CHAPTER - 5

FORMULATION OF LINEAR PROGRAMMING AND FUZZY LINEAR PROGRAMMING MODELS

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5.1 GENERAL

A model is a conceptualization of a system or situation, which retains the essential characteristics of that system for specific purposes.

A number of different types of models that ought to prove useful, particularly in any new type of activity such as conjunctive use planning includes the system state models, the goals and objective models, the strategic models, the system behavior models, the decision response simulation models and optimization models.

System state models involve large class of models needed just to describe the essential characteristics of the situations and systems in which the problems have to be found.

The model, which gives us the essential characteristics of conditions suitable or unsuitable for that particular problem is termed as goals and objective models.

The models that decide the strategy to be adopted, i.e., converting present situation in better and more acceptable situation is known as strategic model. The strategic model could be classified as a hierarchical model in which to make complex process feasible various subsystems being modeled in a simpler, but still rational way.

Some are predictive models with which evaluation of technical, economical, financial, political, legal, moral, environmental and other types of feasibility can be done. Because these models deal with systems, they are called system behavior or system response models.

With simulation models and selection of any compatible set of conditions, one can simulate the corresponding effect on the magnitudes of the quantitative objectives and constraints, with a good digital computer one can run lots of mathematical experiments without the time and expense of actually implementing any of the possibilities.

To use the simulation effectively is very difficult as in water strategy decisions large number of combinations are available. Therefore, need for rational guidance arise which suggests combination of decisions and if that decisions does not suit to fulfill the requirements to suggest next set of decisions and so on. Then optimization models can be used.

5.2 OPTIMISATION MODEL CONSTRAINTS

An "optimum" can be defined as the situation which, no matter how it is changed, the resulting situation would be no better from the point of view of those responsible for making the decision.

There is a single basic process involved in the construction of all the optimization models. The sub groups of available information formed and logical arguments are included which explains why that particular sub group is included or not included in optimal solution. By iteration, optimum solution is found considering all the constraining sub groups.

For optimizing a strategy many techniques are used, such as linear programming, dynamic programming, gradient search, non- linear programming, etc.

Each is based on some specific set of logical arguments (usually mathematical principles). The validity of each procedure depends primarily on the nature of the mathematical form of the system response functions which have to be simulated, particularly the objectives selected in the model to be maximized or minimized.

In this study linear and fuzzy linear programming techniques are used and solution is obtained with the help of simplex method.

5.3 FORMAT TO SOLVE LINEAR PROGRAMMING MODEL

A linear programming model consists of an objective function subjected to various constraints.

Various constraints form a closed solution space having linear boundaries if definite optimum solution of the problem is available. With the help of objective function and type of problem (i.e., maximization or minimization), the optimal solution is obtained. To get the solution simplex method is utilized. It is the characteristics of the optimum solution that is associated with a corner or extreme point of the solution space. The simplex method employs an iterative process that starts at a feasible corner point, normally origin, and systematically moves from one feasible extreme point to another until the optimum point is eventually reached.

5.4 FORMULATION OF OBJECTIVE FUNCTION

In the linear programming model objective function controls the optimum solution. For formulation of objective function, following data are required:

(i) The cropping pattern

(ii)Yield of crops

(iii)Farm harvest price of each crop

(iv)Crop production cost

(v) Surface and ground water consumed

(vi)Unit cost of surface and ground water

The objective function can be formulated as follows:

$$\text{MAXIMIZE } (Z) = \sum_{j=1}^{J} [A_j * NR_j] - \sum_{i=1}^{I} [(Scc + Soc) Sw_i] - \sum_{i=1}^{I} [(Gcc + Goc) Gw_i] \quad (5.1)$$

Where,

Z = Objective function

A_j = Irrigated area of j^{th} crop, ha

NR_j = Net returns from j^{th} crop, Rs. / ha

= (Yield, Quintal/ha) * (Farm harvest price, Rs./Quintal) - (Crop production cost, Rs./Quintal)

Scc = Annual capital cost for surface water, Rs./ha.m

Soc = Annual operational cost for surface water, Rs./ha.m

Sw_i = Surface water supplied to canal diversions for command area in i^{th} month, ha.m

Gcc = Annual capital cost for ground water, Rs./ha.m

Goc = Annual operational cost for ground water, Rs./ha.m

Gw_i = Ground water pumped in command area m in i^{th} month, ha.m

j = 1, 2, 3,......., J (No. of crops)

i = 1, 2, 3,......., I (No. of decision period, i.e., month)

5.5 CONSTRAINTS

The objective function is subjected to the various constraints, discussed subsequently.

