Chapter 5

Stability Analysis of Discrete Systems

Contents

In this chapter, new results for exponential and asymptotic stability of the null solutions of nonlinear discrete-time systems using the concept of (sp) matrix and the concept of generalised subradius are derived in Sections 5.1 and 5.2 respectively. Accurate estimate for the norm of solution of such system is also derived in Section 5.2. The problem of the asymptotic relationship between the solutions of a linear Volterra difference equation and its perturbed equation is proved by using the dichotomic condition of linear Volterra system in Section 5.3. Numerical examples are also given to support the results in each case.

5.1 Stability using (sp) Matrix

5.1.1 Importance of (sp) Matrix

In this chapter, a new result for the exponential stability of the null solution of a nonlinear non-autonomous discrete dynamical system described by

$$x(t+1) = g(t, x(t)), \quad t \in N_0 \tag{5.1.1}$$

where $g : N_0 \times \Omega \to \Omega$, $\Omega \subset R^n$ is a continuous nonlinear function satisfying $g(t,0) = 0 \ \forall t \in N_0$, is derived using the concept of (sp) matrix introduced by Xue and Guo [63]. Consider g in the form

$$g(t, x(t)) = Ax(t) + f(t, x(t))$$

where $x(t) \in \Omega$, $A \in s = \{A = (a_{ij})_{n \times n} : a_{ij} \geq 0, \Sigma_{j=1}^{n} a_{ij} \leq 1, \forall i = 1, 2, ..., n\}$ is a (sp) matrix, and the function $f : N_0 \times \Omega \to \Omega$ satisfies the inequality

$$\| f(t, x(t)) \| \leq a(t) \| x(t) \|, \ t \in N_0$$

where, $\sum a(t)$ is a convergent series of positive numbers. We prove that the null solution of the system is exponentially stable.

It is well known that if the spectral radius of the jacobian $Dg(0)$ of system (5.1.1) is strictly smaller than 1, then the null solution is exponentially stable. In order to check this condition, we have to compute the eigenvalues of the jacobian, which is a difficult task for higher dimensional systems. Checking if a matrix is (sp) can be easily done even for higher dimensional matrices, using a simple algorithm described in the definition of the (sp) matrix (see section 2.6.3, definition 2.6.2). Therefore, the method proposed in this paper is very efficient for numerical computations, as it avoids the evaluation of the eigenvalues of the jacobian.

Recently, Xue and Guo [63] studied asymptotic stability of null solution of

$$x(t+1) = Ax(t) \quad t \in N_0 \tag{5.1.2}$$

by introducing the notion of (sp) matrix (see Section 2.6.3, Theorem 2.6.6) and proved that, for $A \in s$, the zero solution of linear system (5.1.2) is asymptotically stable if and only if A is a (sp) matrix. We perceive system (5.1.1) as a perturbation of system (5.1.2). We provide sufficient conditions on the nonlinear function f to ensure that the null solution of the perturbed system (5.1.1) is exponentially stable.

5.1.2 Exponential Stability of Null Solution of Semi-linear System

Before we prove the main result, we first establish the following theorem. This theorem guarantees that if $A \in s$ is an (sp) matrix then the zero solution of linear system (5.1.2) is not only asymptotically stable but is exponentially stable.

Theorem 5.1.1. *Consider the autonomous linear system (5.1.2), where A is a $n \times n$ (sp) matrix and let $\Phi(t, t_0)$ be the transition matrix for (5.1.2). Then there exist positive constants β and $\eta \in (0, 1)$ such that*

$$\| \Phi(t, t_0) \| \leq \beta \eta^{t-t_0}$$

Proof. Since A is a (sp) matrix, then by Theorem 2.6.6, the null solution of system (5.1.2) is asymptotically stable and hence it is uniformly asymptotically stable as the system is autonomous. Then for every $\epsilon > 0$ there exists $\delta > 0$ and $N(\epsilon) > 0$

such that for $\| x_0 \| < \delta$, we have

$$\| \Phi(t, t_0)x_0 \| < \epsilon$$

for $t \geq t_0 + N(\epsilon)$. Also from the uniform stability of the solution $x = 0$, there exists an $\eta_1 > 0$ such that

$$\| \Phi(t, t_0) \| < \eta_1$$

for $t \geq t_0 + N(\epsilon)$, where η_1 can be chosen arbitrarily small.

Also because the uniform asymptotic stability implies the uniform stability, we obtain $\| \Phi(t, t_0) \|$ is bounded by positive number β_1 for all $t \geq t_0$. We then have for $t \in [t_0 + mN, t_0 + (m+1)N]$, $m > 0$

$$\begin{aligned}
\| \Phi(t, t_0) \| &\leq \| \Phi(t, t_0 + mN) \| \| \Phi(t_0 + mN, t_0 + (m-1)N) \| \ldots \| \Phi(t_0 + N, t_0) \| \\
&< \beta_1 \eta_1^m \\
&\leq \frac{\beta_1}{\eta_1} (\eta_1^{\frac{1}{N}})^{(m+1)N} = \beta \eta^{(m+1)N} \quad \text{where } \beta = \frac{\beta_1}{\eta_1}, \eta = \eta_1^{\frac{1}{N}} < 1 \\
&\leq \beta \eta^{t-t_0} \quad \text{for } mN \leq t - t_0 \leq (m+1)N
\end{aligned}$$

and this proves the theorem. □

Theorem 5.1.2. *Suppose that*

(a) $A \in s$ is a $n \times n$ (sp) matrix.

