Chapter 6

Controllability: By Spectral Method

Contents

In this chapter we investigate the controllability property of a class of nonlinear system described by a first order differential equation. The main objective is to obtain controllability result and develop a computational algorithm for the steering control. Our result depend upon the spectral properties of the controllability Grammian. In this approach we do not compute the inverse of the controllability Grammian and the computation of steering control is comparatively easy, especially when n is large. In section 6.1, we introduce the nonlinear system. We obtain the spectral controllability result for the corresponding linear system, in section 6.2.

In section 6.3 the characrization of the controllability is provided. Section 6.4 deals with the controllability analysis and computation of steering control of the nonlinear system with Lipschitz continuous nonlinearity. We provide examples of both linear and nonlinear system, in section 6.5, to illustrate this approach. Section 6.6 deals with the Summary of the chapter.

6.1 Introduction

We consider the nonlinear system described by following first order differential equation:

$$\begin{aligned} \tfrac{dx(t)}{dt} &= A(t)x(t) + B(t)u(t) + f(t, x(t)) \\ x(t_0) &= x_0. \end{aligned} \tag{6.1.1}$$

where the state $x(t) \in R^n$, the control $u(t) \in R^m$, $A(t)$ and $B(t)$ are matrices of order $n \times n$ and $m \times n$ respectively. $x_0 \in R^n$ is the initial state and $f(t, x(t))$ is a nonlinear function.

For time independent linear system, the controllability can be checked by using the Kalman's Rank condition without computing the controllability Grammian. However, for time dependent systems, controllability is checked by finding the rank of the controllability Grammian. For the computation of steering control it is usually done by computing the inverse of the Grammian, which is often difficult when the dimension of the state space is larger. Here we use eigen values and eigen vectors of the Grammian matrix to compute the steering control. Mathematical packages like MATLAB/Maple/Mathematica can be efficiently used to find the eigen values and eigen vectors. With help of such packages the computation of steering controls is made easy using our approach.

If $\Phi(t, s)$ is the state transition matrix for the homogeneous part of (6.1.1) then the system (6.1.1) can be written as

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds + \int_{t_0}^{t} \Phi(t, s)f(s, x(s))ds \tag{6.1.2}$$

Controllability Grammian matrix of the linear part of system (6.1.1) is given by

$$W(t_0,T)=\int_{t_0}^{T}\Phi(T,s)B(s)B^*(s)\Phi^*(T,s)ds \tag{6.1.3}$$

Some properties of the Grammian matrix are given in the following lemma:

Lemma 6.1.1. *The controllability Grammian $W(t_0,T)$ has the following properties:*

(i)$W(t_0,T)$ is symmetric matrix.

(ii)There exists an orthogonal matrix P such that $P^{-1}W(t_0,T)P=D$, where D is a diagonal matrix.

(iii)There is an orthonormal basis for R^n consisting of eigenvectors of $W(t_0,T)$.

Proof. (i)The first symmetric property of $W(t_0,T)$ follows directly from the definition. Since

$$W(t_0,T)=\int_{t_0}^{T}\Phi(T,s)B(s)B^*(s)\Phi^*(T,s)ds$$

$$\begin{aligned}(W(t_0,T))^* &= \int_{t_0}^{T}(\Phi(T,s)B(s)B^*(s)\Phi^*(T,s))^*ds\\ &= \int_{t_0}^{T}((\Phi(T,s))^*)^*((B(s))^*)^*B^*(s)\Phi^*(T,s)ds\\ &= \int_{t_0}^{T}\Phi(T,s)B(s)B^*(s)\Phi^*(T,s)ds\\ &= W(t_0,T)\end{aligned}$$

Hence $W(t_0,T)$ is symmetric matrix.

(ii) Let the Grammain matrix $W(t_0,T)$ be of order n. We shall prove (ii) by induction on n, the order of matrix. For n = 1, the lemma is obvious, as it is a diagonal matrix. We assume that the lemma holds for all Grammian matrices of order $(n-1)$.

Since $W(t_0,T)$ is symmetric, it has at least one eigenvalue, let it be λ_1. Let x_1 be a unit eigen vector for this eigenvalue, that is $||x_1||=1$ and $W(t_0,T)x_1=\lambda_1x_1$. We construct an orthonormal basis (using Gram Schmidt process) $B_1=\{x_1,v_2,....,v_n\}$ of R^n. Let

$$S_1=[x_1\ \ v_2\ \ v_3\ \ .....\ \ v_n]$$

Note that S_1 is an orthonormal matrix, that is $S_1^* = S_1^{-1}$. Now consider the matrix ${S_1}^{-1}W(t_0, T)S_1$, we have