5.5.1 Surface Water Availability Constraint

$$\sum_{i=1}^{I} SW_i \leq SWA \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

Where,

Sw_i = Surface water supplied to canal diversions for command area in i^{th} month, ha.m

SWA = Surface water available at the head of canal under consideration for a particular year, ha.m

5.5.2 Canal Capacity Constraint

$$SW_i \leq SWC \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

Where,

Sw_i = Surface water supplied to canal diversions for command area in i^{th} month, ha.m

SWC = Discharge carrying capacity of canal for a particular month, ha.m

5.5.3 Ground Water Potential Constraint

$$\sum_{i=1}^{I} GW_i \leq GWP \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

Where,

Gw_i = Ground water pumped in command area m in i^{th} month, ha.m

GWP = Total ground water potential that can be available yearly, ha.m

5.5.4 Pumping Capacity Constraint

$$GW_i \leq WP_i \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5.5)$$

Where,

Gw_i = Ground water pumped in command area m in i^{th} month, ha.m

WP_i = Ground water pumping capacity of the wells located in command area for i^{th} month

5.5.5 Drainage Requirement Constraint

$$\sum_{i=1}^{I} GW_i \geq DR \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5.6)$$

Where,

Gw_i = Ground water pumped in command area m in i^{th} month, ha.m

DR = Minimum ground water quantum that can be exploited to fulfill the drainage requirement, ha.m

5.5.6 Area Availability Constraint

$$\sum_{i=1}^{I} \sum_{j=1}^{J} K_{ij} * A_j \leq TA \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5.7)$$

Where,

K_{ij} = Land use coefficient for j^{th} crop in i^{th} month as per the crop calendar followed in the command area

A_j = Irrigated area of j^{th} crop, ha

TA = Total area available for cultivation, ha

5.5.7 Water Requirement Constraint

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \{ A_j (WR_{ij} - Re_i) - Q_2 [Q_1 (1 + LR_s) SW_i + (1 + LR_g) GW_i] \} = 0 \quad (5.8)$$

Where,

Q_1, Q_2= Used to take into consideration the efficiency of surface water and ground water respectively

Q_1 $= 1 - SR_1 - AR_1 - ET_1$

Q_2 $= 1 - SR_2 - AR_2 - ET_2$

Wr_{ij} = Irrigation water requirement of j^{th} crop in i^{th} month

Re_i = Effective rainfall during i^{th} month

LR_s = Additional surface water requirement in % for leaching

LR_g = Additional ground water requirement in % for leaching

SR_1 = Fraction of water delivered to canals, lost as surface runoff which is assumed as 10%

SR_2 = Fraction of water from irrigated area, lost as surface runoff which is assumed as 10%

AR_1 = Fraction of water diverted to canal, lost as aquifer recharge which is assumed as 20%

AR_2 = Fraction of water from irrigated area, lost as aquifer recharge, which assumed as 20%

ET_1 = Fraction of water diverted to canal, lost as non beneficial evapotranspiration, which is assumed as 10 %

ET_2 = Fraction of water from irrigated area lost as non beneficial evapotranspiration, which is assumed as 10 %

Therefore,

Q_1 $= 1 - 0.1 - 0.2 - 0.1$

$= 0.6$

Q_2 $= 1 - 0.1 - 0.2 - 0.1$

$= 0.6$

5.5.8 Management Constraint

$$A_j \leq Am_j \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.9)$$

Where,

A_j = Irrigated area of j^{th} crop, ha

Am_j = Upper limit of areas under various crops according to management considerations in ha of j^{th} crop.

This constraint provides upper limit for various crops, according to management considerations. Irrigated area of j^{th} crop in ha should be less than or equal to the upper limit of areas under various crops according to management considerations in ha of j^{th} crop.

