CHAPTER II

ON CORRELATION BETWEEN CERTAIN SUMMABILITY METHODS AS APPLIED TO ORTHOGONAL SERIES

2.1 Let the Fourier orthogonal expansion of a function $f(x) \in L^2[a,b]$ with respect to an orthonormal system of functions $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be

$$(2.1.1) \qquad f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)$$

where

$$(2.1.2) \qquad c_n = \int_a^b f(x)\, \phi_n(x)\, dx \, .$$

The Euler means, Cesàro means, Riesz means, and Logarithmic means of the orthogonal series (2.1.1) are designated by $\tau_n(x)$, $\sigma_n(x)$, $\sigma_n(\lambda,x)$ and $L_n(x)$ respectively*. Let $S_n(x)$ denote the nth partial sum of the series (2.1.1).

Meder+ has proved the following result connecting Cesàro means and Euler means of the orthogonal series (2.1.1):

Theorem A: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[\sigma_n(x) - \tau_n(x) \right]^2$$

converges almost everywhere in $[a,b]$.

I propose to extend in this chapter the line of action of Meder regarding the interconnections of the means in various summability methods. The following theorems will be proved which would connect the Riesz-means with the other means.

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* All means are n^{th} means of order unity.

\+ Meder [53].

<u>THEOREM 1:</u> If $\{\lambda_n\}$ is a positive, strictly increasing numerical sequence with $\lambda_0 = 0$ such that

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_{n+1}^2} = O\left(\frac{1}{\lambda_k^2}\right), \tag{2.1.3}$$

then the series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left[\sigma_n(\lambda, x) - \tau_n(x)\right]^2$$

converges almost everywhere in $[a, b]$.

<u>THEOREM 2:</u> Let $\{\lambda_n\}$ be as defined in Theorem 1 satisfying (2.1.3). Then the series

$$\sum_{n=2}^{\infty} \frac{1}{n\log n}\left[\sigma_n(\lambda, x) - L_n(x)\right]^2$$

converges almost everywhere in $[a, b]$.

2.2 <u>Proof of Theorem 1:</u> We have

$$\sum_{n=1}^{\infty} \frac{1}{n}\int_a^b \left[\sigma_n(\lambda, x) - \tau_n(x)\right]^2 dx \le$$

$$(2.2.1) \leq 2 \sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx \; +$$

$$+ 2 \sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \left[\sigma_n(x) - \tau_n(x)\right]^2 dx \; .$$

Now,

$$\sigma_n(\lambda, x) - \sigma_n(x) =$$

$$= \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) c_k \phi_k(x) - \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k \phi_k(x)$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n+1} - \frac{\lambda_k}{\lambda_{n+1}}\right) c_k \phi_k(x) \; .$$

Therefore by the orthonormality property of $\{\phi_n(x)\}$

$$\frac{1}{n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx \; =$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\lambda_k}{\lambda_{n+1}} - \frac{k}{n+1}\right)^2 c_k^2 \; \leq$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k^2\, c_k^2}{\lambda_{n+1}^2} + \frac{1}{n}\sum_{k=1}^{n}\frac{k^2 c_k^2}{(n+1)^2} .$$

Consequently,

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx \leq$$

$$\leq \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k^2\, c_k^2}{\lambda_{n+1}^2} + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2 c_k^2}{(n+1)^2} =$$

(2.2.2) $$= \sum\nolimits_1 + \sum\nolimits_2 \text{, say.}$$

Using (2.1.8) in $\sum_1$, we get,

$$\sum\nolimits_1 = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k^2\, c_k^2}{\lambda_{n+1}^2} = \sum_{k=1}^{\infty}\lambda_k^2\, c_k^2 \sum_{n=k}^{\infty}\frac{1}{n\lambda_{n+1}^2} =$$

(2.2.3) $$= O(1)\sum_{k=1}^{\infty} c_k^2 .$$

Also

$$\sum\nolimits_2 = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2 c_k^2}{(n+1)^2} \leq$$

$$\leq \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

(2.2.4) $$= O(1) \sum_{k=1}^{\infty} c_k^2 .$$

Therefore, from (2.2.2), (2.2.3) and (2.2.4),

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx =$$

(2.2.5) $$= O(1) \sum_{k=1}^{\infty} c_k^2 .$$

Since $f(x) \in L^2[a,b]$, by Bessel's inequality*

$$\sum_{k=0}^{\infty} c_k^2 \leq \int_a^b f^2(x)\, dx < \infty ,$$

which implies

(2.2.6) $$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx < \infty .$$

From (2.2.1), (2.2.6) and Theorem A, our theorem follows.