$$(S_1^{-1}W(t_0,T)S_1)^* = (S_1^*W(t_0,T)S_1)^* = S_1^*W(t_0,T)^*S_1^{**}$$

$$= S_1^*W(t_0,T)^*S_1 = S_1^{-1}W(t_0,T)S_1.$$

implies $S_1^{-1}W(t_0,T)S_1$ is a symmetric matrix.
Now the first column of $S_1^{-1}W(t_0,T)S_1$ is given by $(S_1^{-1}W(t_0,T)S_1)(e_1)$, where e_1 is the standard unit vector in R^n. Since $S_1e_1 = x_1$, we have

$$(S_1^{-1}W(t_0,T)S_1)e_1 = (S_1^{-1}W(t_0,T))(S_1e_1) = S_1^{-1}W(t_0,T)x_1 = S_1^{-1}\lambda_1x_1$$

$$= \lambda_1(S_1^{-1}x_1) = \lambda_1(S_1^{-1}S_1e_1) = \lambda_1e_1$$

$$\implies (S_1^{-1}W(t_0,T)S_1)e_1 = \lambda_1e_1$$

$\implies$ first column of $(S_1^{-1}W(t_0,T)S_1)e_1$ is given by $(\lambda_1\ 0\ \ldots..\ 0)^*$

Hence we can write

$$S_1^{-1}W(t_0,T)S_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix}$$

where A_1 is an $(n-1)\times(n-1)$ symmetric matrix. By induction hypothesis, there exits a $(n-1)\times(n-1)$ orthogonal matrix S_2, such that $S_2^{-1}A_1S_2 = \tilde{D}$, a $(n-1)\times(n-1)$ diagonal matrix. Let

$$\tilde{S_2} = \begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}$$

Then $\tilde{S_2}$ is an $n\times n$ orthogonal matrix.

$$(\tilde{S_2})^{-1}{S_1}^{-1}W(t_0,T)S_1\tilde{S_2} = \tilde{S_2}^*S_1^{-1}W(t_0,T)S_1\tilde{S_2}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & S_2^* \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & S_2^*A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & S_2^* A_1 S_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \tilde{D} \end{pmatrix} = D$$

where D is a diagonal matrix of order $n \times n$. If $P = S_1 \tilde{S}_2$ implies $P^{-1} W(t_0, T) P = D$, where P is orthogonal matrix. Hence $W(t_0, T)$ is diagonalizable.

(iii) Suppose $\lambda_1, \lambda_2, ..., \lambda_q$ be the distinct eigenvalues of $W(t_0, T)$ with multiplicities $m_1, m_2, ..., m_q$. Now, Since $W(t_0, T)$ is symmetric implies $W(t_0, T)$ is diagonalizable. Now

$$det(W(t_0, T) - \lambda I) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} .....(\lambda - \lambda_q)^{m_q}$$

That is $\sum_{j=1}^{q} m_j \leq n$ and $dim\{\lambda_j\} \leq m_j \;\; , 1 \leq j \leq q$ which implies $\sum_{j=1}^{q} dim\{\lambda_j\} \leq \sum_{j=1}^{q} m_j \leq n$

Now the left side of the above equation is the maximal number of independent eigenvectors. $\sum_{j=1}^{q} m_j < n$ or $dim\{\lambda_j\} < m_j$ for some j, that is $W(t_0, T)$ does not have n-linearly independent eigenvectors. But $W(t_0, T)$ is diagonalizable, that is $W(t_0, T)$ has n-linearly independent eigenvectors. Which implies that $\sum_{j=1}^{q} m_j = n$ and $dim\{\lambda_j\} = m_j \;\; , j = 1, 2, ..., q$.

Let A_j be an orthonormal basis for $\{\lambda_j\}$. When $i \neq j$ λ_i and λ_j are two distinct eigen values of $W(t_0, T)$ and corresponding eigenvectors $x_i \in \{\lambda_i\}$ and $x_j \in \{\lambda_j\}$ respectively. A_i and A_j are orthonormal basis for $\{\lambda_i\}$ and $\{\lambda_j\}$ respectively.

$$\begin{aligned}
\lambda_i(x_i.x_j) &= (\lambda_i x_i).x_j \\
&= (W(t_0, T)x_i).x_j \\
&= (W(t_0, T)x_i)^* x_j \\
&= x_i^* W(t_0, T)^* x_j \\
&= x_i^* W(t_0, T) x_j \;\; (Since\ W(t_0, T)\ is\ symmetric) \\
&= x_i^* \lambda_j x_j = x_i.(\lambda_j x_j)
\end{aligned}$$

$$\implies \lambda_i(x_i.x_j) = \lambda_j(x_i.x_j)$$

$$\implies (\lambda_i - \lambda_j) x_i.x_j = 0$$

But $\lambda_i \neq \lambda_j \implies x_i.x_j = 0 \implies$ x_i is orthogonal to x_j. $\implies$ the vectors in A_i

are orthogonal to those in A_j.

