CHAPTER VI

ON THE ABSOLUTE SUMMABILITY (c , 1) OF A LACUNARY FOURIER SERIES AND ITS CONJUGATE SERIES

The following theorem is due to Kennedy[1].

THEOREM A:

If f(x) is of bounded variation and $f(x) \in \text{Lip}\,\alpha$, $o < \alpha < 1$, in some subinterval I, then the series (L) converges absolutely.

We discuss, here, the absolute summability of the series (L) by assuming the only condition on f(x) that the function is of bounded variation in some subinterval I. We prove the following theorems.

THEOREM 26:

If

(1) $$\underline{\lim} \frac{n_{k+1} - n_k}{\log k} = B \ , \ B > o,$$

and if f(x) is of bounded variation in some subinterval I, then the series (L) and (L_1) are everywhere absolutely summable (c , 1).

THEOREM 27:

If

(i) $n_{k+1} - n_k \longrightarrow \infty$,

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1) Kennedy [6]

(ii) $\sum_{k=1}^{\infty} \frac{k}{n_k^2} < \infty$,

and (iii) f(x) is of bounded variation in some subinterval I, then the series (L) and (L_1) are everywhere absolutely summable (c , 1).

PROOF OF THEOREM 26:

If σ_n are the arithmetic means of order n of the series $\sum u_n$, we have,

$$\sigma_n = \frac{s_0 + s_1 + \ldots\ldots + s_n}{n + 1} ,$$

and hence

$$(2) \qquad \sigma_n - \sigma_{n-1} = \frac{u_1 + 2u_2 + \ldots\ldots + nu_n}{n(n+1)} .$$

In the case of the lacunary series, where in calculating Fejér sums, it is necessary to replace the absent terms by zeros, we have,

$$(3) \qquad \sigma_{n_k} - \sigma_{n_k - 1} = \frac{n_1 u_{n_1} + n_2 u_{n_2} + \ldots\ldots + n_k u_{n_k}}{n_k(n_k + 1)} .$$

Here, it may be observed that there are k terms in the numerator of the right hand side of (3).

Now, we take,

$$u_{n_k} = (a_{n_k}\cos n_k x + b_{n_k}\sin n_k x) \text{ in case of the series (L) and}$$

$$= (b_{n_k}\cos n_k x - a_{n_k}\sin n_k x) \text{ in case of the series } (L_1).$$

The lacunarity condition (1) implies that $n_{k+1} - n_k \longrightarrow \infty$ and hence, by theorem A, chapter V, we have

$$u_{n_k} = O(1/n_k), \text{ and hence}$$

(4) $$n_k u_{n_k} = O(1) .$$

Therefore, we have,

(5) $$|\sigma_{n_k} - \sigma_{n_k - 1}| = O(k/n_k^2).$$

Now, under the lacunarity condition (1), it can be concluded that

(6) $$n_k > c\ k \cdot \log k ,$$

where c is an absolute constant.

This can be seen as follows.

From (1), we have

$$n_k - n_{k-1} > c_1 \log k , \quad c_1 > o, (k=1,2,3,\ldots);$$

and hence

$$n_k > n_1 + c_1 \sum_{p=2}^{k} \log p$$

$$> c_1 \int_2^k \log t \, dt$$

$$> c \; k \cdot \log k \;, \; c > o, (k = 1,2,3,\ldots.).$$

Using (5) and (6), we have,

$$|\, \sigma n_k - \sigma \overline{n_k - 1}| \; \leq \frac{A}{k \log^2 k} \, ,$$

where A is an absolute constant, and hence the convergence of $\sum_{k=1}^{\infty} |\, \sigma n_k - \sigma \overline{n_k - 1}|$ follows from the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \log^2 k}$.

This completes the proof of the theorem.

PROOF OF THEOREM 27:

The condition (4) holds under the conditions (i) and (iii) of this theorem. Using (5) and the condition (ii) of this theorem, the convergence of

$$\sum_{k=1}^{\infty} |\, \sigma n_k - \sigma \overline{n_k - 1} \,|$$

follows.