CHAPTER III

THE ORDER OF MAGNITUDE OF COEFFICIENTS OF THE LACUNARY FOURIER SERIES, ITS ABSOLUTE CONVERGENCE AND ITS ALMOST EVERYWHERE CONVERGENCE

1 In the present chapter we obtain some estimates regarding the behaviour of the Fourier coefficients of a lacunary Fourier series and with the help of these estimates, we study the behaviour of the series in respect of its absolute convergence and almost everywhere convergence.

Noble[1)] proved that if the series (L) satisfies the lacunarity condition

$$(1) \qquad \lim \frac{N_k}{\log n_k} = \infty \quad , \text{ as } k \longrightarrow \infty ,$$

where

$$N_k = \min \{ n_k - n_{k-1} , n_{k+1} - n_k \} ,$$

and if the function f satisfies the Lipschitz condition of order α, $0 < \alpha < 1$, in a subinterval

$I = \{ x : |x - x_0| \leq \delta \}$ of $[-\pi , \pi]$, then

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1) Noble [11]

(2) $$a_{n_k} = O(1/n_k^{\alpha}) ,$$

$$b_{n_k} = O(1/n_k^{\alpha}).$$

Subsequently Kennedy[1] improved this result by showing that the conclusion (2) holds under the weaker lacunarity condition that $n_{k+1} - n_k \to \infty$. The same author[2], in another paper showed that even when the subinterval I is replaced by a set E of positive measure, the conclusion (2) of the theorem holds, provided that $n_{k+1} - n_k \to \infty$ is replaced by a more stringent condition. More precisely the author has proved that if the sequence $\{n_k\}$ satisfies the Hadamard lacunarity condition

(3) $$\frac{n_{k+1}}{n_k} > \lambda > 1 ,$$

and if,

(4) $$f \in \text{Lip}\, \alpha(E) , \quad 0 < \alpha < 1 ,$$

where E is a set of positive measure, then (2) holds.

Tomić[3], while retaining the Hadamard lacunarity condition, and replacing the set of positive measure by a single point, studied the behaviour of

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1) Kennedy [6]
2) Kennedy [7]
3) Tomić [19]

the Fourier coefficients. He proved the following theorem.

THEOREM A (Tomić)

If the series (L) is the Fourier series of f(x) satisfying the Hadamard lacunarity condition (3), and

$$(5) \quad \omega(x_o, h) = \sup_{o \leq t \leq h} |f(x_o + t) - f(x_o)| = O(h^{\alpha}),$$

$o < \alpha \leq 1$, then

$$(6) \qquad a_{n_k} = O(1/n_k^{\beta}),$$

$$b_{n_k} = O(1/n_k^{\beta});$$

where $\beta = \frac{\alpha}{2 + \alpha}$.

We shall now examine the behaviour of the Fourier coefficients for the sequence $\{n_k\}$ given by (7) below.

First , we prove the following theorem.

THEOREM 5:

If the series (L) is the Fourier series of a function f(x) satisfying the condition (5), and the sequence $\{n_k\}$ is given by

$$(7) \qquad n_k = [a^{k^r}], \quad a > 1, \text{ and } o < r_1 < r \leq 1,$$

then,

(8) $$a_{n_k} = O(k^{2(1-r)}/ n_k^{\beta}),$$

$$b_{n_k} = O(k^{2(1-r)}/ n_k^{\beta});$$

where $\beta = \frac{\alpha}{2 + \alpha}$.

If $r = 1$, then our theorem gives the result (6) of Tomić.

PROOF OF THEOREM 5:

There is no loss of generality in taking $x_o = o$.

