CHAPTER V

ON THE CONVERGENCE OF A LACUNARY FOURIER SERIES AND ON THE ABSOLUTE CONVERGENCE OF A SERIES ASSOCIATED WITH A LACUNARY FOURIER SERIES

1 In this section, we discuss the convergence of the series (L) and its conjugate series (L_1) in $[-\pi, \pi]$, when the function f satisfies a certain condition in some subinterval I of $[-\pi, \pi]$. The following theorems are due to Kennedy[1).]

THEOREM A:

If f(x) is of bounded variation in some subinterval I, and if $n_{k+1} - n_k \longrightarrow \infty$ as $k \rightarrow \infty$, then

$$a_{n_k} = O(1/n_k),$$

$$b_{n_k} = O(1/n_k).$$

THEOREM B:

If $f(x) \in \text{Lip}\,\alpha$ in some subinterval I, and if $n_{k+1} - n_k \longrightarrow \infty$ as $k \longrightarrow \infty$, then

$$a_{n_k} = O(1/n_k^{\alpha}),$$

$$b_{n_k} = O(1/n_k^{\alpha}).$$

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1) Kennedy [6]

We prove the following theorems.

THEOREM 20:

If $f(x) \in L^2(I)$, then the series (L) and its conjugate series (L_1) are almost everywhere convergent.

THEOREM 21:

If f(x) is of bounded variation in some subinterval I, then the series (L) is convergent to $f(x + o) + f(x - o)/2$ at any point where this expression has a meaning and the conjugate series is convergent to $\bar{f}(x)$ whenever it exists, and when x is a point of the Lebesgue set.

PROOF OF THEOREM 20:

If $f(x) \in L^2(I)$, then by lemma 2, chapter IV, $f(x) \in L^2 [-\pi , \pi]$, and hence by Carleson's[1] theorem, the Fourier series (L) of f converges almost everywhere.

Also, by Riesz - Fischer[2] theorem, the conjugate series (L_1) is the Fourier series of $\bar{f}(x) \in L^2 [-\pi , \pi]$ whenever $f(x) \in L^2 [-\pi , \pi]$ and hence the series (L_1) is almost everywhere convergent by Carleson's theorem.

PROOF OF THEOREM 21:

If S_n are the partial sums and σ_n are the arithmetic means of order n for the series

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1) Carleson [4]

2) Bary ([1] , p.64)

$$u_o + u_1 + u_2 + \ldots\ldots\ldots + u_n + \ldots\ldots ,$$

then,

$$(1) \qquad S_n - \sigma_n = \frac{u_1 + 2u_2 + \ldots\ldots + nu_n}{n + 1} .$$

In case of a lacunary series, where in calculating Fejẽr sums it is necessary to replace the absent terms by zeros, we have,

$$(2) \qquad S_{n_k} - \sigma_{n_k} = \frac{n_1 u_{n_1} + n_2 u_{n_2} + \ldots\ldots + n_k u_{n_k}}{n_k + 1} .$$

Now, we take

$u_{n_k} = a_{n_k} \cos n_k x + b_{n_k} \sin n_k x$ in case of the series (L) and

$u_{n_k} = b_{n_k} \cos n_k x - a_{n_k} \sin n_k x$ in case of the series (L_1).

Under the hypothesis of the theorem , we have,

$$a_{n_k} = O(1/n_k) ;$$

$$b_{n_k} = O(1/n_k) , \text{ by theorem A.}$$

Therefore

$$u_{n_k} = O(1/n_k)$$

and hence,

$$n_k \, u_{n_k} = O(1).$$

Now, the number of terms in the numerator of the right hand side of (2) is k, and hence,

(3) $$|S_{n_k} - \sigma_{n_k}| < \frac{Ak}{n_k},$$

where A is an absolute constant.

Now,

$$\frac{k}{n_k} \longrightarrow o \quad \text{as } k \longrightarrow \infty ,$$

whenever $n_{k+1} - n_k \longrightarrow \infty$.

Therefore

$$|S_{n_k} - \sigma_{n_k}| \longrightarrow o .$$

Now, it is known that the Fourier series (L) is summable (c, 1) to f(x + o) + f(x - o)/2 for every value of x for which this expression has a meaning i.e.

$$\sigma_{n_k} \longrightarrow f(x + o) + f(x - o)/2.$$

Hence $S_{n_k} \longrightarrow f(x + o) + f(x - o)/2$ for every value of x for which this expression has a meaning.

It is also known that the series (L_1) is summable (c , 1) to $\bar{f}(x)$ for every value of x for which $\bar{f}(x)$ exists and when x is a point of the Lebesgue set. Hence by the same argument as used above, (L_1) converges to $\bar{f}(x)$ whenever it exists, and when x is a point of the Lebesgue set.