(b) $f : N_0 \times \Omega \rightarrow \Omega$, $\Omega \subset R^n$ is a continuous nonlinear function satisfying

(i) $f(t, 0) = 0 \quad \forall t \in N_0$

(ii) For every $\delta > 0$, there exists a sequence $a(t)$ satisfying $a(t) > 0$ for all $t \in N_0$ and $\sum_{t=0}^{\infty} a(t) < \infty$ such that

$$\| f(t, x(t)) \| \leq a(t) \| x(t) \| \qquad \text{for } \| x(t) \| < \delta$$

then the zero solution of the non-linear system (5.1.1) is exponentially stable.

Proof. Since A is a (sp) matrix, it follows from Theorem 5.1.1 that

$$\| \Phi(t, m) \| \leq \beta \eta^{t-m} \ for \ t_0 \leq m \leq t, \ forsome \ \beta \geq 0 \ and \ \eta \in (0, 1)$$

By using the variation of constant formula, the solution of equation (5.1.1) is given by

$$x(t,t_0,x_0) = \Phi(t,t_0)x_0 + \Sigma_{j=t_0}^{t-1}\Phi(t,j+1)f(j,x(j))$$

Thus,

$$\begin{aligned} \| x(t) \| &\leq \beta\eta^{t-t_0} \| x_0 \| + \Sigma_{j=t_0}^{t-1}\beta\eta^{t-j-1} \| f(j,x(j)) \| \\ &\leq \beta\eta^{t-t_0} \| x_0 \| + \beta\eta^{t}\Sigma_{j=t_0}^{t-1}\eta^{-j-1} \| f(j,x(j)) \| \\ \text{Hence} \quad \eta^{-t} \| x(t) \| &\leq \beta\{\eta^{-t_0} \| x_0 \| + \Sigma_{j=t_0}^{t-1}\eta^{-j-1}a(j) \| x(j) \| \} \end{aligned}$$

Applying the Gronwall's inequality,

$$\begin{aligned} \eta^{-t} \| x(t) \| &\leq \eta^{-t_0} \| x_0 \| \Pi_{j=t_0}^{t-1}\left[1+\beta\eta^{-1}a(j)\right] \\ &\leq \eta^{-t_0} \| x_0 \| \exp\left[\Sigma_{j=t_0}^{t-1}\beta\eta^{-1}a(j)\right] \\ &\leq \eta^{-t_0} \| x_0 \| \tilde{M}, \quad \text{where} \quad \exp\left[\Sigma_{j=t_0}^{t-1}\beta\eta^{-1}a(j)\right] \leq \tilde{M} < \infty \\ \| x(t) \| &\leq \| x_0 \| \tilde{M}\eta^{t-t_0} \end{aligned}$$

Hence the null solution of (5.1.1) is exponentially stable. □

Corollary 5.1.1. *If A is a (sp) matrix and $\lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0$ then the zero solution of* (5.1.1) *is exponentially stable.*

Proof. It is easy to see that under this condition, $\| f(t,x) \| = o(\| x \|)$ i.e. if given $\epsilon > 0$, there is $\delta > 0$ such that $\| f(t,x) \| \leq \epsilon \| x \|$, whenever $\| x \| < \delta$ and $t \in N_0$ and by applying the above theorem we obtain the result. □

5.1.3 Numerical Examples

Example 5.1.1. *Consider the* 4 *dimensional nonlinear system*

$$x(t+1) = Ax(t) + f(t,x(t))$$

$$where\ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\ and\ f(t,x(t)) = \frac{1}{4t^2}\begin{pmatrix} 5x_1(t)sinx_1(t) \\ x_2(t)cosx_2(t) \\ x_3(t)sinx_1(t)cosx_3(t) \\ x_4(t)cos2x_2(t) \end{pmatrix};$$

$\Omega = R^4$

Here A is a (sp) matrix. The function f is continuous, it satisfies $f(t,0) = 0$ and

$$\begin{aligned}
\| f(t,x(t)) \|^2 &= \frac{1}{16} \left| \frac{x_2(t)cosx_2(t)}{t^2} \right|^2 + \left| \frac{x_4(t)cos2x_2(t)}{t^2} \right|^2 \\
&\quad +25 \left| \frac{x_1(t)sinx_1(t)}{t^2} \right|^2 + \left| \frac{x_3(t)cosx_3(t)sinx_1(t)}{t^2} \right|^2 \\
&\leq \frac{25}{16t^4}\{| x_1(t) |^2 + | x_2(t) |^2 + | x_3(t) |^2 + | x_4(t) |^2\} \\
&\leq \frac{25}{16t^4} \| x(t) \|^2 \\
i.e.\ \| f(t,x(t)) \| &\leq \frac{5}{4t^2} \| x(t) \|
\end{aligned}$$

