Chapter 4

Controllability of Discrete Volterra Systems

Contents

In this chapter, a necessary and sufficient condition is given for controllability of discrete time linear Volterra system. Local controllability result for a semi-linear discrete Volterra system is also proved. Numerical examples are provided to illustrate the results.

4.1 Introduction

Gaishun and Dymkov [11] studied controllability of the linear discrete Volterra system of the form

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t), \quad t \in N_0$$

by a method based on the representation of the Volterra operator generated by the equation in the ring of formal power series. In this paper we study controllability of a non-autonomous linear system of the form :

$$\Sigma_L : x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t), \quad t \in N_0 \tag{4.1.1}$$

and local controllability of a semi-linear discrete Volterra system of the form :

$$\Sigma_N : x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t) + f(x(t), u(t)), \quad t \in N_0 \tag{4.1.2}$$

using a different approach and in much more straightforward manner. Here, $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$ are sequences of real $n \times n$ and $n \times m$ - matrices, respectively, and $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively. $f(.,.) : R^n \times R^m \to R^n$ is a nonlinear function of state and control variables. It follows easily that for a given control sequence $\{u(t)\}_{t\in N_0}$ and initial state $x(t_0) = x_0$, there exists a unique solution to the linear system Σ_L. We make the following definitions to obtain solution of Σ_L. Define the set of linear operators $Q_t : R^n \to R^n$, $t \in N_0$ by

$$Q_0 = I, \quad Q_{t+1} = \sum_{i=0}^{t} A(i)Q_{t-i}, \quad t \in N_0 \tag{4.1.3}$$

Using these operators, the solution of (4.1.1) is given by

$$x(t) = Q_t x_0 + \sum_{i=0}^{t-1} Q_i B(t-i-1)u(t-i-1) \tag{4.1.4}$$

We first obtain the controllability results of the linear system Σ_L.

Definition 4.1.1. ***(Global controllability*** *)(see Elaydi [8]) Let $x_0, x_1 \in R^n$ be given arbitrarily. The system Σ_L is controllable if we can find a sequence of control vectors $\{u(t) \in R^m,\ t \in N_0\}$ such that for some $N \in N_0$ the solution $(x(t))_{t\in N_0}$ of equation (4.1.1) with $x(0) = x_0$ satisfies the desired final state*

$$x(N) = x_1 \tag{4.1.5}$$

In view of (4.1.4), we are looking for a control sequence $(u(t))_{t\in N_0}$ satisfying

$$x_1 - Q_N x_0 = \sum_{i=0}^{N-1} Q_i B(N-i-1)u(N-i-1)$$

or

$$x_1 - Q_N x_0 = \sum_{i=1}^{N} Q_{i-1} B(N-i)u(N-i) \tag{4.1.6}$$

This is a linear system for the unknowns $\{u(0), ..., u(N-1)\} \in R^{mN}$. Define

$$U \triangleq \{\mathbf{u} = (u(0), u(1), ..., u(N-1)) \in R^{mN}\}$$

Equation (4.1.6) shows that there will be an input $\mathbf{u} \in U$ that will transfer a given arbitrary state x_0 to a desired final state x_1 in N time steps if and only if the linear map $L : R^{mN} \to R^n$ defined by

$$L : \mathbf{u} \to \sum_{i=1}^{N} Q_{i-1} B(N-i)u(N-i) \tag{4.1.7}$$

is onto. From (4.1.7), we see that $LL^* : R^n \to R^n$ has a square matrix representation, where the adjoint operator L^* is defined as follows :

For $v \in R^n$, $\mathbf{u} \in U$,

$$\begin{aligned}
< L\mathbf{u}, v > &= < \sum_{i=1}^{N} Q_{i-1}B(N-i)u(N-i), v > \\
&= \sum_{i=1}^{N} < Q_{i-1}B(N-i)u(N-i), v > \\
&= \sum_{i=1}^{N} < B(N-i)u(N-i), Q_{i-1}^{*}v > \\
&= \sum_{i=1}^{N} < u(N-i), B^{*}(N-i)Q_{i-1}^{*}v > \\
&= < \mathbf{u}, (B^{*}Q^{*})v > \\
i.e. \quad < \mathbf{u}, L^{*}v > &= < \mathbf{u}, (B^{*}Q^{*})v > \\
i.e. \quad L^{*}v &= (B^{*}(N-1)Q_0^{*}v, B^{*}(N-2)Q_1^{*}v, ..., B^{*}(0)Q_{N-1}^{*}v) \\
i.e. \quad LL^{*}v &= \sum_{i=1}^{N} Q_{i-1}B(N-i)B^{*}(N-i)Q_{i-1}^{*}v
\end{aligned}$$