$$\implies A = A_1 \cup A_2 \cup A_3........ \cup A_q$$

is an orthonormal basis for R^n. $\square$

6.2 Controllability: Linear System

Let us consider the linear part of the system (6.1.1)

$$\left.\begin{array}{l} \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \\ x(0) = x_0. \end{array}\right\} \tag{6.2.1}$$

where the state $x(t) \in R^n$, the control $u(t) \in R^m$, $A(t)$ and $B(t)$ are matrices of order $n \times n$ and $m \times n$, respectively and $x_0 \in R^n$ is the initial state. The system (6.2.1) can be reduced to the integral form:

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)ds \tag{6.2.2}$$

where $\Phi(t, s)$ is the transition matrix, generated by $A(t)$. A necessary and sufficient condition for the controllability of the system (6.2.1)is given by the following theorem:

Theorem 6.2.1. *The system (6.2.1) is controllable iff $\lambda_i > 0$ for all i, where λ_i is the i^{th} eigenvalue of the Grammian matrix $W(t_0, T)$. Further, the control that steers the system from x_0 to x_1 is given by*

$$u(t) = B^*(t)\Phi^*(T, t) \sum_i \frac{c_i v_i}{\lambda_i} \tag{6.2.3}$$

where $\{v_n\}$ is the orthonormal basis of R^n generated by eigenvectors corresponding to $\{\lambda_i\}$ and $c_i's$ are the coordinates of the vector $x_1 - \Phi(T, t_0)x_0$ with respect to the orthonormal basis $\{v_n\}$ $\square$

Proof. Let us assume that the Grammian matrix $W(t_0, T)$ is of order n and let $\lambda_1, \lambda_2, ..., \lambda_n$ be its eigenvalues with $\lambda_i > 0$ for $i = 1, 2, 3, ..., n$. Since $W(t_0, T)$ is a symmetric matrix, it has n linearly independent eigenvectors $v_1, v_2, ..., v_n$, forming orthonormal basis of R^n (state space). Let

$$x_1 - \Phi(T,t_0)x_0 = \sum_{i=1}^{n} c_i v_i \tag{6.2.4}$$

be the unique representation of $x_1 - \Phi(T,t_0)x_0$ with respect to $\{v_i\}_i^n$. Now we claim that the control defined by (6.2.3), steers the system (6.2.1) from x_0 to x_1 during the time $[t_0, T]$. That is, the system (6.2.1) is controllable. From equation (6.2.2), we have

$$x(t_0) = \Phi(t_0,t_0)x_0 + \int_{t_0}^{t_0} \Phi(t_0,s)B(s)u(s)ds$$
$$\implies x(t_0) = x_0$$

And at $t = T$

$$x(T) = \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)B(s)u(s)ds$$

using the equation (6.2.3), we have

$$\begin{aligned}
x(T) &= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)B(s)B^*(s)\Phi^*(T,s)\sum_{i=1}^{n}\frac{c_i v_i}{\lambda_i}ds \\
&= \Phi(T,t_0)x_0 + \sum_{i=1}^{n} c_i \int_{t_0}^{T} \frac{\Phi(T,s)B(s)B^*(s)\Phi^*(T,s)v_i}{\lambda_i}ds \\
&= \Phi(T,t_0)x_0 + \sum_{i=1}^{n} c_i \frac{W(t_0,T)v_i}{\lambda_i} \\
&= \Phi(T,t_0)x_0 + \sum_{i=1}^{n} c_i \frac{\lambda_i v_i}{\lambda_i} \\
&= \Phi(T,t_0)x_0 + \sum_{i=1}^{n} c_i v_i \\
&= \Phi(T,t_0)x_0 + x_1 - \Phi(T,t_0)x_0 \quad \text{[from equation(6.2.4)]} \\
x(T) &= x_1.
\end{aligned}$$

Hence the system is controllable. □

When the control matrix $B(t)$ satisfies very strong conditions then steering control can be computed directly without the use of Grammian matrix as we see in the following results. We have the following theorem, when $B(t) = I$, identity matrix on R^n.

