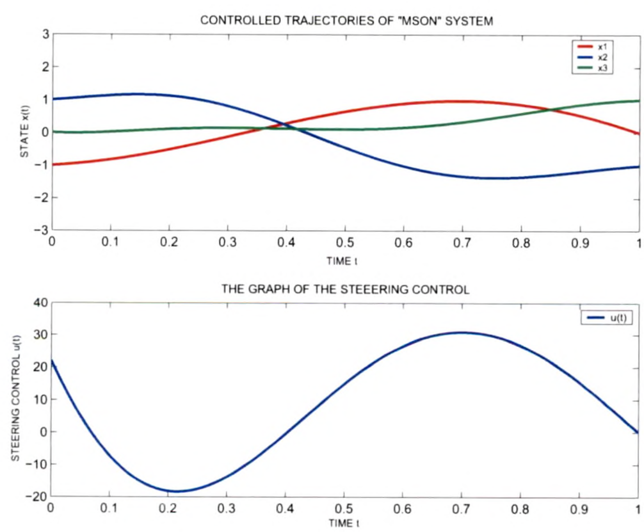
$$b = ||B|| = 1.4142$$

$$p = \sup_{t \in [0,T]} ||P(t)|| = 52.1831$$

Let us take $f_1(x_1, x_2, x_3) = \frac{sin(x_1(t))}{82}$, $f_2(x_1, x_2, x_3) = \frac{cos(x_2(t))}{81}$, and $f_3(x_1, x_2, x_3) = \frac{x_3(t)}{80}$. The nonlinear function $f(t, x(t))$ is Lipschitz continuous with Lipschitz constant $\alpha = 1/80$ and $\alpha a T(1 + abpT) < 1$. Hence, it satisfies all the assumption of the Theorem 3.4.1. Therefore, from the same theorem, the MSON system is controllable. Now using the **Algorithm 3.5.1** with Páde approximation to Sine and Cosine matrices, the controllability Grammian matrix, $W(0, T)$ turns out to be:

$$W = \begin{pmatrix} 0.0682 & 0.1128 & 0.0241 \\ 0.1128 & 0.1998 & 0.0525 \\ 0.0241 & 0.0525 & 0.0994 \end{pmatrix}$$

Now using the equation (3.5.2) one can compute the steering control $u(t)$, steering the state from $x_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ to $x_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ during the time interval $[0, 1]$. The controlled trajectory and the steering control $u(t)$ are for this example are plotted in the following figure.



3.6 Summary

In this chapter we have studied the controllability of Matrix Second Order Linear and Nonlinear Systems in finite dimensional space. We made use of Sine and Cosine matrices to obtain the solutions of the second order systems. An algorithm based on páde approximation to compute Sine and Cosine of a matrix is given. We have also provided an algorithm for the actual computation of steering control of the MSOL. A sufficient condition of controllability of second order nonlinear system has been proved by invoking the fixed point theorem. At the end we have presented numerical experiments for both linear and nonlinear systems.