In virtue of the lacunarity of the Fourier series, we have,

(9) $$S_{n_k} - f(x) = \frac{1}{2\pi(n_{k+1} - n_k)} \int_0^{\pi} \varphi(x,t) \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k\frac{1}{2}t}{\sin^2\frac{1}{2}t} dt,$$

and hence

(10) $$|f(x) - S_{n_k}(x)| \leq \frac{A_1}{n_{k+1} - n_k} \left\{ \int_0^{\frac{\pi}{n_{k+1}-n_k}} + \int_{\frac{\pi}{n_{k+1}-n_k}}^{h(n_k)} + \int_{h(n_k)}^{\pi} \right\}$$

$$|\varphi(x,t)| \frac{|\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k\frac{1}{2}t|}{\sin^2\frac{1}{2}t} dt,$$

$$= I_1 + I_2 + I_3 ,$$

where $h = h(n_k) = n_k^{-1/2+\alpha}$, and $\varphi(x,t) = f(x+t) + f(x-t) - 2f(x)$

The reason why $h(n_k)$ is chosen in this way will be clear later.

Let A denote an absolute constant which may be different at different occurrences.

Now,

$$I_1 \leq \frac{A}{n_{k+1} - n_k} \sup_{\substack{|x| \leq h \\ |t| \leq \frac{\pi}{n_{k+1}-n_k}}} |f(x+t)+f(x-t)-2f(x)|$$

$$\int_0^{\frac{\pi}{n_{k+1}-n_k}} \frac{|\sin^2 n_{k+1}\tfrac{1}{2}t - \sin^2 n_k \tfrac{1}{2}t|}{\sin^2 \tfrac{1}{2}t}\, dt.$$

Using (11) of chapter II, where the sequence $\{n_k\}$ is the same as in (7) above, we find that

$$\frac{1}{n_{k+1} - n_k} \int_0^{\frac{\pi}{n_{k+1}-n_k}} \frac{|\sin^2 n_{k+1}\tfrac{1}{2}t - \sin^2 n_k \tfrac{1}{2}t|}{\sin^2 \tfrac{1}{2}t}\, dt$$

$$= O(\log k).$$

Hence

(11) $$I_1 \leq A\omega(o\ ,\ 2h)\log k.$$

$$I_2 \leq \frac{A}{n_{k+1}-n_k} \sup_{\substack{|x| \leq h \\ \frac{\pi}{n_{k+1}-n_k} \leq |t| \leq h}} |f(x+t)+f(x-t)-2f(x)| \int_{\frac{\pi}{n_{k+1}-n_k}}^{h(n_k)} \frac{1}{t^2}\,dt .$$

$$\leq \frac{A\omega(o\ ,\ 2h)}{n_{k+1}-n_k}(n_{k+1}-n_k)$$

(12) $$\leq A\omega(o\ ,\ 2h).$$

Also $$I_3 \leq \frac{A}{(n_{k+1}-n_k)\, h^2(n_k)}$$

(13) $$\leq \frac{A\, k^{1-r}}{n_k\, h^2\, (n_k)} .$$

Collecting (11), (12) and (13), we get,

(14) $$|f(x)-S_{n_k}(x)| \leq A\omega(o\ ,\ 2h)\log k + \frac{A\, k^{1-r}}{n_k\, h^2\, (n_k)} .$$

Now, it can be seen that

$$(15) \qquad a_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x) - S_{n_{k-1}}) P_{N_k - 1} \cos n_k x \, dx ,$$

$$b_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x) - S_{n_{k-1}}) P_{N_k - 1} \sin n_k x \, dx ,$$

where

$$N_k = \min \left\{ n_k - n_{k-1} , n_{k+1} - n_k \right\} ,$$

and

$$P_{N_k} - 1 = 2K_{N_k} - 1(x) = 1 + 2 \sum_{p=1}^{N_k - 1} (1 - \frac{p}{N_k}) \cos px .$$

Also

$P_{N_k} - 1(x)$ has the following obvious properties:

(Fejér kernal)

$$(16) \qquad (i) \qquad P_{N_k - 1}(x) \geq 0 \quad , \text{ for all } x ,$$

$$(ii) \qquad \left| \int_{0}^{x} P_{N_k} - 1(t) dt \right| \leq \pi , \text{ for all } N_k ,$$

$$(iii) \qquad \sup_{h \leq |x| \leq \pi} |P_{N_k} - 1(x)| = O(\frac{1}{N_k h^2}).$$