Since $\Sigma_{t=1}^{\infty}\frac{5}{4t^2} < \infty$, f satisfies all assumption of Theorem 5.1.2. Hence the zero solution is exponentially stable. If we take initial state as $x_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ *then it reaches to zero state in* 12 *iterations. Figure* 5.1 *shows that the zero solution of the given system is exponentially stable. Note that the verification of the (sp) matrix can be done using Matlab program $P - 5$ given in the Appendix and remaining computation of the data is done using program $P - 6$ of the Appendix.*

Example 5.1.2. *Consider the system*

$$x(t+1) = Ax(t) + f(t,x(t)), \quad t \in N_0$$

with the same matrix A as given in the above example and the nonlinear function

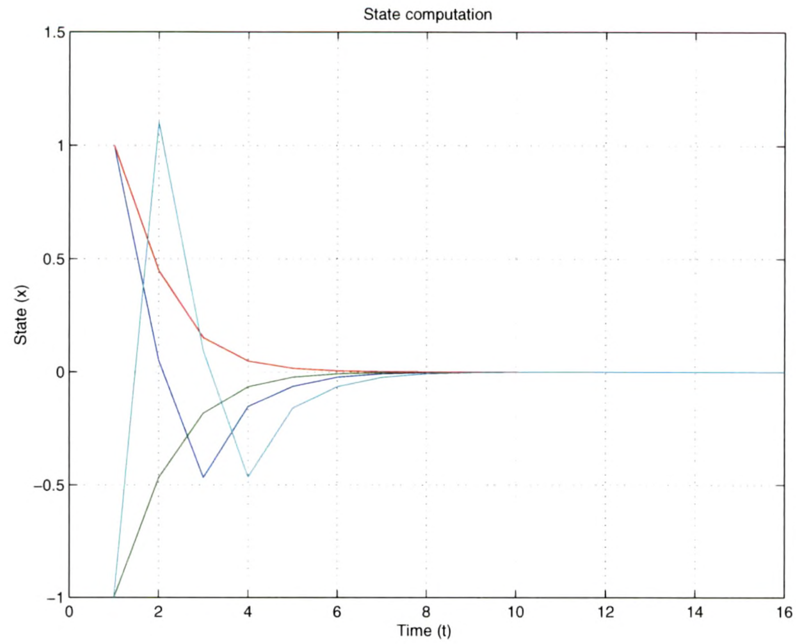

Figure 5.1: Asymptotic behavior of null solution

$f(t, x)$ given by $f(t, x(t)) = \frac{1}{4}\begin{pmatrix} 3x_3(t)x_1(t) \\ x_1(t)x_2(t) \\ x_3(t)x_2(t) \\ x_4(t)x_3(t) \end{pmatrix}$ *Since*

$$\begin{aligned}
\| f(t,x) \|^2 &= \frac{1}{16}\{9 \mid x_1(t)x_3(t) \mid^2 + \mid x_1(t)x_2(t) \mid^2 + \mid x_2(t)x_3(t) \mid^2 + \mid x_3(t)x_4(t) \mid^2\} \\
&\leq \frac{9}{16}\{\mid x_1(t)x_3(t) \mid^2 + \mid x_1(t)x_2(t) \mid^2 + \mid x_2(t)x_3(t) \mid^2 + \mid x_3(t)x_4(t) \mid^2\} \\
&\leq \frac{9}{16}\{\mid x_1(t) \mid^2\mid x_3(t) \mid^2 + \mid x_1(t) \mid^2\mid x_2(t) \mid^2 + \mid x_2(t) \mid^2\mid x_3(t) \mid^2 + \\
&\qquad + \mid x_3(t) \mid^2\mid x_4(t) \mid^2\} \\
&\leq \frac{9}{16}\{\mid x_1(t) \mid^2 (\mid x_3(t) \mid^2 + \mid x_2(t) \mid^2) + \mid x_3(t) \mid^2 (\mid x_2(t) \mid^2 + \mid x_4(t) \mid^2)\} \\
&\leq \frac{9}{16}\{\mid x_1(t) \mid^2 (\mid x_1(t) \mid^2 + \mid x_2(t) \mid^2 + \mid x_3(t) \mid^2 + \mid x_4(t) \mid^2) \\
&\quad + \mid x_3 \mid^2 (\mid x_1(t) \mid^2 + \mid x_2(t) \mid^2 + \mid x_3(t) \mid^2 + \mid x_4(t) \mid^2)\} \\
i.e. \frac{\| f(t,x) \|^2}{\| x \|^2} &\leq \frac{9}{16}\{\mid x_1(t) \mid^2 + \mid x_3(t) \mid^2\}
\end{aligned}$$