Now define the controllability Grammian for the linear Volterra system (4.1.1) by

$$W(0,N) = \sum_{i=1}^{N} Q_{i-1}B(N-i)B^{*}(N-i)Q_{i-1}^{*} \tag{4.1.8}$$

In Section 4.2, we give two different conditions, for the global controllability of (4.1.1), namely

(i) condition using controllability Grammian and

(ii) Kalman type rank condition.

In Section 4.3, we prove a local controllability theorem for the semi-linear system (4.1.2) and numerical examples illustrating the results are included in Section 4.4.

4.2 Controllability of Linear Volterra system

4.2.1 Controllability using Controllability Grammian

In this section we prove necessary and sufficient condition for the controllability of the linear Volterra system using controllability Grammian.

Theorem 4.2.1. *Let $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$ be sequences of real $n\times n$ and $n\times m$ - matrices, respectively and let L be the operator defined as in (4.1.7). Then the following statements are equivalent.*

1. *The non-autonomous Volterra system (4.1.1) is controllable on $[0, N]$*
2. *$range(L) = R^n$.*
3. *$range(LL^*) = R^n$.*
4. *$detW(0, N) \neq 0$, where W is the Controllability Grammian defined by (4.1.8).*

Proof. The solution of the system (4.1.1) is given by

$$x(t) = Q_t x_0 + \sum_{i=0}^{t-1} Q_i B(t-i-1)u(t-i-1)$$

We now prove (1) $\Leftrightarrow$ (2).
The system (4.1.1) is controllable on $[0, N]$ if and only if for every x_1 and $x_0 \in R^n$ there exists a control sequence $\mathbf{u} \in U$ satisfying

$$x_1 = Q_N x_0 + \sum_{i=0}^{N-1} Q_i B(N-i-1)u(N-i-1)$$

or

$$x_1 - Q_N x_0 = \sum_{i=1}^{N} Q_{i-1} B(N-i)u(N-i)$$

Thus the system (4.1.1) is controllable if and only if the operator $L : R^{mN} \rightarrow R^n$ defined by (4.1.7) is surjective. Thus statement (1) is equivalent, as noted above to the surjectivity of L, which is equivalent to $range(L) = R^n$.
(2) $\Leftrightarrow$ (3)

Let $range(L) = R^n$.
$\Leftrightarrow range(LL^*) = R^n$.
(3) $\Leftrightarrow$ (4)
From (4.1.8), it follows that $LL^* = W$ is a square matrix and hence $range(LL^*) = R^n$,
$\Leftrightarrow W(0, N)$ is invertible.
$\Leftrightarrow det(W(0, N)) \neq 0$. Hence the proof. □

4.2.2 Controllability using Kalman type Rank Condition

Let us define the controllability matrix by

$$W_c = \left[B(N-1) \mid Q_1B(N-2) \mid ... \mid Q_{N-1}B(0)\right] \tag{4.2.1}$$

and assume that

$$rank(W_c) = rank(\left[B(N-1) \mid Q_1B(N-2) \mid ... \mid Q_{N-1}B(0)\right]) = n \tag{4.2.2}$$

Theorem 4.2.2. *If the rank condition (4.2.2) is satisfied for some $N \in N_0$, then for every pair $x_0, x_1 \in R^n$ there exists a control sequence $\{u\}$ steering x_0 to x_1 in N time steps.*

Proof. Let us assume that for some $N \in N_0$, it is true that $rank\ W_c = n$. Then the system

$$x_1 - Q_N x_0 = \sum_{i=1}^{N} Q_{i-1}B(N-i)u(N-i) \tag{4.2.3}$$

has a solution $u(0), ..., u(N-1) \in R^m$ for every choice of $x_0,\ x_1 \in R^n$. If the rank condition (4.2.2) is satisfied, the system (4.2.3) has infinitely many solutions. Now pick up a special one which is defined uniquely. For that purpose we define for every $k = 1, ..., N$ an $n \times m$ matrix C^k by

$$C^k = Q_{k-1}B(N-k)\ \ for\ \ k = 1, 2, 3, ..., N$$

The condition (4.2.2), then implies the existence of n linearly independent columns in matrix W_c.

