CHAPTER II

SHADOWING PROPERTY ON G–SPACES

Studying the available literature, it appeared to us that the shadowing property which is one of the very useful dynamical property of continuous self–maps on metric spaces has not been defined and studied on metric G–spaces. Analyzing the definition of shadowing property on metric spaces, in this chapter we introduce and study this notion for continuous self maps on metric G–spaces. Let X be a metric space with metric d and let $f: X \to X$ be a continuous map. Recall that for a positive real number δ, a sequence of points $\{x_i : a<i<b\}$ is said to be a *δ–pseudo orbit* for f if for each i, $a<i<b-1$, $d(f(x_i), x_{i+1}) < \delta$. If a topological group G acts trivially on X i.e. $g\,x = x$, for each $g \in G$ then this means for each i, there exists a $g_i \in G$ such that $d\,(g_i\, f(x_i), x_{i+1}) < \delta$ (here $g_i\, f(x_i) = f(x_i)$) i.e. x_{i+1} may not be δ–close to $f(x_i)$ but it is δ–close to some point in the G–orbit of $f(x_i)$. Similarly for $\varepsilon > 0$, δ–pseudo orbit $\{x_i : a<i<b\}$ is said to be *ε–traced* by a point x of X if for each i, $a<i<b$, $d(f^i(x), x_i) < \varepsilon$, where $-\infty \le a < b \le \infty$, if f is bijective otherwise $0 \le a < b \le \infty$ Under the trivial action of G on X this condition means there exists $g_i \in G$, such that $d(f^i(x), g_i\, x_i) < \varepsilon$ i.e. $f^i(x)$ is ε–close to some point in the G–orbit of x_i, for each i. The above observations motivate us to define and study the above notions in G–setting.

In 2003, Pilyguin and Tikhomirov [35] have studied the concept of shadowing property of continuous actions of some abelian groups like Z^p and $Z^p \times R^p$ on a metric space X. Their concept does not involve self–maps on X whereas our notion of shadowing is defined and studied for continuous self–maps on metric G–spaces.

In Section 1 we define the notion of shadowing property for a continuous map f on a metric G–space X and term it as the G–shadowing property for f. Under the trivial action of G on X the notions of shadowing property and G–shadowing property for f coincide. However, examples provided justify that both notions for f are independent under a non – trivial action of G on X. It is also observed through examples that the notion of G–shadowing property depends on the action of the group G. In Section 2, we study several properties of maps possessing G–shadowing property and give necessary examples to strengthen the hypothesis. In Section 3, we obtain a characterization for the identity map on a compact metric G–space to possess the G–shadowing property. In Section 4, we obtain conditions under which G–shadowing property of a map f on a metric G–space X implies the shadowing property of the induced map $\hat{f}$ on the orbit space X/G and vice versa.

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1. G– shadowing property: Definitions and Examples.

In this section, we define the notion of G–shadowing property for a continuous map f on a metric G–space and study various examples.

Definitions 2.1.1. Let (X, d) be a metric G–space and $f: X \to X$ be a continuous map.

(a) For a positive real number δ, a sequence of points $\{x_i : a<i<b\}$ in X is said to be *$\delta - G$ pseudo orbit* for f if for each i, $a<i<b-1$, there exists a $g_i \in G$ such that $d\,(g_i f(x_i), x_{i+1})<\delta$.

(b) For a given $\varepsilon > 0$, a $\delta - G$ pseudo orbit $\{x_i : a<i<b\}$ for f is said to *ε–traced* by a point x of X if for each i, $a<i<b$, there exists a $p_i \in G$ such that $d(f^i(x), p_i x_i)<\varepsilon$.

Note that if f is bijective we take $-\infty \le a<b \le \infty$, other wise $0 \le a<b \le \infty$.

(c) Map f is said to have the *G–shadowing property* (termed as the *G–pseudo orbit tracing property* in **[42]**) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\delta - G$ pseudo orbit for f is ε–traced by a point of X.

Examples 2.1.2. (a) Consider the closed unit interval I=[0, 1] of the real numbers with the usual metric. Let $G = \mathbb{Z}_2$, the additive group of integers modulo 2, act on I by the action defined by $1x = x$ and $-1x = 1-x$, for each $x \in \mathrm{I}$. Then the map $f: \mathrm{I} \to \mathrm{I}$ defined by $f(x) = \frac{x}{2}$ has the shadowing property as well as $\mathbb{Z}_2$–shadowing property.

2.1.2. (b) Consider the subspace $X=\left\{\frac{1}{n},1-\frac{1}{n}:n\in \mathbb{N}\right\}$ of the usual metric space R of real numbers. Let $G=\mathbb{Z}_2$ act on X by the action $1x=x$ and $-1x=1-x$, for each $x\in X$. Then the natural left shift f on X fixing 0 and 1 has shadowing property as well as $\mathbb{Z}_2$–shadowing property.