Theorem 6.2.2. *The system* $\frac{dx}{dt} = A(t)x(t) + u(t)$ *is controllable. And the steering control is given by*

$$u(t) = \frac{\Phi(t,T)x_1 - \Phi(t,t_0)x_0}{T - t_0} \tag{6.2.5}$$

□

Proof. Substituting $t = T$ and $u(t)$ as defined in (6.2.5), in the system (6.2.2), we have

$$\begin{aligned}
x(T) &= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)\frac{\Phi(s,T)x_1 - \Phi(s,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \frac{\Phi(T,s)\Phi(s,T)x_1 - \Phi(T,s)\Phi(s,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \frac{\Phi(T,T)x_1 - \Phi(T,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \frac{x_1 - \Phi(T,t_0)x_0}{T-t_0}\int_{t_0}^{T} ds \\
&= \Phi(T,t_0)x_0 + \frac{x_1 - \Phi(T,t_0)x_0}{T-t_0}(T-t_0) \\
&= \Phi(T,t_0)x_0 + x_1 - \Phi(T,t_0)x_0 \\
x(T) &= x_1
\end{aligned}$$

Hence the system is controllable. □

In case the control matrix $B(t)$ is invertible or right invertible, that is $\exists B^+(t)$ such that $B(t)B^+(t) = I$, then we can find steering control without the computation of Grammian matrix $W(t_0,T)$ and the eigenvalues or eigen vectors of $W(t_0,T)$. However, in this case, the dimension of the control space needs to be as large as the state space.

Theorem 6.2.3. *The system (6.2.1) is controllable if* $B(t)$ *is right invertible. In this case the steering control is given by*

$$u(t) = B^+(t)\frac{\Phi(t,T)x_1 - \Phi(t,t_0)x_0}{T - t_0} \tag{6.2.6}$$

□

Proof. Substituting $t = T$ and $u(t)$ as defined in (6.2.6), in the system (6.2.2), we have

$$\begin{aligned}
x(T) &= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \Phi(T,s)B(s)B^{+}(s)\frac{\Phi(s,T)x_1 - \Phi(s,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \frac{\Phi(T,s)\Phi(s,T)x_1 - \Phi(T,s)\Phi(s,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \int_{t_0}^{T} \frac{\Phi(T,T)x_1 - \Phi(T,t_0)x_0}{T-t_0}ds \\
&= \Phi(T,t_0)x_0 + \frac{x_1 - \Phi(T,t_0)x_0}{T-t_0}\int_{t_0}^{T} ds \\
&= \Phi(T,t_0)x_0 + \frac{x_1 - \Phi(T,t_0)x_0}{T-t_0}(T-t_0) \\
&= \Phi(T,t_0)x_0 + x_1 - \Phi(T,t_0)x_0 \\
x(T) &= x_1
\end{aligned}$$

Hence the system is controllable. □

6.3 Characterization of controllability

Definition 6.3.1. *(Brokett [14]) A bounded linear operator $C : L^2([I, R^m]) \to R^n$ is defined by*

$$Cu = \int_{t_0}^{T} \Phi(T,s)B(s)u(s)ds$$

The corresponding adjoint operator $C^* : R^n \to L^2(I, R^m)$ is given by

$$\begin{aligned}
< Cu, v >_{R^n} &= < \int_{t_0}^{T} \Phi(T,s)B(s)u(s)ds, v >_{R^n} \\
&= \int_{t_0}^{T} < \Phi(T,s)B(s)u(s), v >_{R^n} ds \\
&= \int_{t_0}^{T} < u(s), B^*(s)\Phi^*(T,s)v >_{R^n} ds \\
&= < u, C^*v >_{L^2}
\end{aligned}$$

Thus $C^*v = B^*(t)\Phi^*(T,t)v$, that is $W(t_0,T) = CC^*$. Now we have the following theorem:

Theorem 6.3.1. *Let E_0 be the eigenspace corresponding to the eigen value $\lambda = 0$ of the Grammian matrix $W(t_0, T)$. Then we have $E_0 = ker(C^*)$.* □

Proof. Let $y \neq 0$ and $y \in ker(C^*)$

$$\begin{aligned} \implies C^*y &= 0 \\ \implies CC^*y &= 0 \\ \implies CC^*y &= \lambda y \ \ for \ \ \lambda = 0 \end{aligned}$$

$\implies \lambda = 0$ is an eigen value of CC^* and $y \in E_0$.

$$\implies ker(C^*) \subset E_0$$

Now $y \in E_0$, implies $CC^*y = \lambda y$. Therefore

$$\begin{aligned} < CC^*y, y > &= \lambda < y, y > \\ < C^*y, C^*y > &= 0 \\ ||C^*y||^2 = 0 & \\ C^*y = 0 & \end{aligned}$$

$\implies y \in ker(C^*)$. That is $E_0 \subset ker(C^*)$. Hence $E_0 = ker(C^*)$. □

Lemma 6.3.1. *The controllable space of the system (6.2.1) is invariant under the mapping $\Phi(T, t_0)$.*

Proof. Let S be the space of all controllable states during $[t_0, T]$.

$$S = \{z : z = \Phi(T, t_0)x_0 + \int_{t_0}^{T} \Phi(T, s)B(s)u(s)ds, \ \ u \in L^2(I, R^n), x_0 \in S\}$$