We define a $n \times n$ matrix C by these linearly independent columns,

$$C = \begin{pmatrix} C^{k_1}_{1j_{k_1}} & . & . & . & C^{k_n}_{1j_{k_n}} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ C^{k_1}_{nj_{k_1}} & . & . & . & C^{k_n}_{nj_{k_n}} \end{pmatrix}$$

Now define a control sequence $u \in R^n$ by

$$u = \begin{pmatrix} u_{j_{k_1}}(k_1 - 1) \\ . \\ . \\ . \\ u_{j_{k_n}}(k_n - 1) \end{pmatrix}$$

and make

$$u_j(k-1) = 0 \;\; for \;\; k \neq k_l, \;\; j \neq j_{k_l}, \;\; l = 1, ..., n$$

then we obtain from equation (4.2.3)

$$x_1 = Q_N x_0 + Cu$$

Since C is invertible we have

$$u = C^{-1}(x_1 - Q_N x_0) \tag{4.2.4}$$

Obviously this control steers the system from x_0 to x_1 in N time steps. □

4.2.3 Another Steering Control for Linear Volterra System

We now provide another steering control using controllability Grammian.

Theorem 4.2.3. *If the system Σ_L is controllable, in N time steps then $\forall x_0, x_1 \in R^n$, $\exists$ a control sequence $u : N_0 \to R^m$ defined by*

$$\{u(t-i)\}_{i=1}^{t} := \{B^*(t-i)Q^*_{i-1}W^{-1}(0,N)(x_1 - Q_N x_0)\}_{i=1}^{t}, t = 1, 2, 3, ..., N \tag{4.2.5}$$

steers the initial state x_0 to the desired final state x_1 in N time steps.

Proof. Since the linear system (4.1.1) is controllable, we have by Theorem 4.2.1, $detW(0,N) \neq 0$, where $W(0,N)$ is given by (4.1.8). To prove that control given by (4.2.5) steers the state x_0 to x_1, we substitute this control in (4.1.6) to obtain :

$$x(t) = Q_t x_0 + \sum_{i=1}^{t} Q_{i-1} B(t-i) u(t-i)$$

$$x(t) = Q_t x_0 + \sum_{i=1}^{t} Q_{i-1} B(t-i) B^*(t-i) Q_{i-1}^* W^{-1}(0,t)(x_1 - Q_N x_0)$$

It can be easily verified that at $t = 0$, $x(0) = x_0$ and at $t = N$, $x(N) = x_1$. Hence the proof. □

Remark 4.2.1. *It can be shown that the control obtained in above theorem is a minimum norm control.*

4.3 Controllability of Semi-linear System

Also to prove local controllability of the semi-linear system (4.1.2), we use the notion of higher order functions, inverse function theorem and implicit function theorem (see Section 2.6). For the nonlinear system Σ_N, represented by (4.1.2), the following definition of local controllability is relevant.

Definition 4.3.1. ***(Local controllability)*** *A system is locally controllable if there exists a neighborhood $\Omega$of the origin such that, for any $x_0, x_1 \in \Omega$ there is a sequence of inputs $\mathbf{u} = (u(0), u(1), ..., u(N-1))$ that steers the system from x_0 to x_1.*

Now we prove the following result for the local controllability of (4.1.2) under the assumption that Σ_L is controllable and the nonlinear function f is of "higher order".

Theorem 4.3.1. *If the linear system Σ_L is controllable and $f \in H$, then the semi-linear system Σ_N is locally controllable.*

Proof. We will show the existence of a control sequence $\mathbf{u}$ that transfers the state from x_0 to x_1 in N time steps. The properties of higher order functions will be repeatedly used in the following derivations.