2.1.2. (c) Consider the subspace $X=\left\{\pm\frac{1}{n},\pm\left(1-\frac{1}{n}\right):n\in \mathbb{N}\right\}$ of R. For $x\in X$, let x_+ denote the element of X which is immediately right to x and x_- that element of X which is immediately left to x. Suppose $G=\mathbb{Z}_2$ acts on X by the action $1x=x$ and $-1x=-x$, for each $x\in X$. Consider the map $f:X\to X$ defined by $f(x)=\begin{cases} x, & \text{if } x\in\{-1,0,1\} \\ x_+, & \text{if } x<0 \\ x_-, & \text{if } x>0 \end{cases}$.

We first observe that g_i in the definition of $\delta-G$pseudo orbit for f need not be the identity element of the group. For $\delta=\frac{1}{10}$, consider the sequence $\theta=\{x_i:i\geq 0\}=\left\{-\frac{1}{4},\frac{1}{4},-\frac{1}{5},\frac{1}{5},-\frac{1}{6},\frac{1}{6},.........\right\}$. Then θ is not a δ–pseudo orbit for f. For $i=0$, $|f(x_0)-x_1|=\left|-\frac{1}{5}-\frac{1}{4}\right|=\frac{9}{20}\not<\frac{1}{10}$. Now, $\mathbb{Z}_2(f(x_0))=\left\{-\frac{1}{5},\frac{1}{5}\right\}$. There is $g_i=-1\in\mathbb{Z}_2$ such that $|-1f(x_0)-x_1|=\left|\frac{1}{5}-\frac{1}{4}\right|=\frac{1}{20}<\frac{1}{10}$. Next, we show that f does not have the shadowing property. Let ε be such that $0<\varepsilon<\frac{1}{5}$. For $\delta>0$, choose the δ–pseudo orbit θ for f as follows: Choose

$x_0, x_1, \ldots\ldots, x_{n-1}$ such that $-1 \le x_i < 0$, for each $i \in \{0,1,2,\ldots n-1\}$, satisfying $|0 - x_i| > \varepsilon$ for some i. Take x_n=0. Further, choose $x_{n+k} \ge 0$ for each $k \in \{1,2,\ldots\ldots\}$ and satisfying $|0 - x_{n+k}| > \varepsilon$, for some k. Then θ is not ε–traced by any point of X. For observe that every element in the f–orbit of a negative point is negative whereas every element in the f–orbit of a positive point is positive. But elements of θ are both negative as well as positive. Also there are elements in θ, both negative and positive which are at a distance greater than ε from 0. Therefore θ cannot be ε–traced by any point of X. Further, we show that f has the Z_2–shadowing property. Observe that if $X_1 = \left\{\frac{1}{n}, 1 - \frac{1}{n} : n \in \mathbf{N}\right\}$, then Example 2.1.2(b) shows $f_{|X_1}$ has the shadowing property. Therefore for given $\varepsilon > 0$ there is $\delta > 0$ such that every δ–pseudo orbit in X_1 is ε–traced by a point of X_1. Let θ be a $\delta - Z_2$ pseudo orbit for f. Then for each $i \ge 0$, there exists $g_i \in Z_2$ such that $|g_i f(x_i) - x_{i+1}| < \delta$. If for some $j+1$, x_{j+1}, is such that $-1 \le x_{j+1} < 0$, then we replace x_{j+1} by $-1x_{j+1}$ and obtain a new δ–pseudo orbit $\alpha = \{x_i' \ : i \ge 0\}$ in X_1. Since $f_{|X_1}$ has the shadowing property α is ε–traced by a point of X_1, say, x. This implies for each $i \ge 0$, $|f^i(x) - x_i'| < \varepsilon$. But this further implies $|f^i(x) - (-1x_i)| < \varepsilon$. Therefore θ is ε–traced by the point x of X. Therefore f has the Z_2-shadowing property. Consider the homeomorphism $h : X \to X$ defined by

$$h(x)=\begin{cases} x, & \text{if } x\in\{-1,0,1\} \\ -x_{-}, & \text{if } x<0 \\ -x_{+}, & \text{if } x>0 \end{cases}.$$

and natural action of $G_1=\{h^n : n\in Z\}$ on X defined by $h^n x = h^n(x)$. Then the orbit space $X/G_1=\{G_1(0), G_1(1), G_1(-1), G_1(\frac{1}{2}), G_1(\frac{1}{3})\}$ and f has the G_1–shadowing property. Thus a map may have the G–shadowing property with respect to more than one group.

2.1.2. (d) Consider the subspace $X_1=\left\{\pm\frac{1}{n}, \pm\left(1-\frac{1}{n}\right) : n\in\mathbf{N}\right\}$ of R. Suppose G=Z$_2$ acts on X by the action $1x=x$ and $-1x=-x$, for all $x\in X$. Consider the map $f_1 : X_1 \to X_1$ defined by

$$f_1(x)=\begin{cases} x, & \text{if } x\in\{-1,0,1\} \\ x_{+}, & \text{if } x<0 \\ x_{-}, & \text{if } x>0 \end{cases}$$

Let $X_2=X_1, f_2=f_1$. Then by example 2.1.2(c) both $f_1=f_2$ has Z$_2$–shadowing property. But the product map $f_1\times f_2$ does not have the Z$_2$–shadowing property. By Theorem 2.2.8, $f_1\times f_2$ has the Z$_2\times$Z$_2$–shadowing property.