It shows that $\lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0$. *Hence according to Corollary 5.1.1, the zero solution* *is exponentially stable. If we take the initial state* $x_0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, *it reaches to null state within 11 iterations.*

5.2 Stability using Generalized Subradius

5.2.1 Introduction

Let R^n be the set of n-real vectors endowed with the Euclidean norm $\| \, . \, \|$. Consider the n dimensional discrete system

$$x(t+1) = A(t)x(t) + f(t, x(t)), \;\; t \in N_0 = \{0, 1, 2, ...\} \qquad (5.2.1)$$

where $(A(t))_{t \in N_0}$ is a sequence of real $n \times n$ matrices and $\{f(t, x(t))\}_{t \in N_0}$ is a sequence of nonlinear continuous vector functions. In [43], Medina and Gil derived accurate estimates for the norms of solutions using the approach based on the "freezing" method for difference equations and on recent estimates for the powers of a constant matrix.

Also many authors have studied the asymptotic stability of the null solution of such systems. A well-known result of Perron which dates back to 1929 (see Gordon [50]), (Ortega [14] , page 270) and (LaSalle [15] , Theorem 9.14) states that system (5.2.1) is asymptotically stable with $A(t) \equiv A$ (constant matrix) provided that spectral radius $\rho(A)$ of A is less than 1 and $f(t, x) = o(\| \; x \; \|)$. In [37], we established asymptotic stability of the null solution using a concept of (sp) matrix and taking some growth condition on f.

To obtain the estimates for the norms of solution and asymptotic stability of the null solution of system (5.2.1), we use the recent concept of generalized subradius.

In (Czornik 2005 [5]), the ideas of generalized spectral subradius and the joint spectral subradius are introduced and shown the relationship between generalized spectral radii and the stability of discrete time-varying linear system. (refer Section

2.6.4).

Let

$$\Phi(t,0) = A(t-1)A(t-2)...A(0), \;\; \Phi(t,t) = I$$

denotes the transition matrix of the system

$$x(t+1) = A(t)x(t) \quad , t \in N_0 \tag{5.2.2}$$

and

$$x(t,0,x_0) = \Phi(t,0)x_0$$

is the unique solution of equation (5.2.2) with initial condition $x(0,0,x_0) = x_0$. Let Σ denotes a nonempty set of all real $n \times n$ matrices and $\rho_*(\Sigma)$ denotes the generalized subradius of Σ (refer Section 2.6.4).

In [5], Czornik, proved that the discrete time-varying linear system (5.2.2) with $(A(t))-$ a sequence of matrices taken from Σ and any initial state $x_0 \in R^n$, $\lim_{t\to\infty} x(t) = 0$ if and only if $\rho_*(\Sigma) < 1$ (refer Theorem 2.6.7).

Note that if $\rho_*(\Sigma) < 1$, then for fixed $d \in (\rho_*(\Sigma), 1)$ there exist matrices $A_{i_1}, A_{i_2}, ..., A_{i_{n_0}} \in \Sigma$ such that $\| A_{i_1}A_{i_2}...A_{i_{n_0}} \| < d$.

We say that the zero solution of (5.2.1) is stable if for any $\epsilon > 0$ there exist $\delta > 0$ and $t_0 \in N_0$ such that

$$\| x(t) \| < \epsilon$$

whenever $t \geq t_0$ and $\| x_0 \| < \delta$.

We say that the zero solution of (5.2.1) is absorbing if for any $x_0 \in R^n$,

$$\lim_{t\to\infty} \| x(t) \| = 0.$$

We say that the zero solution of (5.2.1) is asymptotically stable if it is both stable and absorbing.

We extend the result proved by Czornik [5] for the nonlinear system (5.2.1) under the following assumptions.

Assumptions :

[A] Let $\rho_*(\Sigma) < 1$ and

$$\| \Phi(t,j) \| \leq \beta_j, \forall \quad 0 \leq j \leq t < \infty \quad where \quad \sum_{j=0}^{\infty} \beta_j \leq \eta, \quad for \quad some \quad constant \quad \eta >$$

[B] There exist constants $q, \mu \geq 0$ such that

$$\| f(t,x) \| \leq q \| x \| + \mu, \quad \forall x \in B_r = \{x \in R^n : \| x \| < r\} \quad for \quad some \quad r > 0.$$

[C] The constants q and η defined above satisfy $q\eta < 1$.

5.2.2 Accurate Estimate of Solution

Now we derive accurate estimate for the norm of solution of system (5.2.1).