Choosing zero control, that is $u = 0$, we have $\Phi(T, t_0)x_0 = z \in S$. Thus the controllable space is invariant under the mapping $\Phi(T, t_0)$. □

Theorem 6.3.2. *If $\lambda = 0$ is a eigen value of $W(t_0, T)$, then $E_0^{\perp}$ is a controllable space of the system (6.2.1).* □

Proof. Let $\lambda = 0$ be a zero of order k. That is $\lambda_1 = \lambda_2 = \lambda_3 = ... = \lambda_k = 0$. Since $W(t_0, T)$ is symmetric matrix. Algebraic multiplicity and geometric multiplicity are same. Hence there exits orthonormal basis $\{v_1, v_2, ..., v_k\}$ of E_0. Let $\{v_{k+1}, v_{k+2}, ..., v_n\}$ be the corresponding orthonormal basis of $E_0^\perp$. Now for any $x_0, x_1 \in E_0^\perp$, $x_1 - \Phi(T, t_0)x_0$ has a unique representation

$$x_1 - \Phi(T, t_0)x_0 = \sum_{i=k+1}^{n} c_i v_i$$

.

Now, if we use a control

$$u(t) = B^*(t)\Phi^*(T, t) \sum_{i=k+1}^{n} \frac{c_i v_i}{\lambda_i} \; (\lambda_i \neq 0)$$

Then the state satisfies

$$x(T) = \Phi(T, t_0)x_0 + \int_{t_0}^{T} \Phi(T, s)B(s)B^*(s)\Phi^*(T, s) \sum_{i=k+1}^{n} \frac{c_i v_i}{\lambda_i} ds \;\; for \;\; (\lambda_i \neq 0)$$

Then

$$\begin{aligned} x(T) &= \Phi(T, t_0)x_0 + W(t_0, T) \sum_{i=k+1}^{n} \frac{c_i v_i}{\lambda_i} \\ &= \Phi(T, t_0)x_0 + \sum_{i=k+1}^{n} \frac{c_i W(t_0, T)v_i}{\lambda_i} \\ &= \Phi(T, t_0)x_0 + \sum_{i=k+1}^{n} c_i v_i \\ &= \Phi(T, t_0)x_0 + x_1 - \Phi(T, t_0)x_0 \\ x(T) &= x_1 \end{aligned}$$

That is the system is controllable. Since we have chosen x_0 and x_1 arbitrarily from $E_0^\perp$, it gives $E_0^\perp$ is a controllable space of the system (6.2.1) □

We now prove the controllability results of the nonlinear system (6.1.1).

6.4 Controllability: Nonlinear System

In this section we prove the controllability of the nonlinear system (6.1.1). Without loss of generality we assume the following:

[**B**] Let $b = \sup_{t_0 \leq t \leq T} ||B(t)|| < \infty$.

[**A**] $||\Phi(t,s)|| \leq m$, for all $t, s \in [t_0, T]$.

[**F**] The function f satisfies Caratheodory conditions, that is, $f(t,x)$ is measurable with respect to t for all $x \in R^n$ and continuous with respect to x for almost all $t \in [t_0, T]$. Further f is a Lipschitz continuous, that is, there exists a constant $\alpha \geq 0$ such that $||f(t,x) - f(t,y)|| \leq \alpha||x - y||$ for all $x, y \in R^n$.

We shall reduce the controllability of (6.1.1) into solvability problem. For $x \in C([t_0, T]; R^n)$, let us define

$$x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,\tau)f(\tau,x(\tau))d\tau = \sum_{i=1}^{n} c_{x_i} v_i \tag{6.4.1}$$

where $\{v_n\}$ is the orthonormal basis of R^n generated by the eigenvectors corresponding to the eigenvalues $\{\lambda_i\}$ of $W(t_0, T)$. Here $c'_{x_i}s$ are coordinates of the vector $(x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T} \Phi(T,t)f(t,x(t))dt)$ with respect to the orthonormal basis $\{v_n\}$. Now, the control defined by

$$u(t) = B^*(t)\Phi^*(T,t)\sum_i \frac{c_{x_i}v_i}{\lambda_i} \tag{6.4.2}$$

steers the system (6.1.1) from x_0 to x_1 during $[t_0, T]$, provided x in (6.4.1) satisfies

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)B^*(s)\Phi^*(T,s)\sum_i \frac{c_{x_i}v_i}{\lambda_i}ds + \int_{t_0}^{t} \Phi(t,s)f(s,x(s))ds \tag{6.4.3}$$

We apply Banach fixed point theorem (refer Limaye [55]), for establishing the solvability of the equation (6.4.3). We define a mapping $F : C([t_0, T]; R^n) \rightarrow$