2.1.2. (e) Consider the unit circle S^1 of the plane and suppose group $G\equiv U_4$ of fourth roots of unity acts on S^1 by the usual action of complex multiplication. Define map $f:S^1\to S^1$ by $f(z)=z^2=e^{2i\theta}$, $z=e^{i\theta}\in S^1$. Then we show that f does not have the G-shadowing property. Observe that for $z=e^{i\theta}$, $G(e^{i\theta})=\left\{e^{i\theta}, e^{i(\theta+\frac{\pi}{2})}, e^{i(\theta+\pi)}, e^{i(\theta+\frac{3\pi}{2})}\right\}$ and $f^k(z)=e^{i2^k\theta}$. Let ε be such that $0<\varepsilon<d(e^{i0}, e^{\frac{i\pi}{4}})$. In order to show that f does not have the G-shadowing property we show that for every $\delta>0$ there is a $\delta-G$ pseudo orbit $\{z_i : i\geq 0\}$ which is not $\varepsilon-$traced by any point of S^1. Take $p_0=(1,0)$. Choose $p_1\in U_\delta((1,0))$ such that $f^k(p_i)\in U_\delta\left(e^{\frac{\pi}{2}i}\right)$, for some k. Consider the sequence $\{z_i : i\geq 0\}=\{p_0, p_1, f(p_1),\ldots\ldots, f^{k-1}(p_1), p_0, p_1, f(p_1),\ldots\ldots\}$, i.e. for each $n\geq 0$, $z_{kn}=p_0$ and $z_{kn+j}=f^{j-1}(p_1)$, $1\leq i\leq k-1$. Then $\{z_i : i\geq 0\}$ is a $\delta-G$ pseudo orbit for f. For when $i=0$, $d(f(z_0),z_1)=d(f(p_0),p_1)<\delta$, for $1\leq i\leq k-2$, $d(f(z_i),z_{i+1})=d(f^i(p_1),f^i(p_1))=0<\delta$ and for $i=k-1$, $d(f(z_i),z_{i+1})=d(e^{\frac{3\pi}{2}i}f^k(p_1),z_{k+1})<\delta$ as $f^k(p_1)\in U_\delta(e^{\frac{\pi}{2}i})$ will imply $e^{\frac{3\pi}{2}i}f^k(p_1)\in U_\delta(e^{0i})$. Thus for each $i\geq 0$ there is $g_i\in G$ such that $d(g_i f(z_i),z_{i+1})<\delta$. Therefore $\{z_i : i\geq 0\}$ is a $\delta-G$ pseudo orbit for f. We complete the proof by showing that $\{z_i : i\geq 0\}$ is not $\varepsilon-$traced by any point of S^1. Obviously, $\{z_i | i\geq 0\}$ is not $\varepsilon-$traced by any point of $S^1-U_\varepsilon((1,0))$, because if $z\in S^1-U_\varepsilon((1,0))$, $d((1,0),z)>\varepsilon$, i.e. $d(f^0(z),z_0)>\varepsilon$. Now for $z\in U_\varepsilon((1,0))$ there exists m such that $f^m(z)$ is at a distance greater than ε from $e^{\frac{\pi}{2}i}$. But for any i, z_i lies on the arc between $(1,0)$ and $(0,1)$. Therefore $\{z_i : i\geq 0\}$ is $\varepsilon-$traced by any point of S^1. Therefore f does not have the $G-$shadowing property. Since f is positively expansive open map, f has the shadowing property by Theorem 1.13.

2.1.2. (f) For each $n \in \mathbb{N}$, let X_n denote the circle centered at origin and of radius $\frac{1}{n}$. Consider the subspace $X = \bigcup_{n=1}^{\infty} X_n \cup \{(0, 0)\}$ of the plane and let $f : X \to X$ be the identity map. Suppose the group $G = SO(2)$ acts on X by the usual action of matrix multiplication. For a given $\varepsilon > 0$, choose δ such that $0 < \delta < \min\{\frac{\varepsilon}{2}, \frac{1}{6}\}$. Let $\theta = \{z_i : i \in \mathbf{Z}\}$ be a δ-G pseudo orbit for f. Then by the choice of δ either for each $i \in \mathbf{Z}$, $z_i \in X_m$, for some $m \in \mathbf{N}$, $m \leq 6$ or $z_i \in X_m$ and $z_j \in X_k$ for some $m, k \in \mathbf{N}$, $m \neq k, m, k \geq 6$. But in each case θ is ε-traced by the point $x \in X_m$. Therefore f has the G-shadowing property. Observe that space X is compact and the orbit space X/G of X is totally disconnected.

Remark 2.1.3. (i) In each of the examples 2.1.2(a) and (b) the map has both, shadowing property as well as the G–shadowing property.

(ii) In example 2.1.2(c) the map has the G–shadowing property with respect to two different groups. Also it does not have the shadowing property.

(iii) In example 2.1.2(d) the map has the G–shadowing property with respect to one group but does not have with respect to another group. Thus the notion of G–shadowing property depends on the action of G.

(iv) In example 2.1.2(e) the map has the shadowing property but does not have the G–shadowing property.

(v) Example 2.1.2(c) and (e) justify that the notions of shadowing property and G–shadowing property are independent of each other.