$$\begin{aligned}
x(1) &= A(0)x(0) + B(0)u(0) + f(x(0), u(0)) \\
&= Q_1 x_0 + B(0)u(0) + f(x_0, u(0)) \\
x(2) &= (A^2(0) + A(1))x_0 + A(0)B(0)u(0) + B(1)u(1) \\
&\quad + A(0)f(x_0, u(0)) + f(A(0)x_0 + B(0)u(0) + f(x_0, u(0)), u(1)) \\
&\equiv Q_2 x_0 + Q_1 B(0)u(0) + B(1)u(1) + f_2(x_0, u(0), u(1)), \ taking \\
&\quad A(0)f(x_0, u(0)) + f(A(0)x_0 + B(0)u(0) + f(x_0, u(0)), u(1)) = f_2(x_0, (u(0), u(1))) \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
x(N) &\equiv Q_N x_0 + [B(N-1) \mid Q_1 B(N-2) \mid ... \mid Q_{N-1}B(0)] \begin{bmatrix} u(N-1) \\ u(N-2) \\ . \\ . \\ . \\ u(0) \end{bmatrix} \\
&\quad + f_N(x_0, (u(0), u(1), ..., u(N-1)))
\end{aligned}$$

Since

$$W_c \equiv \left[B(N-1) \mid Q_1 B(N-2) \mid ... \mid Q_{N-1}B(0)\right]$$

and

$$\mathbf{u} \equiv [u(0), u(1), ..., u(N-1)]^T$$

we have

$$x(N) = Q_N x_0 + W_c \mathbf{u} + f_N(x_0, \mathbf{u})$$

Since we require $x(N) = x_1$,

$$x_1 = Q_N x_0 + W_c \mathbf{u} + f_N(x_0, \mathbf{u})$$

where $f_2(.), ..., f_N(.)$ are all properly defined higher order functions. Let W_c^* be the transpose of W_c and let $\mathbf{u} = W_c^*\mathbf{v}$. Therefore, we have

$$x_1 = Q_N x_0 + W_c W_c^* \mathbf{v} + f_N(x_0, W_c^* \mathbf{v})$$

Since the linear system is controllable, W_c is of full rank and hence $W_c W_c^*$ is an invertible matrix. By inverse function theorem and implicit function theorem (see Corollary 2.8.1), if x_0, x_1 in a neighborhood Ω of the origin, there exist $\mathbf{v}$ given by

$$\mathbf{v} = (W_c W_c^*)^{-1}(x_1 - Q_N x_0) + g(x_0, x_1), \quad for \quad some \quad g(.) \in H.$$

Thus the control sequence

$$\mathbf{u} = W_c^* \mathbf{v}$$

steers x_0 to x_1 for all x_0, x_1 in a neighborhood of origin. Hence the theorem. □

4.4 Numerical Examples

Example 4.4.1. *Consider a 2-dimensional discrete linear Volterra system of the form*

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t), \quad t \in N_0$$

with $A(t) = \begin{pmatrix} cos(t) & sin(t) \\ \frac{t}{5} & 2cos(t) \end{pmatrix}$ *and* $B(t) = \begin{pmatrix} .5 \\ .5t \end{pmatrix}$. *Let us take* $N = 5$ *and initial state* $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and the final state* $x_1 = \begin{pmatrix} -20 \\ 1 \end{pmatrix}$. *Then using the Matlab program* $P - 3$ *in Appendix, we compute the value of controllability matrix*

$$W_c = \begin{pmatrix} 0.5000 & 0.5000 & 1.6116 & 2.5491 & 0.6294 \\ 2.0000 & 3.0000 & 5.1806 & 6.2451 & 1.7621 \end{pmatrix}$$

This shows that rank of W_c *is* 2 *and hence the given system is controllable by Theorem 4.2.2. Hence we can compute the control that steers initial state* x_0 *to desired final state* x_1. *Computation of this control is done using the formula (4.2.4), where matrix* C *is computed as*

$$C = \begin{pmatrix} 0.5000 & 0.5000 \\ 2.0000 & 3.0000 \end{pmatrix}.$$

The matrix C is invertible and C^{-1} is given by

$$C^{-1} = \begin{pmatrix} 6 & -1 \\ -4 & 1 \end{pmatrix}.$$

Also Q_N is computed as

$$Q_N = \begin{pmatrix} 0.8969 & 25.4421 \\ 10.3115 & 53.9320 \end{pmatrix}.$$

Hence using equation (4.2.4), we obtain control $u = \begin{pmatrix} -17.3487 \\ 26.4393 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