Hence

$$(17) \quad |a_{n_k}| \le \left(\int_{-h(n_k)}^{h(n_k)} + \int_{-\pi}^{-h(n_k)} + \int_{h(n_k)}^{\pi} \right) |f(x) - \bar{S}_{n_{k-1}}| \, |P_{N_k - 1}(x)| \, |\cos n_k x| \, dx$$

$$= J_1 + J_2 + J_3 \text{ , say.}$$

Using (14) and the property (16 (ii) of $P_{N_k}-1$, we obtain,

$$(18) \quad J_1 \le A\omega(o \, , \, 2h) \log k + \frac{A \, k^{1-r}}{n_k \, h^2(n_k)} \; .$$

Now, it is known[1] that

$$(19) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} | \sigma n - f(x)| \, dx = o(1) \, ,$$

where σ_n denotes the (c , 1) means of the general Fourier series without any lacunarity condition.

With our lacunarity condition, we have,

$$(20) \quad |S_{n_k} - s| \, dx = \frac{1}{n_{k+1} - n_k} \int_{-\pi}^{\pi} |(n_{k+1} + 1)(\sigma \bar{n}_{k+1} - f(x)) - (n_k + 1)(\sigma \bar{n}_k - f(x))| dx$$

1) Zygmund ([22], p.144)

$$\leq A\, k^{1-r} \int_{-\pi}^{\pi} |\sigma_{n_{k+1}} - f(x)|\,dx + A\, k^{1-r} \int_{-\pi}^{\pi} |\sigma_{n_k} - f(x)|\,dx$$

(21) $$= o(k^{1-r}) .$$

From the property 16(iii) of $P_{N_k - 1}$, and (21), and using the fact that

$$\frac{1}{N_k} \leq \frac{A\, k^{1-r}}{n_k} ,$$

we obtain,

(22) $$J_2 = o\left(\frac{k^{2(1-r)}}{n_k\, h^2(n_k)} \right) .$$

It can be shown similarly that

(23) $$J_3 = o\left(\frac{k^{2(1-r)}}{n_k\, h^2(n_k)} \right).$$

Collecting (18) , (22) and (23), we obtain,

(24) $$|a_{n_k}| \leq A\, \omega(o , 2h) \log k + A \left(\frac{k^{2(1-r)}}{n_k\, h^2(n_k)} \right) ,$$

since $\omega(o , 2h) = O(h^{\alpha}(n_k))$ and $h(n_k) = n_k^{-1/2+\alpha}$,

we get from (24),

$$| a_{n_k} | \leq A\, h^{\alpha} \log k + \frac{A\, k^{2(1-r)}}{n_k\, h^2(n_k)}$$

$$= \frac{A \log k}{n_k^{\alpha/2+\alpha}} + \frac{A\, k^{2(1-r)}}{n_k^{\alpha/2+\alpha}} ,$$

and hence,

$$(25) \qquad a_{n_k} = O\left(\frac{k^{2(1-r)}}{n_k^{\beta}}\right), \quad \beta = \frac{\alpha}{2+\alpha} .$$

Similarly, we obtain,

$$b_{n_k} = O\left(\frac{k^{2(1-r)}}{n_k^{\beta}}\right) .$$

<u>COROLLARY</u>:

Under the conditions of theorem 5, the series (L) is absolutely convergent.

PROOF:

It can be seen that for the sequence $\{n_k\}$ as in (7), we have, for all sufficiently large k, and for any positive number m,

$$(26) \qquad n_k > k^m .$$

Hence, using the estimates in (8), we have, for all sufficiently large k,

$$a_{n_k} = O(1/k^s),$$
$$b_{n_k} = O(1/k^s);$$

where $s > 1$.

Consequently

$$\sum_{k=1}^{\infty} (|a_{n_k}| + |b_{n_k}|)$$

is convergent.

2 In this section, we shall apply the estimates of a_{n_k} and b_{n_k} obtained above in examining the questions of absolute convergence and the almost everywhere convergence of the lacunary Fourier series (L).

Zygmund[1] proved the following theorem.