Theorem 5.2.1. *Under Assumptions [A],[B] and [C] any solution $\{x(t)\}_{t=0}^{\infty}$ of (5.2.1) satisfies the inequality*

$$\sup_{t \geq 0} \| x(t) \| \leq \frac{\beta_0 \| x_0 \| + \mu\eta}{1 - q\eta}$$

provided that

$$\| x_0 \| \leq \frac{r(1-q\eta) - \mu\eta}{\beta_0}, \quad for \ some \ r > 0$$

Proof. By inductive arguments, it is easy to see that the unique solution $\{x(t)\}_{t=0}^{\infty}$ of (5.2.1) under initial condition $x(0) = x_0$ is given by

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1) f(j, x(j))$$

i.e.

$$\| x(t) \| \leq \| \Phi(t,0) \| \| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| (q \| x(j) \| + \mu) \quad if \quad \| x \| < r$$

Hence,

$$\| x(t) \| \leq \beta_0 \| x_0 \| + q \max_{i=0,1,\ldots t-1} \| x(i) \| \sum_{j=0}^{t-1} \| \Phi(t, j+1) \| + \mu \sum_{j=0}^{t-1} \| \Phi(t, j+1) \|$$

i.e.

$$\| x(t) \| \leq \beta_0 \| x_0 \| + q \max_{i=0,1,\ldots t-1} \| x(i) \| \sum_{j=0}^{\infty} \| \Phi(t, j+1) \| + \mu \sum_{j=0}^{\infty} \| \Phi(t, j+1) \|$$

i.e.

$$\max_{i=0,1,\ldots t} \| x(i) \| \leq \beta_0 \| x_0 \| + q\eta \max_{i=0,1,\ldots t} \| x(i) \| + \mu\eta$$

$$\sup_{i \geq 0} \| x(i) \| (1 - q\eta) \leq \beta_0 \| x_0 \| + \mu\eta$$

i.e.

$$\sup_{i \geq 0} \| x(i) \| \leq \frac{\beta_0 \| x_0 \| + \mu\eta}{(1 - q\eta)}$$

provided that

$$\| x_0 \| \leq \frac{r(1 - q\eta) - \mu\eta}{\beta_0}$$

Hence the theorem. □

5.2.3 Asymptotic Behavior of Null Solution

We now prove the asymptotic stability of the null solution of system (5.2.1).

Theorem 5.2.2. *Suppose that the system (5.2.1) satisfies*

(i) $\lim_{t \to \infty} \sum_{j=0}^{t-1} \| \Phi(t, j+1) \| = 0$

(ii) $\rho_*(\Sigma) < 1$ *and*

(iii) *Assumption [B]*

Then the zero solution of (5.2.1) is asymptotically stable.

Proof. The unique solution $\{x(t)\}_{t=0}^{\infty}$ of (5.2.1) under initial condition $x(0) = x_0$ is given by

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j))$$

Hence

$$\begin{aligned}
\| x(t) \| &\leq \| \Phi(t,0) \| \| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| \| f(j,x(j)) \| \\
&\leq \| \Phi(t,0) \| \| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| (q \| x(j) \| + \mu) \\
i.e. \| x(t) \| &\leq \{\| \Phi(t,0) \| \| x_0 \| + \mu \sum_{j=0}^{t-1} \| \Phi(t,j+1) \|\} + \sum_{j=0}^{t-1} q \| \Phi(t,j+1) \| \| x(j) \|
\end{aligned}$$

Now applying the discrete Gronwall's inequality,

$$\| x(t) \| \leq \{\| \Phi(t,0) \| \| x_0 \| + \sum_{j=0}^{t-1} \mu \| \Phi(t,j+1) \|\} \prod_{j=0}^{t-1} [1 + q \| \Phi(t,j+1) \|]$$

Since condition (ii) and Theorem 2.6.7 implies that

$$\| \Phi(t,0) \| \| x_0 \| \to 0, \ \ as \ \ t \to \infty.$$

Hence by using condition (i), we prove

$$\| x(t) \| \to 0 \ as \ t \to \infty$$

Hence the proof. □

5.2.4 Numerial Example

Example 5.2.1. *Consider the non-autonomous system given by the following equation.*

$$x(t+1) = A(t)x(t) + f(t,x(t)) \tag{5.2.3}$$

where the sequence $A(t) = \begin{pmatrix} \frac{1}{3}t & 0 \\ 1 & .04t \end{pmatrix}$, *Let* $\Sigma = \{A(h) : h = 0.002, 0.004, .006, ...\}$ *and*

$f = \frac{1}{6}\begin{pmatrix} \frac{1}{2}x_2(t) + \epsilon x_2^2(t) \\ \frac{1}{2}x_1(t) \end{pmatrix}$. *Let* $\epsilon \in (0, \frac{1}{2})$. *For* $n = 10$, *we can verify that the generalized spectral radius* $\rho_*(\Sigma) = 0.008 < 1$ *and* $\eta = 1$. *Also*