$C([t_0, T]; R^n)$ by

$$(Fx)(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)B^*(s)\Phi^*(T,s)\sum_i \frac{c_{x_i}v_i}{\lambda_i}ds + \int_{t_0}^{t}\Phi(t,s)f(s,x(s))ds \tag{6.4.4}$$

The solvability of (6.4.3) follows if we prove that F has a fixed point. We first prove the following lemmas:

Lemma 6.4.1. *Under the Assumptions [A] and [F] we have the following inequality:*

$$||(\sum_{i=1}^{n}\frac{c_{x_i}v_i}{\lambda_i} - \sum_{i=1}^{n}\frac{c_{y_i}v_i}{\lambda_i})|| \leq \frac{1}{|\lambda|}m\alpha\int_{t_0}^{T}||y(s)-x(s)||ds$$

where $|\lambda| = min\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\} \neq 0$, $x, y \in C([t_0, T]; R^n)$ *and* $\{v_n\}$ *is the orthonormal basis of* R^n *generated by the eigenvectors corresponding to the eigenvalues* $\{\lambda_i\}$ *of* $W(t_0, T)$. *Hence* $c'_{x_i}s$ *are coordinates of the vector* $x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T}\Phi(T,t)f(t,x(t))dt$ *with respect to the orthonormal basis* $\{v_n\}$ *and* $c'_{y_i}s$ *are coordinates of the vector* $x_1 - \Phi(T,t_0)x_0 - \int_{t_0}^{T}\Phi(T,t)f(t,y(t))dt$ *with respect to the orthonormal basis* $\{v_n\}$.

Proof. We have

$$\begin{aligned}
||(\sum_i\frac{c_{x_i}v_i}{\lambda_i} - \sum_i\frac{c_{y_i}v_i}{\lambda_i})|| &\leq ||\sum_i\frac{(c_{x_i}v_i - c_{y_i}v_i)}{\lambda_i}|| \\
&\leq \frac{1}{|\lambda|}||\sum_i(c_{x_i}v_i - c_{y_i}v_i)|| \\
&= \frac{1}{|\lambda|}||\int_{t_0}^{T}\Phi(T,s)f(s,y(s))ds - \int_{t_0}^{T}\Phi(T,s)f(s,x(s)ds|| \\
&\leq \frac{1}{|\lambda|}\int_{t_0}^{T}||\Phi(T,s)||\ ||f(s,y(s)) - f(s,x(s)||ds \\
&\leq \frac{1}{|\lambda|}m\int_{t_0}^{T}\alpha||y(s)-x(s)||ds \\
&\leq \frac{1}{|\lambda|}m\alpha\int_{t_0}^{T}||y(s)-x(s)||ds
\end{aligned}$$

□

Lemma 6.4.2. *Under the Assumptions* [A], [B] *and* [F] *and* $\alpha m(T-t_0)(\frac{m^2b^2}{|\lambda|}(T-t_0)+1) < 1$, *the operator* F *is a contraction.*

Proof. Let x and y be two solutions of the system (6.1.1). Therefore, by using the assumptions $[A]$, $[B]$, $[F]$ and Lemma 6.4.1 we have:
$||Fx - Fy||$