5.5.9 Socio – Economic Constraint

$$A_j \geq Ase_j \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.10)$$

Where,

A_j = Irrigated area of j^{th} crop, ha

Ase_j = lower limit of area under particular crops according to socio-economic needs in ha of j^{th} crop

5.5.10 Non – Negativity Constraint

Area under a crop, A_j, surface water releases, SW_i and ground water drawal GW_i cannot be negative quantities.

$$A_j \geq 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.11)$$

$$SW_i \geq 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.12)$$

$$GW_i \geq 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.13)$$

5.6 FUZZY LINEAR PROGRAMMING MODEL

In describing the fuzzy linear programming, the concept of *satisfaction of criteria* is important. The criteria can either be the constraints or the objectives. In general, the objective is to find $\mathbf{X}$ that would satisfy the set of criteria or equations in Eq. 5.14.

$$-\mathbf{C}^T\mathbf{X} \leq -\mathbf{Z} \quad \ldots \quad (5.14)$$

$$\mathbf{AX} \leq \mathbf{b} \quad \ldots \quad (5.15)$$

$$\mathbf{X} \geq \mathbf{0} \quad \ldots \quad (5.16)$$

The value of $\mathbf{Z}$ represents the aspiration level of the objective function. The goal of fuzzy linear programming is to determine a solution that would satisfy the criteria mostly when uncertainty or fuzziness exists in Eq. 5.14.

There are two major streams in solving Eq. 5.14 using the fuzzy logic extension theory. The first represents the objective function and the constraints using fuzzy sets that will be aggregated to derive at a maximizing decision. The second describes the coefficients ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{Z}$, and $\mathbf{C}^T$) as fuzzy functions and solves the fuzzy functions using fuzzy logic extension theory. The first work has an advantage where the formulated problem can be solved using linear programming model. However, as the technique stresses on fuzziness in the inequality signs, the individual effect of the coefficients ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{Z}$, and $\mathbf{C}^T$) may not be properly accounted for. Since the second work describes the coefficients as fuzzy functions, the individual effects of the coefficients are taken into account during the decision making. However, the resulted formulation is nonlinear in the constraint matrix. To solve the problem, an iterative procedure is needed to search for the optimal solution.

5.7 PROCEDURE TO SOLVE FUZZY LINEAR PROGRAMMING

In many practical situations, it is not reasonable to require that the constraints or the objective function in linear programming problems be specified in precise crisp terms. In such situations it is desirable to use some type of fuzzy linear programming.

The most general type of fuzzy linear programming is formulated as follows

$$\text{Max} \sum_{j=1}^{n} C_j X_j \quad \ldots \quad (5.17)$$

$$\text{S.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \ (i \ \varepsilon \ N_m) \quad \ldots \quad (5.18)$$

$$X_j \geq 0 \ (j \ \varepsilon \ N_n) \quad \ldots \quad (5.19)$$

Where A_{ij}, B_i, C_j are fuzzy numbers, and X_j are variables whose states are fuzzy numbers ($i \ \varepsilon \ N_m$, $j \ \varepsilon \ N_n$); the addition and multiplication are operations of fuzzy arithmetic, and ≤ denotes the ordering of fuzzy numbers. Instead of discussing this general type, one can exemplify the issues involved by the following case of fuzzy linear programming problems.

<u>Case</u>: - Fuzzy linear programming problems in which only the right hand side numbers B_i are fuzzy numbers

$$\text{Max} \sum_{j=1}^{n} C_j X_j \quad \ldots \quad (5.20)$$

$$\text{S.t.} \sum_{j=1}^{n} A_{ij} X_j \leq B_i \ (i \ \varepsilon \ N_m) \quad \ldots \quad (5.21)$$

$$X_j \geq 0 \ (j \ \varepsilon \ N_n) \quad \ldots \quad (5.22)$$

In general, fuzzy linear programming problems are first converted into equivalent crisp linear or nonlinear problems, which are then solved by standard methods. The final results of a fuzzy linear programming problem are thus real numbers, which represent a compromise in terms of the fuzzy numbers involved.