$$\begin{aligned}
\| f(t,x) \|^2 &= \frac{1}{36}\{\frac{1}{4}x_2^2 + \epsilon x_2^2 + \epsilon^2 x_2^4 + \frac{1}{4}x_1^2\} \\
&\le \frac{1}{36}\{\frac{1}{4}x_2^2 + \epsilon x_2^2 + \epsilon^2 x_2^2 + \frac{1}{4}x_1^2\} \\
&\le \frac{1}{36}\{(\frac{1}{2} + \epsilon)^2 x_2^2 + \frac{1}{4}x_1^2\} \\
&\le \frac{1}{36}(\frac{1}{2} + \epsilon)^2 \| x \|^2 \\
&\le \frac{1}{36}d^2 \| x \|^2 \text{ taking } d = (\frac{1}{2} + \epsilon) < 1 \\
i.e., \| f(t,x) \| &\le \frac{d}{6} \| x \| \\
\| f(t,x) \| &\le q \| x \| \text{ taking } q = \frac{d}{6}
\end{aligned}$$

i.e. f *satisfies the Assumption* $[B]$ *with* $q = \frac{d}{6} = \frac{0.7}{6} = 0.1167 < 1$ *taking* $\epsilon = 0.2$ *without loss of generality and* $\mu = 0$. *The function* f *is continuous and it satisfies* $f(0) = 0$. *Note that* $q\eta = 0.1167 < 1$. *Hence all the assumptions of Theorem (5.2.2) are satisfied. We can see in Figure* 5.2 *that initial state* $x_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ *reaches to null state in* 10 *iterations. Hence the null solution of equation (5.2.3) is asymptotically stable.*

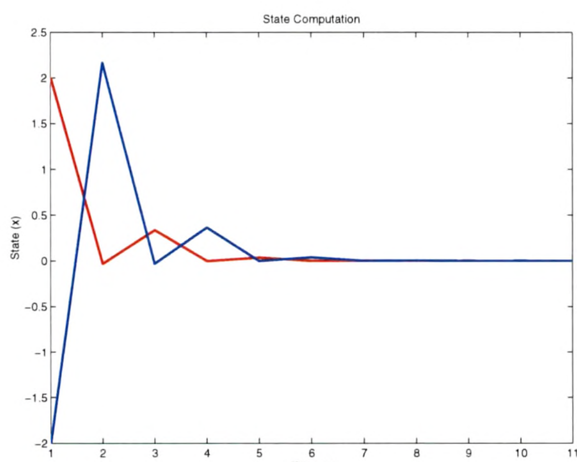

Figure 5.2: Asymptotic behavior of null solution

5.3 Asymptotic Equivalence of Discrete Volterra Systems

Now we show that under certain conditions there exists a homeomorphism between solutions of linear discrete Volterra system and its nonlinear perturbation. This correspondence is elevated to asymptotic equivalence under suitable condition. Numerical example is given to illustrate the result.

5.3.1 Introduction

Two systems of differential or difference equations are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions eventually tends to zero.

The problem of the asymptotic relationship between the solutions of a linear Volterra difference equation and its perturbed nonlinear equation is studied by many authors in several papers (see [13], [4],[53], [31]). The authors Choi and Koo [53], have studied this problem by using resolvent matrix and comparison principle. Pinto [31] studied the asymptotic equivalence between the solutions of linear difference system given by

$$x(t+1) = A(t)x(t) \tag{5.3.1}$$

and its perturbed nonlinear system

$$y(t+1) = A(t)y(t) + f(t, y(t)) \tag{5.3.2}$$

by means of the concept of (h, k) dichotomy. Let $N_0 \triangleq \{0, 1, 2, ...\}$ and $t_0 \in N_0$. Here we consider a linear Volterra system of convolution type

$$x(t+1) = A(t)x(t) + \sum_{r=t_0}^{t} B(t-r)x(r), \;\; x(t_0) = x_0, \;\; t \in N_{t_0} \triangleq \{t_0, t_0+1, ...\} \tag{5.3.3}$$

and its nonlinear perturbation

$$y(t+1) = A(t)y(t) + \sum_{r=t_0}^{t} B(t-r)y(r) + f(t, y(t)), \;\; y(t_0) = x_0, \;\; t \in N_{t_0} \quad (5.3.4)$$

where $A(t)$ is a $n \times n$ nonsingular matrix function, $B(t)$ is a $n \times n$ matrix function and $f : N_{t_0} \times R^n \to R^n$ is a continuous nonlinear function. It can be proved that

$$x(t, t_0, x_0) = X(t)x_0 \quad (5.3.5)$$

is a unique solution of equation (5.3.3) with $x(t_0) = x_0$. where $X(t)$ is a $n \times n$ matrix, called the fundamental matrix of system (5.3.3) and satisfies the following equation

$$X(t+1) = A(t)X(t) + \sum_{r=t_0}^{t} B(t-r)X(r)$$

It is easily proved that the solution of the perturbed equation (5.3.4) is given by

$$y(t, t_0, x_0) = X(t)x_0 + \sum_{r=t_0}^{t-1} X(t-r-1)f(r, y(r)) \quad (5.3.6)$$

Mathematically, systems (5.3.3) and (5.3.4) are said to be asymptotically equivalent if, for every solution $x(t)$ of (5.3.3), there exists a solution $y(t)$ of (5.3.4) such that

$$y(t) = x(t) + o(1) \;\; as \;\; t \to \infty.$$

In this chapter, we investigate the asymptotic equivalence between two Volterra systems by using the concept of ordinary dichotomy. In [8], Elaydi defined the ordinary dichotomy and provided some applications.