$$\begin{aligned}
&= \sup_{t\in[t_0,T]} ||(Fx)(t) - (Fy)(t)|| \\
&= \sup_{t\in[t_0,T]} || \int_{t_0}^{t} \Phi(t,s)B(s)B^*(s)\Phi^*(T,s)(\sum_i \frac{c_{x_i}v_i}{\lambda_i} - \sum_i \frac{c_{y_i}v_i}{\lambda_i})ds \\
&\quad + \int_{t_0}^{t} \Phi(t,s)(f(s,x(s)) - f(s,y(s))ds|| \\
&\leq \sup_{t\in[t_0,T]} || \int_{t_0}^{t} \Phi(t,s)B(s)B^*(s)\Phi^*(T,s)(\sum_i \frac{c_{x_i}v_i}{\lambda_i} - \sum_i \frac{c_{y_i}v_i}{\lambda_i})ds|| \\
&\quad + \sup_{t\in[t_0,T]} || \int_{t_0}^{t} \Phi(t,s)(f(s,x(s)) - f(s,y(s))ds|| \\
&\leq \sup_{t\in[t_0,T]} \int_{t_0}^{t} ||\Phi(t,s)||\ ||B(s)||\ ||B^*(s)||\ ||\Phi^*(T,s)||\ ||(\sum_i \frac{c_{x_i}v_i}{\lambda_i} - \\
&\quad \sum_i \frac{c_{y_i}v_i}{\lambda_i})||ds + \sup_{t\in[t_0,T]} \int_{t_0}^{t} ||\Phi(t,s)||||(f(s,x(s)) - f(s,y(s))||ds \\
&\leq \sup_{t\in[t_0,T]} m^2b^2 \int_{t_0}^{t} ||(\sum_i \frac{c_{x_i}v_i}{\lambda_i} - \sum_i \frac{c_{y_i}v_i}{\lambda_i})||ds \\
&\quad + \sup_{t\in[t_0,T]} m \int_{t_0}^{t} ||(f(s,x(s)) - f(s,y(s))||ds \\
&\leq m^2b^2 \sup_{t\in[t_0,T]} \int_{t_0}^{t} ||(\sum_i \frac{c_{x_i}v_i}{\lambda_i} - \sum_i \frac{c_{y_i}v_i}{\lambda_i})||ds + m \sup_{t\in[t_0,T]} \int_{t_0}^{t} \alpha||x(s) - y(s)||ds \\
&\leq m^2b^2 \sup_{t\in[t_0,T]} \int_{t_0}^{t} \frac{1}{|\lambda|} m\alpha \int_{t_0}^{T} ||y(\tau) - x(\tau)||d\tau ds + m \sup_{t\in[t_0,T]} \int_{t_0}^{t} \alpha||x(s) - y(s)||ds \\
&\leq m^2b^2 \frac{1}{|\lambda|} m\alpha ||y - x|| \int_{t_0}^{T}\int_{t_0}^{T} d\tau ds + m\alpha||x - y|| \int_{t_0}^{T} ds \\
&\leq m^2b^2 \frac{1}{|\lambda|} m\alpha (T - t_0)^2 ||x - y|| + m\alpha(T - t_0)||x - y|| \\
&\leq m\alpha(T - t_0)(\frac{m^2b^2}{|\lambda|}(T - t_0) + 1)||x - y||
\end{aligned}$$

Since

$$\alpha \leq \frac{1}{m(T - t_0)(\frac{m^2b^2}{|\lambda|}(T - t_0) + 1)}$$

F is a contraction. □

Theorem 6.4.1. *The system (6.1.1) is controllable if $A(t)$ and $B(t)$ satisfy the assumptions $[A], [B]$ and the function f satisfies the assumption $[f]$ and is Lipschitz continuous with the Lipschitz constant*

$$\alpha \leq \frac{1}{m(T-t_0)(\frac{m^2b^2}{|\lambda|}(T-t_0)+1)}$$

An algorithm for the computation of steering control and controlled trajectory is given by:

$$u^n(t) = B^*(t)\Phi^*(T,t)\sum_i \frac{c_{x_i^n} v_i}{\lambda_i}$$

$$x^{n+1}(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u^n(s)ds + \int_{t_0}^{t} \Phi(t,s)f(s,x^n(s))ds \qquad (6.4.5)$$

$x^0(t) = x_0, \;\; n = 1,2,3,4,......$ □

Proof. By the Lemma 6.4.1, the operator F, given by the equation (6.4.4) is a contraction. Hence by Banach contraction principle, F has a unique fixed point. Thus the equation(6.4.3) is solvable. And thus the nonlinear system(6.1.1) is controllable. The computational algorithm follows directly from Banach contraction principle. □

Now in the next section we give examples of both linear and nonlinear systems to illustrate our results.

6.5 Computational Algorithm

Linear System: The control which steers the initial state x_0 of the system (6.2.1) to a desired state x_1 during $[t_0, T]$ is given by (See Theorem 6.2.1)

$$u(t) = B^*(t)\Phi^*(T,t)\sum_i \frac{c_i v_i}{\lambda_i}$$

Example 6.5.1. *Consider a 3-dimensional Linear system*

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x(t) \in R^3$$

with initial conditions

$$x(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

where,

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The controllability matrix is given by

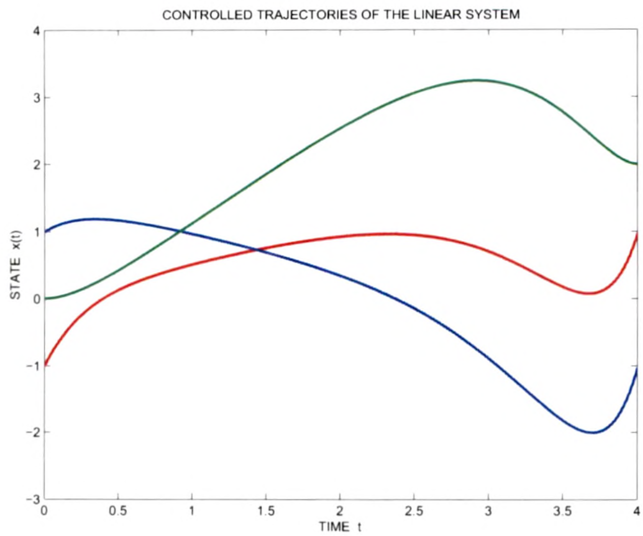
$$Q = [B \ \ AB \ \ (A^2)B] = \begin{pmatrix} 1 & 3 & 13 \\ 1 & 4 & 13 \\ 0 & 2 & 7 \end{pmatrix}$$