(vi) In example 2.1.2(f) one can analyze conditions under which the identity map has the $G-$ shadowing property.

The following result gives a class of maps on a metric G–space X having the G–shadowing property.

Proposition 2.1.4. *Let (X, d) be a metric G–space, where G is compact and d is an invariant metric on X. Then a pseudoequivariant contraction map f on X has the G–shadowing property.*

Proof. Since f is a contraction map, there exists $0<c<1$ such that $d(f(x), f(y)) \le c\, d(x, y)$, for all $x, y \in X$. Let $\varepsilon > 0$ be given. Choose δ such that $0 < \frac{\delta}{1-c} < \frac{\varepsilon}{2}$. We show that every $\delta-G$pseudo orbit $\theta = \{x_i : i \ge 0\}$ for f is ε–traced by a point of X. Infact, we show that θ is ε–traced by the point x_0 i.e. for each i, $i \ge 0$, there exists $p_i \in G$ such that

$$d(f^i(x_0), p_i\, x_i) < \varepsilon \qquad \textbf{(I)}$$

For $i=0$, $d(f^0(x_0), x_0) = 0 < \varepsilon$. Therefore **(I)** holds for $p_0 = e \in G$, where e is the identity in G. Since θ is a $\delta-G$pseudo orbit for each i, $i \ge 0$, there exists $g_i \in G$ such that

$$d(g_i\, f(x_i),\, x_{i+1}) < \delta \qquad \textbf{(II)}$$

For $i=1$ from **(II)**, $d(g_0\, f(x_0),\, x_1) < \delta$. Therefore by invariancy of metric d there is $p_1 = g_0^{-1} \in G$ such that $d(f(x_0), p_1\, x_1) < \delta < \frac{\varepsilon}{2} < \varepsilon$. Therefore **(I)** holds for $i=1$. By **(II)**,

$$d(g_1\, f(x_1),\, x_2) < \delta \Rightarrow d(f(x_1),\, g_1^{-1} x_2) < \delta$$

$$\Rightarrow d(f(p_1 x_1), p_2 x_2) < \delta ,$$

where $p_2 = p_1 g_1^{-1} \in G$. Note that

$$\begin{aligned} d(f^2(x_0), p_2 x_2) &\leq d(f^2(x_0), f(p_1 x_1)) + d(f(p_1 x_1), p_2 x_2) \\ &\leq c\, d(f(x_0), p_1 x_1) + d(f(p_1 x_1), p_2 x_2) \\ &< c\,\delta + \delta = \delta(c+1) < \varepsilon \end{aligned}$$

Therefore (I) holds for $i = 2$. Using (II) for $i = 2$, we obtain $d(g_2\, f(x_2), x_3) < \delta \Rightarrow d(f(x_2), g_2^{-1} x_3) < \delta \Rightarrow d(f(p_2 x_2), p_3 x_3) < \delta$, where $p_3 = p_2 g_2^{-1} \in G$. Therefore,

$$\begin{aligned} d(f^3(x_0), p_3 x_3) &\leq d(f^3(x_0), f(p_2 x_2)) + d(f(p_2 x_2), p_3 x_3) \\ &\leq c\, d(f^2(x_0), p_2 x_2) + d(f(p_2 x_2), p_3 x_3) \\ &< c\,(c\delta + \delta) + \delta = \delta(c^2 + c + 1) < \varepsilon \end{aligned}$$

Hence (I) holds for $i = 3$. Assume that for some $p_k \in G$, $d(f^k(x_0), p_k x_k) < \delta\,(c^{k-1} + c^{k-2} + \ldots\ldots + c + 1) < \varepsilon$. Again by (II), $d(g_k\, f(x_k), x_{k+1}) < \delta \Rightarrow d(f(p_k x_k), p_{k+1} x_{k+1}) < \delta$, where $p_{k+1} = p_k g_k^{-1} \in G$.

Consider,

$$\begin{aligned} d(f^{k+1}(x_0), p_{k+1} x_{k+1}) &\leq d(f^{k+1}(x_0), f(p_k x_k)) + d(f(p_k x_k), p_{k+1} x_{k+1}) \\ &\leq c\, d(f^k(x_0), p_k x_k) + d(f(p_k x_k), p_{k+1} x_{k+1}) \\ &< c\,\delta(c^{k-1} + c^{k-2} + \ldots + c + 1) + \delta < \varepsilon \end{aligned}$$

Therefore (I) holds for $i = k+1$, whenever it holds for $i = k$. Hence by the Principle of Mathematical Induction (I) holds for each $i, i \geq 0$. Since θ is an arbitrary $\delta - G$ pseudo orbit for f, it follows that every $\delta - G$ pseudo orbit for f is ε-traced by a point of X. Therefore f has the G-shadowing property

We recall here that a contraction map on a metric space has the shadowing property.

2. Properties of maps possessing the G–shadowing property .

We now observe some properties of the maps which are G–shadowing maps and provide necessary examples to strengthen the hypothesis. The following result shows that the notion of G–shadowing property is independent of the choice of metric for X, compatible with the topology of X if the space is compact.

