Chapter 6

Optimal Control of Discrete Volterra System - a Classical Approach

Contents

In this chapter, optimal control problem described by discrete-time linear Volterra system is studied by the conventional minimization method of Lagrange multipliers.

6.1 Introduction

The problem of optimal control for discrete Volterra system have received a great deal of attention and have attracted many researchers. Gaishun and Dymkov [11] studied linear-quadratic optimization and feedback control problems for linear discrete Volterra system using operator approach. Belbas and Schmidt [48] studied an optimal control problem for a system governed by a Volterra integral equation with impulsive terms. Here we adapt independent approach of classical minimization technique called method of Lagrangian multipliers to find the optimal control of the following linear Volterra system.

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t), \;\; t \in N_0 \tag{6.1.1}$$

where $A(i)'s, i = 0, 1, ...t$ are $n \times n$ nonsingular matrices and B is $n \times m$ matrix. We consider a quadratic performance index for the finite time process $(0 \leq t \leq N)$ as

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}\left[x^*(t)Qx(t) + u^*(t)Ru(t)\right] \tag{6.1.2}$$

where S, Q are $n \times n$ positive definite or positive semidefinite Hermitian matrices (or real symmetric matrices). R is an $m \times m$ positive definite Hermitian or real symmetric matrix.

We find a controller which minimizes J as given by equation (6.1.2), when it is subjected to the constraint equation specified by (6.1.1) and when initial condition on state vector is specified as

$$x(0) = c. \tag{6.1.3}$$

6.2 Solution by the Classical Minimization Method using Lagrange Multipliers

We minimize J defined by

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}\left[x^*(t)Qx(t) + u^*(t)Ru(t)\right]$$

subjected to the constraint equation

$$x(t+1) = \sum_{i=0}^{t} A_i x(t-i) + Bu(t), \quad t \in N_0$$

where $t = 0, 1, 2, ..., N-1$, and when initial condition on state vector is specified as

$$x(0) = c$$

Now by using a set of Lagrange multipliers $\lambda(1), \lambda(2), ...\lambda(N)$, $\lambda(i)'s \in R^n$, $for \quad i = 1, 2, ..., N$, we define a new performance index L as follows.

$$\begin{aligned} L \;=\; & \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}\Big\{\left[x^*(t)Qx(t) + u^*(t)Ru(t)\right] \\ & +\lambda^*(t+1)\Big[\sum_{i=0}^{t} A_i x(t-i) + Bu(t) - x(t+1)\Big] \\ & +\Big[\sum_{i=0}^{t} A_i x(t-i) + Bu(t) - x(t+1)\Big]^*\lambda(t+1)\Big\} \end{aligned} \tag{6.2.1}$$

Obviously $L = L^*$.
To minimize the functional L, we differentiate L with respect to each component of vectors $x(t)$,$u(t)$ and $\lambda(t)$ and set the results equal to zero. i.e.

$$\begin{aligned} \frac{\partial L}{\partial x(t)} &= 0, \quad t = 1, 2, ..., N \\ \frac{\partial L}{\partial u(t)} &= 0, \quad t = 0, 1, 2, ..., N-1 \\ \frac{\partial L}{\partial \lambda(t)} &= 0, \quad t = 1, 2, ..., N \end{aligned}$$

Using the following equalities (refer Ogata [16], page 670),

$$\frac{\partial}{\partial x}x^*Ax = 2Ax, \ \ if \ \ A = A^*,$$

and

$$\frac{\partial}{\partial x}x^*Ay = Ay$$

we obtain the following partial derivatives

$$\frac{\partial L}{\partial x(t)} = 0$$

This implies that

$$Qx(t) + \sum_{i=0}^{N-t-1} A^*(i)\lambda(t+i+1) - \lambda(t) = 0 \ \ , t = 1, 2, ..., N-1 \tag{6.2.2}$$

Now

$$\frac{\partial L}{\partial x(N)} = 0$$

implies

$$Sx(N) - \lambda(N) = 0 \tag{6.2.3}$$

Similarly

$$\frac{\partial L}{\partial u(t)} = 0$$

$\Rightarrow$

$$Ru(t) + B^*\lambda(t+1) = 0 \tag{6.2.4}$$

$$\frac{\partial L}{\partial \lambda(t)} = 0$$

$\Rightarrow$

$$\sum_{i=0}^{t-1} A(i)x(t-i-1) + Bu(t-1) - x(t) = 0 \tag{6.2.5}$$

Equation (6.2.5) is simply the system state equation. Equation (6.2.3) specifies the final value of the Lagrange multiplier.

Now we shall simplify the equations just obtained. From Equation (6.2.2), we have

$$\lambda(t) = Qx(t) + \sum_{i=0}^{N-t-1} A^*(i)\lambda(t+i+1), \ \ t = 1, 2, ..., N-1 \tag{6.2.6}$$

with the final condition $\lambda(N) = Sx(N)$.
By solving (6.2.4) for $u(t)$ and noting that R^{-1} exists, we obtain,

$$u(t) = -R^{-1}B^*\lambda(t+1), t = 0, 1, ..., N-1 \tag{6.2.7}$$

Equation (6.2.5) can be rewritten as

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t), t = 0, 1, ..., N-1 \tag{6.2.8}$$

This is the state equation. Substitution of (6.2.7) into (6.2.8) results in

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) - BR^{-1}B^*\lambda(t+1), t = 0, 1, ..., N-1 \tag{6.2.9}$$

with the initial condition $x(0) = c$.

To obtain the solution to the minimization problem, we need to solve (6.2.6) and (6.2.9) simultaneously. Note that for the system (6.2.9),the initial condition $x(0)$ is specified, while for the Lagrange multiplier equation (6.2.6), the final condition $\lambda(N)$ is specified. Thus,

$$u(t) = -R^{-1}B^*\lambda(t+1), t = 0, 1, ..., N-1$$

where $\lambda(t)$ is solution of boundary value problem. Hence we proved the following theorem.

Theorem 6.2.1. *The minimization problem for (6.1.1),(6.1.2) has a optimal solution given by*

$$u(t) = -R^{-1}B^*\lambda(t+1), t = 0, 1, ..., N-1$$

where $\lambda(t)$ is the solution of the boundary value problem (6.2.6), (6.2.9).

6.3 Summary

Optimal control problem of discrete-time linear Volterra system is studied using classical technique of Lagrange multipliers.