Definition 5.3.1. *The linear system (5.3.3) has an ordinary dichotomy if there exists a projection matrix P and a positive constant M such that*

$$\begin{aligned} \| X(t)PX^{-1}(s) \| &\leq M \;\; for \;\; t \geq s \geq t_0 \\ \| X(t)(I-P)X^{-1}(s) \| &\leq M \;\; for \;\; s \geq t \geq t_0. \end{aligned}$$

Let $\| x \| = \sup_{t \geq t_0} \| x(t) \|$ Define

$$C(t_0) := \{x : N_{t_0} \to R^n : \| x \| < \infty\} \tag{5.3.7}$$

Obviously $C(t_0)$ is a Banach space. Our result generalizes the theorem proved by Pinto [31]. We make the following assumptions.

Assumptions :

[A] The linear system (5.3.3) has an ordinary dichotomy on N_{t_0}.

[B] $f : N_{t_0} \times R^n \to R^n$ is a continuous function such that

$$\| f(t,x) - f(t,y) \| \leq \mu(t) \| x - y \|$$

where $\mu \in l^1([t_0, \infty])$ and $\sum_{s=t_0}^{\infty} \| f(s,0) \| < \infty$.

[C] There exists $a \in N_{t_0}$ be so large that $M \sum_{t=a}^{\infty} \mu(t) = \alpha < 1$

Under asymptotic condition $X(t)P \to 0 \quad as \quad t \to \infty$, we will prove that

$$y(t) = x(t) + o(1) \quad as \quad t \to \infty$$

We remark that for $B(t) = 0, \forall t \in N_{t_0}$, system (5.3.3) and (5.3.4) reduces to the particular case of system (5.3.1) and (5.3.2) respectively.

5.3.2 Asymptotic Equivalence

Theorem 5.3.1. *Let Assumptions [A], [B] and [C] hold true. Then there exists a homeomorphism between the bounded solutions $x(t) \in C(t_0)$ of linear system (5.3.3) and bounded solution $y(t) \in C(t_0)$ of the perturbed nonlinear system (5.3.4). Moreover,*

$$y(t) = x(t) + o(1)$$

provided that $X(t)P \to 0$ as $t \to \infty$.

Proof. According to Assumption [C], there is $a \in N_{t_0}$ be such that

$$\alpha = M \sum_{s=a}^{\infty} \mu(s) < 1.$$

Consider Banach space $C(a)$ as defined in (5.3.7) with the

$$\| x \| = \sup_{t \geq a} \| x(t) \|$$

On this space we define the operator T as follows :

$$(Ty)(t) = x(t) + \sum_{s=a}^{t-1} X(t) P X^{-1}(s) f(s, y(s)) - \sum_{s=t}^{\infty} X(t)(I-P) X^{-1}(s) f(s, y(s)) \tag{5.3.8}$$

The infinite sum is obviously convergent and since

$$\| Ty(t) \| \leq \| x(t) \| + M \sum_{s=a}^{\infty} \| f(s, y(s)) \|$$

Therefore,

$$\| Ty(t) \| \leq \| x(t) \| + M \sum_{s=a}^{\infty} \{ \mu(s) \| y(s) \| + \| f(s, 0) \| \}$$

T maps $C(a)$ into itself. Moreover, for all $y_1(t), y_2(t) \in C(a)$, we have

$$\begin{aligned}
\| Ty_1(t) - Ty_2(t) \| &\leq \sum_{s=a}^{t-1} \| X(t) P X^{-1}(s) \{ f(s, y_1(s)) - f(s, y_2(s)) \} \\
&\quad + \sum_{s=t}^{\infty} \| X(t)(I-P) X^{-1}(s) \{ f(s, y_2(s)) - f(s, y_1(s)) \} \\
&\leq M \sum_{s=a}^{\infty} \| f(s, y_1(s)) - f(s, y_2(s)) \| \\
&\leq M \sum_{s=a}^{\infty} \mu(s) \| y_1(s) - y_2(s) \| \\
&\leq \sup_{t \geq a} \| y_1(t) - y_2(t) \| M \sum_{s=a}^{\infty} \mu(s) \\
\| Ty_1 - Ty_2 \| &\leq \alpha \| y_1 - y_2 \|, \quad \textit{with} \quad \alpha < 1
\end{aligned}$$

Therefore Banach's fixed point theorem implies that T has a unique fixed point $y(t) \in C(a)$ i.e.

$$y(t) = Ty(t)$$

This $y(t)$ is a solution of equation (5.3.4). Hence, if $x(t)$ is a solution of (5.3.3), then $y(t)$ is a solution of (5.3.4) with $y(t) \in C(a)$.