Theorem 2.2.1. *Let X be a compact metric G–space and let d and d_1 be two equivalent metrics on X. If a continuous map $f:X\to X$ has the G–shadowing property with respect to metric d then f has the G–shadowing property with respect to d_1.*

Proof. Let $\varepsilon>0$ be given. Since d and d_1 are equivalent metrics, there is an $\eta>0$ such that for each $x\in X$, $U_\eta^d(x)\subset U_\varepsilon^{d_1}(x)$, where $U_\varepsilon^{d_1}(x)$ denotes the open ball centered at x and of radius ε under the metric d_1. Note that compactness of X guarantees that choice of η is independent of the choice of x. G–shadowing property of f with respect to metric d implies there is $\delta>0$ such that every $\delta-G$pseudo orbit for f is η–traced by a point of X. Again, equivalency of metrics d and d_1 implies there is a $\gamma>0$ such that for each $x\in X$, $U_\gamma^{d_1}(x)\subset U_\delta^d(x)$. In order to show that f has the G–shadowing property with respect to metric d_1 we show that every $\gamma-G$ pseudo orbit for f with respect to metric d_1 is ε–traced by a point of X. Let $\theta=\{x_i : i\geq 0\}$ be a $\gamma-G$ pseudo orbit for f with respect to metric d_1. Then for each i, $i\geq 0$, there exits $g_i\in G$ such that

$$\begin{aligned}
&d_1(g_i\ f(x_i),\ x_{i+1})<\gamma \\
&\Rightarrow g_i f(x_i)\in U_\gamma^{d_1}(x_{i+1})\subset U_\delta^d(x_{i+1}) \\
&\Rightarrow d(g_i\ f(x_i),\ x_{i+1})<\delta
\end{aligned}$$

Therefore θ is a $\delta-G$ pseudo orbit for f with respect to metric d. But f has the G–shadowing property with respect to metric d. Hence θ is η–traced by a point of X, say , x. This implies for each i, $i\geq 0$, there exists $p_i\in G$ such that

$$\begin{aligned}
&d(f^i(x_0),\ p_i\ x_i)<\eta \\
&\Rightarrow f^i(x)\in U_\eta^d(p_i x_i)\subset U_\varepsilon^{d_1}(p_i x_i) \\
&\Rightarrow d_1(f^i(x_0),\ p_i\ x_i)<\varepsilon
\end{aligned}$$

Therefore θ is ε–traced by the point x_0 of X. Thus, f has the G–shadowing property with respect to metric d_1.

Consider the map f defined on R, the usual space of real numbers, by $f(x)=\frac{x}{2}$. Then f being contraction map, f has the Z_2–shadowing property. Also its inverse has the Z_2–shadowing property. But the composition of f with its inverse which is the identity on R does not have the Z_2–shadowing property. The following result gives condition under which self composition of maps possess the G–shadowing property if the map has the G–shadowing property.

Theorem 2.2.2. *Let X be a metric G–space and $f: X \to X$ be a continuous map. If f has the G–shadowing property then f^k has the G–shadowing property for each $k > 0$.*

Proof. Choose $k > 0$ and fix it. Let $\varepsilon > 0$ be given. Since f has the G–shadowing property, there exists a $\delta > 0$ such that every $\delta - G$pseudo orbit for f is ε–traced by a point of X. In order to show that f^k has the G–shadowing property we show that every $\delta - G$pseudo orbit $\theta = \{x_i : i \geq 0\}$ for f^k is ε–traced by a point of X. Since θ is a $\delta - G$pseudo orbit for f^k, for each i, $i \geq 0$, there exists $g_i \in G$ such that $d(g_i\, f^k(x_i),\, x_{i+1}) < \delta$. Consider $\{y_i : i \geq 0\} = \{x_0, f(x_0), \ldots, f^{k-1}(x_0), x_1, \ldots, f^{k-1}(x_n), \ldots\}$ i.e. $y_{kn+j} = f^j(x_n)$, $0 \leq j \leq k-1, n \geq 0$. We show that $\{y_i : i \geq 0\}$ is a $\delta - G$pseudo orbit for f. For $n \geq 0, 0 \leq j \leq k-2$, $i = kn + j$ and $e \in G$,

$$d(e\, f(y_{kn+j}),\, y_{kn+j+1}) = d(f^{j+1}(x_n),\, f^{j+1}(x_n)) = 0 < \delta .$$

If $i = kn + k - 1$, then $d(g_n f(y_i),\, y_{i+1}) = d(g_n\, f^k(x_n),\, x_{n+1}) < \delta$. Thus $\{y_i : i \geq 0\}$ is a $\delta - G$pseudo orbit for f. But f has the G–shadowing property, therefore $\{y_i : i \geq 0\}$ is ε–traced by a point of X, say, x. This implies that for each i, $i \geq 0$, there exists $p_i \in G$ such that $d(f^i(x),\, p_i\, x_i) < \varepsilon$. But $y_{kn} = x_n$, for each $n \geq 0$. Therefore, for each $n \geq 0$, there is $p_{kn} \in G$ such that

$d(f^{kn}(x),\ p_{kn}x_n) < \varepsilon$. Hence θ is ε–traced by the point x of X and thus f^k has the G–shadowing property.

In the following result we relate the G–shadowing property of a homeomorphism f with its inverse.