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* Natanson [67] p.177.

An immediate corollary of theorem 1 will be following:

Corollary 1: If the orthogonal series (2.1.2) satisfying (2.1.3), is summable almost everywhere by (C,1)-method to s(x), then

$$\sum_{k=1}^{n}\left[\sigma_k(\lambda,x)-s(x)\right]^2 = o(n)$$

almost everywhere in [a,b].

Proof of the Corollary 1: By Minkowski's inequality*

$$\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\sigma_k(\lambda,x)-s(x)\right]^2\right\}^{\frac{1}{2}} \leq$$

$$\text{(2.2.7)} \qquad \leq \left\{\frac{1}{n}\sum_{k=1}^{n}\left[\sigma_k(\lambda,x)-\tau_k(x)\right]^2\right\}^{\frac{1}{2}} +$$

$$+ \left\{\frac{1}{n}\sum_{k=1}^{n}\left[\tau_k(x)-s(x)\right]^2\right\}^{\frac{1}{2}} =$$

$$= \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

* Natanson [67] p.166

From the hypothesis it is evident that

$$\sum_{2} \longrightarrow 0$$

Also, from the result of theorem 1, the series

$$\sum_{k=1}^{\infty} \frac{1}{k} \left[\sigma_k(\lambda, x) - \tau_k(x) \right]^2$$

is convergent almost everywhere in $[a, b]$. Applying an extension of Kronecker's theorem we get

$$(2.2.8) \qquad \sum_{k=1}^{n} \left[\sigma_k(\lambda, x) - \tau_k(x) \right]^2 = o(n)$$

almost everywhere in $[a, b]$.

The result follows from (2.2.7) and (2.2.8).

2.3 Proof of Theorem 2: We have

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(\lambda, x) - L_n(x) \right]^2 dx \leq$$

$$\leq 2 \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x) \right]^2 dx +$$

$$+ 2 \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(x) - L_n(x) \right]^2 dx \leq$$

$$\leq 2 \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx +$$

$$+ 4 \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(x) - S_n(x)\right]^2 dx +$$

$$+ 4 \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[S_n(x) - L_n(x)\right]^2 dx =$$

$$(2.3.1) \quad = 2 \sum\nolimits_1 + 4 \sum\nolimits_2 + 4 \sum\nolimits_3 , \text{ say .}$$

From theorem 1

$$(2.3.2) \quad \sum\nolimits_1 = \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(\lambda, x) - \sigma_n(x)\right]^2 dx < \infty .$$

Using orthonormality property

$$\sum\nolimits_2 = \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sigma_n(x) - S_n(x)\right]^2 dx$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k \phi_k(x) - \sum_{k=0}^{n} c_k \phi_k(x)\right]^2 dx$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \log n} \sum_{k=1}^{n} \frac{k^2 c_k^2}{(n+1)^2}$$

$$< \sum_{n=2}^{\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 c_k^2$$

$$< \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}$$

(2.3.3) $$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

To establish our theorem it remains to show that

$$\sum\nolimits_3 = \sum_{n=2}^{\infty} \frac{1}{n \log n} \int_a^b \left[S_n(x) - L_n(x) \right]^2 dx < \infty .$$

Since

$$0 < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n < \ell \quad \text{for } n=1,2,3,\ldots,$$

the inequality

$$(a + b)^2 \leq 2(a^2 + b^2)$$

implies

(2.3.4) $$\left(\sum_{k=1}^{n} \frac{1}{k} \right)^2 \leq 2 \left(\ell^2 + \log^2 n \right) \quad \text{for } n=1,2,3,\ldots$$