Figure 4.1 *shows the controlled trajectories using this control.*

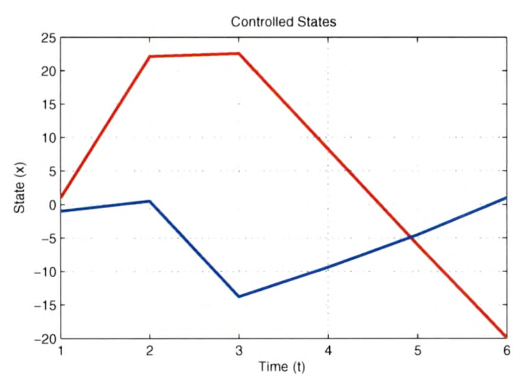

Figure 4.1: Controlled trajectories computed using control given by equation (4.2.4)

Example 4.4.2. *In this example we take all the data similar to the previous Example 4.4.1 and use the techniques of Theorems 4.2.1 and 4.2.3 to compute steering control. According to this, if $detW(0, N) \neq 0$ then the system Σ_L is controllable and the control sequence which steers the initial state to final state is given by equation (4.2.5).*

For the same data, we get controllability Grammian matrix as

$$W(0,5) = \begin{pmatrix} 9.9913 & 27.8775 \\ 27.8775 & 81.9444 \end{pmatrix}.$$

Its determinant is $|W(0,5)| = 41.5806 \neq 0$.
Hence linear system is controllable. The control sequence computed using (4.2.5) is given by

$$\boldsymbol{u} = (4.8698, 12.5442, 5.9818, -5.4964, 0.3316)$$

Using this control, we see that $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *is steered to the final state* $x_1 = \begin{pmatrix} -20 \\ 1 \end{pmatrix}$ *in 5 time steps, see Figure 4.2. Note that the control using the Grammian matrix is a minimum norm control. Note that for computation of the data we use program* $P-4$ *given in the Appendix.*

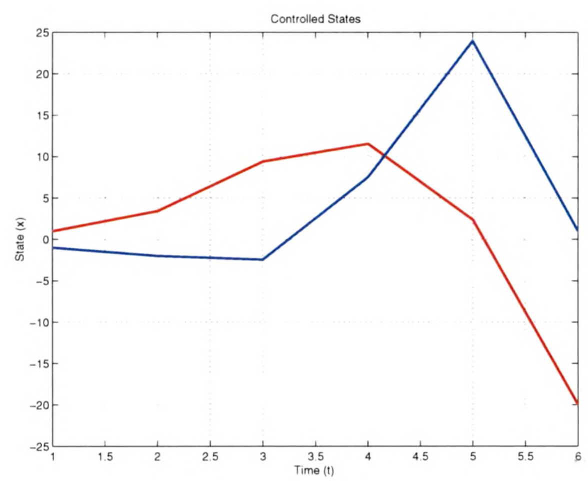

Figure 4.2: Controlled trajectories using controller given in equation (4.2.5)

Example 4.4.3. *Consider the nonlinear discrete Volterra system*

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t) + f(x(t), u(t)) \quad t \in N_0$$

where $A(t)$ *and* $B(t)$ *are as defined in Example 4.4.1 and*

$$f(x(t), u(t)) = \begin{pmatrix} x_2(t)sin(x_1(t))u(t) \\ x_1(t)(1 - cos(x_2(t)))u(t) \end{pmatrix}$$

Since

$$f(0,0)=0 \ \ and \ \ \frac{\partial f}{\partial x}=\begin{pmatrix} x_2(t)cos(x_1(t))u(t) & sin(x_1(t))u(t) \\ (1-cos(x_2(t)))u(t) & x_1(t)sin(x_2(t))u(t) \end{pmatrix},$$

obviously $(\frac{\partial f}{\partial x})_{x=0}=0$. *Thus the nonlinear function* f *is of higher order. As discussed in Example 4.4.1, the linear system is controllable, hence by using Theorem 4.3.1, we conclude that the semi-linear system is also controllable in the neighborhood of the origin.*

4.5 Summary

In this chapter, global controllability of linear discrete-time Volterra system is studied using controllability Grammian and Kalman type rank condition. Also using inverse function theorem and implicit function theorem, sufficient condition for the local controllability of semi-linear Volterra system is obtained. Numerical examples are also given to understand the concepts derived.