THEOREM B: (Zygmund)

If $f(x)$ is of bounded variation and

(27) $\omega(x, h) \leq A \log^{-2-\eta}\left(\frac{1}{|h|}\right), (\eta > 0, -\pi \leq x < \pi),$

then the Fourier series of $f(x)$ converges absolutely.

This theorem does not require any lacunarity condition. Salem[2] has proved that this theorem is best possible in the sense that η cannot be replaced by zero.

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1) Zygmund [21]
2) Salem [13]

In view of Salem's result the question naturally arises as to whether η can be replaced by zero in (27) by imposing certain lacunarity conditions. In this connection we shall need the following theorem of Szidon[1].

THEOREM C: (Szidon)

If (L) is a Fourier series of a bounded function and satisfies the Hadamard lacunarity condition (3), then, it converges absolutely.

It is clear from theorem C that we have to look for a weaker condition than Hadamard's lacunarity condition (3). In the following theorem 6 we study this problem and show that for a sequence $\{n_k\}$ of the type defined in (7), the conclusion of theorem B will hold when $\eta = o$ and even more is true.

THEOREM 6:

Let the sequence $\{n_k\}$ be as in (7).

Let

$$(28) \quad \omega(x_o, h) = O(\log \frac{1}{h})^{-p}, \ h > o, \ 1 < p \leq 2.$$

Then, the series (L) is absolutely convergent provided that

$$(29) \quad \frac{1}{p} < r \leq 1 .$$

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1) Szidon ([15], [16])

PROOF OF THEOREM 6:

Without loss of generality we choose $x_o = o$.

Using (24), we have,

$$(30) \qquad |a_{n_k}| \leq A \left\{\log\left(\frac{1}{h} \right)\right\}^{-p} \log k + A \left(\frac{k^{2(1-r)}}{n_k \, h^2(n_k)} \right) .$$

Let

$$h = h(n_k) = n_k^{-1/2+\epsilon} , \quad \epsilon > o , \text{ we get,}$$

from (30) ,

$$(31) \qquad |a_{n_k}| \leq \frac{A \log k}{k^{rp}} + \frac{A \, k^{2(1-r)}}{n_k^{\epsilon/2+\epsilon}} .$$

Similarly, we have,

$$|b_{n_k}| \leq \frac{A \log k}{k^{rp}} + \frac{A \, k^{2(1-r)}}{n_k^{\epsilon/2+\epsilon}} .$$

Now

$$\sum_{k=1}^{\infty} \frac{\log k}{k^{rp}} ,$$

is convergent for $rp > 1$ i.e. for $r > \frac{1}{p}$, and by using (26), we have,

$$\sum_{k=1}^{\infty} \frac{k^{2(1-r)}}{n_k^{\epsilon/2+\epsilon}}$$

convergent for $r > o$, $\epsilon > o$.

Hence the convergence of

$$\sum_{k=1}^{\infty}(| a_{n_k} | + | b_{n_k} |)$$

follows for $\frac{1}{p} < r \leq 1$.

In the theorem 6, we have to consider $p > 1$, otherwise the range $\frac{1}{p} < r \leq 1$ has no meaning. In the following theorem we consider $p \leq 1$ and examine the almost everywhere convergence of the series (L).

In this connection, we prove the following theorem.

<u>THEOREM 7</u>:

Let the sequence $\{n_k\}$ be as in (7).

Let the condition (28) be satisfied for $\frac{1}{2} < p \leq 1$. Then the Fourier series (L) converges almost everywhere for $\frac{1}{2p} < r \leq 1$.

PROOF:

From (31), we have,

$$a_{n_k} = O(\frac{\log k}{k^{rp}}) \quad ,$$

$$b_{n_k} = O(\frac{\log k}{k^{rp}}) \quad ;$$

and hence

$$\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2)$$

is convergent for $rp > \frac{1}{2}$ i.e. for $r > \frac{1}{2p}$.

Therefore $f \in L_2 [-\pi , \pi]$ and by Carleson[1] theorem the Fourier series (L) converges almost everywhere. This proves the theorem.

In the above theorem, we cannot take $p \leq \frac{1}{2}$, because then the range $\frac{1}{2p} < r \leq 1$ has no meaning.

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1) Carleson [4]