Theorem 2.2.3. *Let X be a metric G–space, where G is compact and d be an invariant metric. If a uniformly continuous pseudoequivariant homeomorphism $f : X \to X$ has the G–shadowing property then so does its inverse f^{-1}.*

Proof. Let $\varepsilon > 0$ be given. Since f has the G–shadowing property, there is a $\delta > 0$ such that every $\delta - G$ pseudo orbit for f is ε–traced by a point of X. Uniform continuity of f implies there is an $\eta > 0$ such that $d(x, y) < \eta \Rightarrow d(f(x), f(y)) < \delta$. In order to show that f^{-1} has the G–shadowing property, we show that every $\eta - G$ pseudo orbit for f^{-1} is ε–traced by a point of X. Let $\theta = \{y_i : i \in \mathbb{Z}\}$ be an $\eta - G$ pseudo orbit for f^{-1}. Then for each i, $i \in \mathbb{Z}$, there is a $g_i \in G$ such that

$$d(g_i\, f^{-1}(y_i),\ y_{i+1}) < \eta$$
$$\Rightarrow d(g_{-i}\, f^{-1}(y_{-i}),\ y_{-i+1}) < \eta \quad \text{(replacing } i \text{ by } -i\text{)}$$

Put $x_i = y_{-i}$, for each $i \in \mathbb{Z}$. Therefore, for each i, $i \in \mathbb{Z}$, there is $g_{-i} = g_i' \in G$ such that

$$d(g_i'\, f^{-1}(x_i),\ x_{i-1}) < \eta$$
$$\Rightarrow d(x_i,\ g_i'^{-1} f(x_{i-1})) < \delta$$

Thus $\{x_i : i \in Z\}$ is a $\delta-G$pseudo orbit for f. But f has the G–shadowing property, therefore $\{x_i : i \in Z\}$ is ε–traced by a point of X, say, x. This implies that for each i, $i \in Z$, there is a $p_i \in G$ such that

$$d(f^i(x), p_i x_i) < \varepsilon$$

$\Rightarrow d((f^{-1})^i(x), p_{-i} y_i) < \varepsilon$, for some $p_{-i} \in G$ (by replacing i by $-i$)

Thus θ is ε–traced by the point x of X. Since θ is an arbitrary $\eta-G$pseudo orbit for f^{-1} it follows that every $\eta-G$pseudo orbit for f^{-1} is ε–traced by a point of X. Therefore f^{-1} has the G–shadowing property.

We now find some condition under which the map f has the G–shadowing property whenever f^k has the G–shadowing property, for some $k > 0$. We first observe the following lemma.

Theorem 2.2.4. *Let X be a metric G–space, where G is compact and d is an invariant metric on X. If $f : X \to X$ is a uniformly continuous pseudoequivariant map and $k \in N$, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that each finite $\delta-G$ pseudo orbit $\theta = \{x_i : 0 \le i \le k\}$ for f is ε–traced by x_0 i.e. for each i, $0 \le i \le k$, there is a $p_i \in G$ such that*

$$d(f^i(x_0), p_i x_i) < \varepsilon \qquad (*)$$

Proof. Let $\theta = \{x_i : 0 \le i \le k\}$ be a finite $\delta-G$pseudo orbit for f. We prove the result by applying the Principle of Mathematical Induction on k. Let $k = 1$, then $i = 0, 1$. For a given $\varepsilon > 0$, choose δ such that $0 < \delta < \varepsilon$. Then

$\delta - G$pseudo orbit $\theta = \{x_0, x_1\}$ satisfies (*) for $i = 0, 1$. Infact, for $i = 0$, $d(f^0(x_0), p_0 x_0) < \varepsilon$, where $p_0 = e$ and for $i = 1$ $d(f(x_0), p_1 x_1)$ = $d(g_0 f(x_0), x_1) < \delta < \varepsilon$, where $p_1 = g_0^{-1}$. Next, using uniformly continuity of f we find an $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f(x), f(y)) < \frac{\varepsilon}{2}$. We assume the result to be true for $k-1$. Thus there exists $\gamma > 0$ such that every $\gamma -$ Gpseudo orbit $\{y_i : 0 \le i \le k-1\}$ for f is η–traced by y_0. Consider a $\frac{\gamma}{2} - G$pseudo orbit $\theta = \{x_i : 0 \le i \le k\}$ for f. Then $\{x_i : 0 \le i \le k-1\}$ is also a $\frac{\gamma}{2} - G$pseudo orbit (hence $\gamma - G$pseudo orbit) for f. Thus for each i, $0 \le i \le k-1$, there is a $p_i \in G$ such that $d(f^i(x_0), p_i x_1) < \eta$. In particular for $i = k-1$, $d(f^{k-1}(x_0), p_{k-1} x_{k-1}) < \eta$ which implies $d(f^k(x_0), f(p_{k-1} x_{k-1})) < \frac{\varepsilon}{2}$, and therefore $d(p'_{k-1} f^k(x_0), f(x_{k-1})) < \frac{\varepsilon}{2}$, where $p'_{k-1} = p_{k-1}^{-1}$. Again, θ is a $\gamma -$ Gpseudo orbit for f. Therefore for $i = k-1$, there exists $g_k \in G$ such that $d(g_k x_k, f(x_{k-1})) < \frac{\varepsilon}{2}$. Finally observe that

$$d(p'_{k-1} f^k(x_0), g_k x_k) \le d(p'_{k-1} f^k(x_0), f(x_{k-1})) + d(g_k x_k, f(x_{k-1})) < \varepsilon.$$

Therefore, (*) holds for all i, $0 \le i \le k$. Thus result holds for k whenever it holds for $k-1$.