We have

$$L_k(x) - S_k(x) =$$

$$= \frac{1}{\log k} \sum_{i=1}^{k} \frac{S_i(x)}{i} - \sum_{v=1}^{k} c_v \phi_v(x)$$

$$= \frac{1}{\log k} \sum_{i=1}^{k} \frac{1}{i} \sum_{v=1}^{i} c_v \phi_v(x) - \sum_{v=1}^{k} c_v \phi_v(x)$$

$$= \frac{1}{\log k} \sum_{v=1}^{k} c_v \phi_v(x) \sum_{i=v}^{k} \frac{1}{i} - \sum_{v=1}^{k} c_v \phi_v(x)$$

$$= \frac{1}{\log k} \sum_{v=1}^{k} c_v \phi_v(x) \left[\sum_{i=1}^{k} \frac{1}{i} - \sum_{i=1}^{v-1} \frac{1}{i} \right] - \frac{\log k}{\log k} \sum_{v=1}^{k} c_v \phi_v(x)$$

$$(2.3.5) \quad = \frac{1}{\log k} \left(\sum_{i=1}^{k} \frac{1}{i} - \log k \right) \sum_{v=1}^{k} c_v \phi_v(x) -$$

$$- \frac{1}{\log k} \sum_{v=1}^{k} c_v \phi_v(x) \sum_{i=1}^{v-1} \frac{1}{i} .$$

By (2.3.4), (2.3.5) and on orthonormality property

$$\sum_{k=4}^{\infty} \frac{1}{k \log k} \int_a^b \left[S_k(x) - L_k(x) \right]^2 dx \leq$$

$$\leq \sum_{k=4}^{\infty} \frac{4\ell^2}{k(\log k)^3} \left[\sum_{v=1}^{k} c_v^2 + \sum_{v=1}^{k} c_v^2 \log^2 v \right]$$

$$(2.3.6) \quad \leq 4\ell^2 \left[\sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} \sum_{v=1}^{k} c_v^2 + \sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} \sum_{v=1}^{k} c_v^2 \log^2 v \right]$$

As

$$\sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} \sum_{v=1}^{k} c_v^2 < \sum_{v=1}^{\infty} c_v^2 \sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} =$$

$$(2.3.7) \quad = O(1) \sum_{v=1}^{\infty} c_v^2 < \infty$$

and

$$\sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} \sum_{v=1}^{k} c_v^2 (\log v)^2 \leq$$

$$\leq \sum_{v=2}^{\infty} c_v^2 + \sum_{k=4}^{\infty} \frac{1}{k(\log k)^3} \sum_{v=4}^{k-1} c_v^2 (\log v)^2 + A ,$$

where A is an absolute constant,

$$\leq \sum_{v=2}^{\infty} c_v^2 + \sum_{v=4}^{\infty} c_v^2 (\log v)^2 \sum_{k=v+1}^{\infty} \frac{1}{k(\log k)^3} + A$$

$$\leq \sum_{v=2}^{\infty} c_v^2 + \sum_{v=4}^{\infty} c_v^2 (\log v)^2 \cdot \frac{1}{2(\log v)^2} + A =$$

(2.3.8) $$= O(1) \sum_{v=1}^{\infty} c_v^2 < \infty .$$

Using (2.3.7) and (2.3.8) in (2.3.6), we get

(2.3.9) $$\sum\nolimits_3 = O(1) \sum_{v=1}^{\infty} c_v^2 < \infty .$$

In view of (2.3.2), (2.3.3) and (2.3.9) our theorem get established.

<u>Corollary 2:</u> <u>If the orthogonal series</u> (2.1.1) <u>satisfying</u> (2.1.3), <u>is summable almost everywhere by the first logarithmic method to</u> S(x), <u>then</u>

$$\sum_{k=2}^{n} \left[\sigma_k(\lambda, x) - S(x)\right]^2 = o\,(n \log n)$$

almost everywhere in $[a,b]$.

Proof follows on the lines of corollary 1.