Conversely, if $y(t)$ is a solution of (5.3.4) with $y(t) \in C(a)$, then $x(t)$ defined by (5.3.8) with $y(t) = Ty(t)$ is a bounded solution of (5.3.3).

Therefore (5.3.8) with $y(t) = Ty(t)$ establishes a one-to-one correspondence between the bounded solutions of (5.3.3) and (5.3.8) for $t \in N_a$.

Consider now for $t \in N_a$, $x^o(t)$ a bounded solution of (5.3.3) and $y^o(t)$ the corresponding bounded solution of (5.3.4). Then from (5.3.8) with $y(t) = Ty(t)$ and the corresponding equation with the replacement of $x(t)$ and $y(t)$ by $x^o(t)$ and $y^o(t)$, we obtain

$$\| y(t) - y^o(t) \| \leq \| x(t) - x^o(t) \| + \alpha \| y(t) - y^o(t) \|$$

and

$$\| x(t) - x^o(t) \| \leq \| y(t) - y^o(t) \| + \alpha \| y(t) - y^o(t) \| \leq (1+\alpha) \| y(t) - y^o(t) \|$$

Thus, it follows that

$$(1+\alpha)^{-1} \| x(t) - x^o(t) \| \leq \| y(t) - y^o(t) \| \leq (1+\alpha)^{-1} \| x(t) - x^o(t) \|$$

which shows that the one-to-one correspondence between the bounded solutions of (5.3.3) and (5.3.4) for $t \in N_a$ is continuous and its inverse is continuous, so it is a homeomorphism. But the solutions of (5.3.3) and (5.3.4) are defined for all $t \in N_a$ and are uniquely determined by the initial data, so we have a homeomorphism on N_a.

Now let $\epsilon > 0$ be given and choose $t_1 \in N_a$ so large that

$$M \sum_{s=t_1}^{\infty} \| f(s, y(s)) \| \leq M \sum_{s=t_1}^{\infty} \{\mu(s) \| y(s) \| + \| f(s,0) \|\} < \epsilon$$

Thus, if $X(t)P \to 0$ as $t \to \infty$, then we find

$$\| x(t) - y(t) \| \leq \| X(t)P \| \sum_{s=a}^{t_1-1} \| X^{-1}(s)f(s,y(s)) \| + \epsilon \leq 2\epsilon$$

for all large t. Hence $x(t) - y(t) \to 0$ as $t \to \infty$. □

5.3.3 Numerical Example

We use Matlab program $P-7$ for the computations involved in the following example.

Example 5.3.1. *Consider the Volterra system*

$$y(t+1) = A(t)y(t) + \sum_{r=t_0}^{t} B(t-r)y(r) + f(t,y(t)),$$

with $A(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & (t+1) \end{pmatrix}$ *and* $B(t) = \begin{pmatrix} 4^{-t-1} & 0 \\ 0 & 3^{-t-1} \end{pmatrix}$.
Let ϵ *be a small constant and* $f(t,y) = \frac{\epsilon}{t^5}\begin{pmatrix} sin y_1(t) \\ 1 - cos y_2(t) \end{pmatrix}$ *and* $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{aligned} \| f(t,x) - f(t,y) \| &= \frac{\epsilon}{t^5} \left\| \begin{pmatrix} sin x_1 - sin y_1 \\ cos x_2 - cos y_2 \end{pmatrix} \right\| \\ &\leq \frac{2\epsilon}{t^5} \| x - y \| \end{aligned}$$

Hence f satisfies Assumption [B] with $\mu(t) = \frac{2\epsilon}{t^5}$. It is easily verified that the linear system satisfies the dichotomy condition with $M = 1$. We also verify Assumption [C] with $\alpha = 0.0104 < 1$ in 5 time steps. So all the Assumptions of theorem (5.3.1) are satisfied. The solutions of linear system and nonlinear system in 5 time steps are as follows.

$$y = 1.0e{+}003 \begin{pmatrix} 0.0010 & 0.0006 & 0.0003 & 0.0001 & 0.0001 & 0.0000 \\ -0.0010 & -0.0023 & -0.0079 & -0.0345 & -0.1848 & -1.1747 \end{pmatrix} \text{ and}$$

$$x = 1.0e + 003 \begin{pmatrix} 0.0010 & 0.0006 & 0.0003 & 0.0001 & 0.0001 & 0.0000 \\ -0.0010 & -0.0023 & -0.0079 & -0.0345 & -0.1849 & -1.1750 \end{pmatrix}$$

Hence

$$y(t) = x(t) + o(1) \;\; as \;\; t \to \infty.$$

5.4 Summary

In this chapter asymptotic stability of discrete time nonlinear system is studied using the concepts of (sp) matrix and generalized radius. Also accurate estimate for the bound of solution of nonlinear system is obtained. Asymptotic equivalence of solutions of linear and nonlinear discrete Volterra systems is discussed using the notion of ordinary dichotomy. Numerical examples are included to give more clarity and understanding in each case.