Theorem 2.2.5. *Let (X, d) be a metric G–space, where G is compact and d is an invariant metric on X and let $f : X \to X$ be a uniformly continuous pseudoequivariant map such that for each $i > 0$, f^i is uniformly continuous. Suppose for some $k > 0$, f^k has the G–shadowing property, then f has the G–shadowing property.*

Proof. Let $k > 0$ be such that f^k has the G–shadowing property and let $\varepsilon > 0$ be given. Then there is an η_1, $0 < \eta_1 < \frac{\varepsilon}{2}$ such that every $\eta_1 - G$ pseudo orbit $\{x_i : 0 \le i \le k\}$ for f satisfies: for every $i, 0 \le i \le k$, there exists $g_i \in G$ such that $d(f^i(x_0), p_i x_i) < \frac{\varepsilon}{2}$. Since each f^i, $0 \le i \le k$, is uniformly continuous there is $\eta_2 > 0$ such that $d(x, y) < \eta_2 \Rightarrow d(f^i(x), f^i(y)) < \frac{\varepsilon}{2}$. G–shadowing of f^k implies there is a $\tau > 0$ such that every $\tau - G$ pseudo orbit for f^k is η–traced by a point of X. Here η=min{η_1, η_2}. Let $\beta = \min\{\eta, \tau\}$. Then by Lemma 2.2.4 there is a $\delta > 0$ such that every finite $\delta - G$ pseudo orbit $\{z_i : i \ge 0\}$ for f is β–traced by z_0. In order to show that f has the G–shadowing property we show that every $\delta - G$ pseudo orbit for f is ε–traced by a point of X. Let $\theta = \{x_i : i \ge 0\}$ be a $\delta - G$ pseudo orbit for f. For $n \ge 0$, put $y_n = x_{kn}$ and for fixed n, consider a finite $\delta - G$ pseudo orbit $\{x_{kn+j} : 0 \le j \le k\}$. Then by Lemma 2.2.4 for each j, $0 \le j \le k$, there is a $p_{kn+j} \in G$ such that $d(f^j(x_{kn}), p_{kn+j} x_{kn+j}) < \tau$. In particular, for $j = k$, $d(f^k(x_{kn}), p_{kn+k} x_{kn+k}) < \tau$. Since n is arbitrary, this further implies that for each $n \ge 0$, there is $g_{kn+j} = p_{kn+j}^{-1} \in G$, such that $d(g_{kn+j} f^k(y_n), y_{n+1}) < \tau$.

Therefore $\{y_n : n \geq 0\}$ is a $\tau - G$ pseudo orbit for f^k. But f^k has the $G-$ shadowing property. Therefore $\{y_n : n \geq 0\}$ is η–traced by a point of X, say, y. Hence for each $n \geq 0$, there is $t_n \in G$ such that

$$d(f^n(y_n), t_n y_n) < \eta \qquad \textbf{(I)}$$

Again, consider finite $\delta - G$ pseudo orbit $\{x_{kn+j} : 0 \leq j \leq k\}$. Then $\{x_{kn+j} : 0 \leq j \leq k\}$ is β–traced (hence is η–traced) by the point x_{kn} of X. This implies for that each $j,\ 0 \leq j \leq k$, there is a $t'_{kn+j} \in G$ such that

$$d(f^j(x_{kn}), t'_{kn+j}\, x_{kn+j}) < \eta < \frac{\varepsilon}{2} \qquad \textbf{(II)}$$

Now, by **(I)** $d(f^{kn}(y_n), t_{kn}\, y_{kn}) < \eta$ implies that for each $j,\ 0 \leq j \leq k$, $d(f^{kn+j}(y), f^j(t_{kn}\, y_{kn})) < \frac{\varepsilon}{2}$. Therefore for $0 \leq j \leq k$, using invariancy of d, we get

$$d(t_{kn+j} f^{kn+j}(y), f^j(x_{kn})) < \frac{\varepsilon}{2}, \text{ for some } t_{kn+j} \in G \qquad \textbf{(III)}$$

Hence form **(I)**, **(II)** and **(III)**

$$d(t_{kn+j} f^{kn+j}(y),\ t'_{kn+j} x_{kn+j}) \leq d(t_{kn+j} f^{kn+j}(y), f^j(x_{kn})) +$$

$$d(f^j(x_{kn}), t'_{kn+j}\, x_{kn+j}) \ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus for $i \geq 0$, there is $p_i \in G$, such that $d(f^i(y), p_i x_i) < \varepsilon$. Therefore θ is ε–traced by the point y of X. Hence f has the G–shadowing property.

Following result gives the condition under with the conjugate maps has the G–shadowing property.