2 In this section we shall be concerned with the series

$$(4) \qquad \sum_{k=1}^{\infty} \left(\frac{S_{n_k} - s}{n_k} \right) ,$$

where $$S_{n_k} = \sum_{p=1}^{k} u_{n_p} ,$$

$$u_{n_p} = a_{n_p} \cos n_p x + b_{n_p} \sin n_p x ,$$

and s is an appropriate number independent of n_k.

Let

$$\varphi(t) = f(x + t) + f(x - t) - 2s/2.$$

We prove the following theorems.

<u>THEOREM 22:</u>

If f(x) is bounded and if

$$(5) \qquad \sum_{k=1}^{\infty} \left(\frac{1}{n_{k+1} - n_k} \right)$$

is convergent, then the series (4) is absolutely convergent.

THEOREM 23:

If

(i) $\frac{n_{k+1}}{n_k} \longrightarrow 1$, as $k \longrightarrow \infty$,

(ii) $\omega(\frac{\pi}{n_{k+1} - n_k}) \log(1 - \frac{n_k}{n_{k+1}}) = O(1)$,

(iii) $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent,

then the series (4) is absolutely convergent.

THEOREM 24:

If $f(x)$ is of bounded variation in some subinterval I, and if

(6) $$\sum_{k=1}^{\infty} \frac{\log n_k}{n_k}$$

is convergent, then the series (4) is absolutely convergent.

THEOREM 25:

If $f(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$, in some subinterval I, and if

$$\sum_{k=1}^{\infty} \frac{1}{n_k^{\alpha}}$$

is convergent, then the series (4) is absolutely convergent.

PROOF OF THEOREM 22:

We have in virtue of the lacunary Fourier series

$$(7) \qquad \frac{S_{n_k} - s}{n_k} = \frac{1}{2\pi n_k(n_{k+1}-n_k)} \int_0^{\pi} \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt.$$

Therefore

$$\left|\frac{S_{n_k} - s}{n_k}\right| \leq \frac{1}{2\pi(n_{k+1} - n_k)} \cdot \frac{n_{k+1}}{n_k} \cdot \frac{1}{n_{k+1}} \int_0^{\pi} |\varphi(t)| \frac{\sin^2 n_{k+1}\frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt$$

$$+ \frac{1}{2\pi(n_{k+1} - n_k)} \cdot \frac{1}{n_k} \int_0^{\pi} |\varphi(t)| \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt.$$

Now, using $\frac{n_{k+1}}{n_k} = O(1)$, and also using the result that

$$\frac{1}{n_k} \int_0^{\pi} | \varphi(t)| \frac{\sin^2 n_k t}{\sin^2 t}\, dt$$

is bounded, whenever f(x) is bounded, we get,

$$\frac{S_{n_k} - s}{n_k} = O\left(\frac{1}{n_{k+1} - n_k}\right) + O\left(\frac{1}{n_{k+1} - n_k}\right)$$

$$= O\left(\frac{1}{n_{k+1} - n_k}\right).$$

Hence the convergence of the series (4) follows from the convergence of the series (5).

PROOF OF THEOREM 23:

By the method similar to one which is used in theorem 3, chapter II , we have,

$$\left| \frac{S_{n_k} - s}{n_k} \right| = O(1/n_k) .$$

Hence the convergence of the series (4) follows from the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$.

PROOF OF THEOREM 24:

(8) $$\frac{S_{n_k} - s}{n_k} = \frac{u_{n_1} + u_{n_2} + \dots + u_{n_k} - s}{n_k} .$$

We have

$$u_{n_k} = O(1/n_k) , \text{ by theorem A,}$$

and hence,

$$\frac{S_{n_k} - s}{n_k} \leq \frac{A\left(\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k} \right) + |s|}{n_k} ,$$

where A is an absolute constant.

$$= O\left(\frac{\log n_k}{n_k}\right).$$

Hence the convergence of the series (4) follows from the convergence of the series (6).

PROOF OF THEOREM 25:

Under the hypothesis of the theorem, we have

$$u_{n_k} = O\left(\frac{1}{n_k^{\alpha}}\right), \quad \text{by theorem B.}$$

Using (8), we get

$$\frac{s_{n_k} - s}{n_k} \leq \frac{A\left(\frac{1}{n_1^{\alpha}} + \frac{1}{n_2^{\alpha}} + \ldots + \frac{1}{n_k^{\alpha}}\right) + |s|}{n_k}$$

$$= O\left(\frac{n_k^{1-\alpha}}{n_k}\right)$$

$$= O\left(\frac{1}{n_k^{\alpha}}\right).$$

Hence the convergence of the series (4) follows from the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{n_k^{\alpha}} .$$

Theorems analogous to theorems 24 and 25 can be stated for the conjugate series (L_1).