and the Rank(Q) $= 3$. Hence the system is controllable. The controllability Grammian matrix, $W(0,T)$ is given by:

$$W = 10^{12} \begin{pmatrix} 1.0840 & 1.1848 & 0.6058 \\ 1.1848 & 1.2949 & 0.6622 \\ 0.6058 & 0.6622 & 0.3386 \end{pmatrix}$$

taking $T = 4$.
Now using the algorithm given in (6.2.3) we compute the control $u(t)$, steering the state from $x_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ to $x_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ during the time interval $[0, 4]$.

Furthermore, we computed the controlled trajectories as depicted as in the following Figure:



Nonlinear System:

The steering control and controlled trajectories of the nonlinear system (6.1.1) steering from x_0 to x_1 during $[0, T]$ can be approximated from the following algorithm: (See Theorem 6.4.1)

$$u^n(t) = B^*(t)\Phi^*(T,t)\sum_i \frac{c_{x_i^n} v_i}{\lambda_i}$$

$$x^{n+1}(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u^n(s)ds + \int_{t_0}^{t} \Phi(t,s)f(x^n(s))ds \qquad (6.5.1)$$

$x^0(t) = x_0, \quad n = 1, 2, 3, 4, ...$

Example 6.5.2. *Consider the nonlinear system described by:*

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(t, x(t))$$

where,

$$x(t) \in R^3 \text{ and } f(t,x(t)) = \begin{pmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{pmatrix}$$

with the initial conditions

$$x(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The corresponding linear system is controllable as we have seen in the previous example. The controllability Grammian matrix, W turns out to be:

$$W = \begin{pmatrix} 188.5112 & 207.9346 & 101.6571 \\ 207.9346 & 229.3638 & 112.1220 \\ 101.6571 & 112.1220 & 54.9380 \end{pmatrix}$$

Eigenvalues of W are:

$$\begin{pmatrix} 472.7018 \\ 0.0016 \\ 0.1096 \end{pmatrix}$$

All the eigenvalues of W are positive. Now we have the following numerical estimate, for the parameters given in Lemma 6.4.2, taking $T = 1$,

$$m = \sup_{t\in[0,1]} ||e^{A(t-s)}|| \leq 43.2067$$

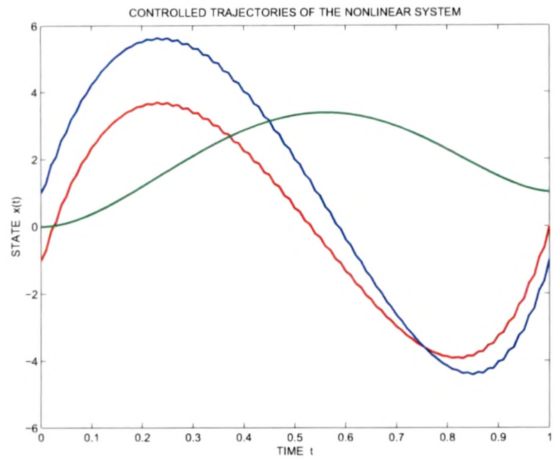
$$b = ||B|| \leq 1.4142$$

$$\lambda = 0.0016$$

Let us take $f_1(x_1, x_2, x_3) = \frac{sin(x_1(t))}{a_1}$, $f_2(x_1, x_2, x_3) = \frac{cos(x_2(t))}{a_2}$, and $f_3(x_1, x_2, x_3) = \frac{x_3(t)}{a_3}$. Here a_1, a_2 and a_3 are chosen in such a way that the nonlinear function $f(t, x(t))$ is Lipschitz continuous with Lipschitz constant α satisfying the inequality $\alpha mT(1 + \frac{m^2b^2T}{\lambda}) < 1$. Now, the above nonlinear system satisfies all the assumptions of the Theorem 6.4.1, hence the system is controllable.

Now using the algorithm given in (6.4.5), we compute the control $u(t)$, steering the state from $x_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ to $x_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ during the time interval $[0, 1]$.

Furthermore, we computed the controlled trajectories which are shown as in the following Figure:



6.6 Summary

In this chapter, we have developed an computational algorithm for the computation of steering control using spectral analysis. In the begining of the chapter we have proved the controllability result of the linear first order system and then the algorithm for computing the steering control of linear system is provided. We have also proved the controllability result for nonlinear system and developed a computational algorithm. The chapter concludes with the numerical examples for both the linear and nonlinear systems.