One can discuss fuzzy linear programming problems as shown in figure 5.1. In this case, fuzzy numbers B_i (i ε N_m) typically have the form where x ε R

For each vector x = (x_1, x_2, ..., x_n), first calculate the degree, D_i (x), to which x satisfies the i^{th} constraint (i ε N_m) by the formula

$$D_i(x) = B_i \left(\sum_{j=1}^{n} A_{ij} X_j \right) \qquad \ldots \quad \ldots \quad \ldots \qquad (5.23)$$

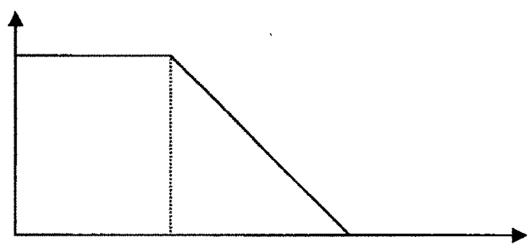
These degrees are fuzzy sets on R^n, and their intersection $\bigcap_{i=1}^{m} D_i$ is a *fuzzy feasible set*

Next, one can determine the fuzzy set of optimal values, by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values, Z_l, is obtained by solving the standard linear programming problem

$$\text{Max } Z = c\,x \qquad \ldots \quad \ldots \quad \ldots \qquad (5.24)$$

$$\text{S.t. } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \ (i \ \varepsilon \ N_m) \qquad \ldots \quad \ldots \quad \ldots \qquad (5.25)$$

$$x_j \geq 0 \ (j \ \varepsilon \ N_n) \qquad \ldots \quad \ldots \quad \ldots \qquad (5.26)$$



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0 bi b_i+p_i

Fuzzy numbers

Figure 5.1: Type of fuzzy numbers employed in fuzzy programming problem

The upper bound of the values, Z_n, is obtained by a similar linear programming problem in which each b_i is replaced with $b_i + p_i$:

$$\text{Max } Z = c\,x \qquad \ldots \quad \ldots \quad \ldots \quad (5.27)$$

$$\text{S.t. } \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i \ (i\ \varepsilon\ N_m) \qquad \ldots \quad \ldots \quad \ldots \quad (5.28)$$

$$x_j \geq 0\ (j\ \varepsilon\ N_n) \qquad \ldots \quad \ldots \quad \ldots \quad (5.29)$$

Then, the fuzzy set of optimal values, G, which is a fuzzy subset of R^n, is defined by

$$G(x) = \frac{cx - z_l}{z_u - z_l} \qquad \text{When } z_l < cx < z_l \qquad \ldots \quad \ldots \quad \ldots \quad (5.30)$$

$$= 1 \qquad \text{When } z_u \leq cx \qquad \ldots \quad \ldots \quad \ldots \quad (5.31)$$

$$= 0 \qquad \text{When } cx \leq z_l \qquad \ldots \quad \ldots \quad \ldots \quad (5.32)$$

Now, the problem becomes the following classical optimization problem

$$\text{Max } \lambda \qquad \ldots \quad \ldots \quad \ldots \quad (5.33)$$

$$\text{S.t } \lambda(Z_u - Z_l) - CX \leq - Z_l \qquad \ldots \quad \ldots \quad \ldots \quad (5.34)$$

$$\lambda\, p_i + \sum_{i=1}^{n} a_{ij}\, x_j \leq b_i + p_i \qquad \ldots \quad \ldots \quad \ldots \quad (5.35)$$

$$\lambda,\ x_j \geq 0\ (j\ \varepsilon\ N_n) \qquad \ldots \quad \ldots \quad \ldots \quad (5.36)$$

The above problem is actually a problem of finding $x\ \varepsilon\ R^n$ such that

$$[(\bigcap_{i=1}^{m} D_i) \cap G]\,(x) \qquad \ldots \quad \ldots \quad \ldots \quad (5.37)$$

Reaches the maximum value, that is, a problem of finding a point that satisfies the constraints and goal with the maximum degree. The credit for this idea goes to Bellman and Zadeh (1970). The method employed here is called a *symmetric method* (i.e., the constraints and the goal are treated symmetrically). There are also nonsymmetric methods.