Theorem 2.2.6. *Let X, Y be compact metric G–spaces with metric d and ρ respectively and $h: X \to Y$ be a pseudoequivariant homeomorphism. Then a continuous map $f: X \to X$ has the G–shadowing property if and only if the $f_1 = hfh^{-1}: Y \to Y$ has the G–shadowing property.*

Proof. Suppose f has the G–shadowing property and let $\varepsilon > 0$ be given. Then by uniform continuity of h there is $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow \rho(h(x), h(y)) < \varepsilon \qquad \textbf{(I)}$$

By G–shadowing property of f, there is an $\eta > 0$ such that every $\eta - G$ pseudo orbit for f is δ–traced by a point of X. Also, h^{-1} is uniformly continuous. Therefore there is $\gamma > 0$ such that for all $y_1, y_2 \in Y$

$$\rho(y_1, y_2) < \gamma \Rightarrow d(h^{-1}(y_1), h^{-1}(y_2)) < \eta \qquad \textbf{(II)}$$

In order to show that f_1 has the G–shadowing property we show that every $\gamma - G$ pseudo orbit for f_1 is ε–traced by a point of Y. Let $\theta = \{y_i : i \geq 0\}$ be a $\gamma - G$ pseudo orbit for f_1. Then for each $i \geq 0$, there exists $u_i \in G$ satisfying

$$\begin{aligned} &\rho(u_i f_1(y_i), y_{i+1}) < \gamma \\ &\Rightarrow d(h^{-1}(u_i f_1(y_i)), h^{-1}(y_{i+1})) < \eta \\ &\Rightarrow d(u_i' f(h^{-1}(y_i)), h^{-1}(y_{i+1})) < \eta, \text{ for some } u_i' \in G \end{aligned}$$

Let $x_i = h^{-1}(y_i)$, for each $i \geq 0$. Then from (II) it follows that $\{x_i : i \geq 0\}$ is an $\eta - G$ pseudo orbit for f. Therefore $\{x_i : i \geq 0\}$ is δ–traced by a point of X, say, x. Hence for each $i \geq 0$, there is $p_i \in G$ such that $d(p_i x_i, f^i(x)) < \delta$, which implies $\rho(h(p_i x_i), hf^i(x)) < \varepsilon$, by (I). this further implies

$\rho(p_i' y_i, f_1^i(h(x))) < \varepsilon$, for some $p_i' \in G$. Therefore θ is ε–traced by the point $h(x)$. Since θ is an arbitrary $\gamma-G$pseudo orbit for f_1, it follows that every $\gamma-G$pseudo orbit for f_1, is ε–traced by a point of Y. Therefore f_1 has the G–shadowing property.

We observe that in the above theorem pseudoequivariancy of h is not a necessary condition.

Example 2.2.7. Consider the usual Z_2–space I with the usual metric. Define a map $h: I \to I$ by $h(x) = \sqrt{x}$. Suppose $p = \{h^n : n \in Z\}$ acts on I by the usual action. Then the map defined on I by $f(x) = \frac{x}{2}$ has the Z_2–shadowing property. Also $f_1 = hfh^{-1}$ given by $f_1(x) = \frac{x}{\sqrt{2}}$ has the Z_2–shadowing property. Observe that h is not a pseudoequivariant map.

In the following theorem we obtain the condition for the product of two maps possessing G–shadowing property to possess the G–shadowing property.

Theorem 2.2.8. *Let (X, d_1) and (Y, d_2) be metric G–spaces and $X \times Y$ the product space with metric $d((x_1, y_1), (x_2, y_1)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$. Suppose $f: X \to X$ and $h: Y \to Y$ are continuous maps. Then the product map $f \times h: X \times Y \to X \times Y$ defined by $(f \times h)(x, y) = (f(x), h(y))$; $(x, y) \in X \times Y$ has the $G \times G$–shadowing property if and only if each of f and h has the*

G–shadowing property. Here $G\times G$ acts on $X\times Y$ by the action $(g,k)(x,y)=(gx,ky)$.

Proof. Suppose f and h has the G–shadowing property. Let $\varepsilon>0$ be given Since f has the G–shadowing property, there is $\delta_1>0$ such that every δ_1-G pseudo orbit for f is $\frac{\varepsilon}{2}$–traced by a point of X. Similarly G–shadowing property of h implies there is a $\delta_2>0$ such that every δ_2-G pseudo orbit for h is $\frac{\varepsilon}{2}$–traced by a point of Y. Let $\delta=\min\{\delta_1,\delta_2\}$ Then in order to show that $f\times h$ has the $G\times G$–shadowing property we show that every δ–$(G\times G)$ pseudo orbit for $f\times h$ is ε–traced by a point of $X\times Y$. Let $\theta=\{z_i=(x_i,y_i):i\geq 0\}$ be a δ–$(G\times G)$ pseudo orbit for $f\times h$ in $X\times Y$. Then for each $i\geq 0$, there is $(t_i,g_i)\in G\times G$ such that

$$d((t_i,g_i)(f\times h)(z_i),z_{i+1}))<\delta$$
$$\Rightarrow d((t_i,g_i)(f(x_i),h(y_i),(x_{i+1},y_{i+1}))<\delta$$
$$\Rightarrow \max\{d_1(t_if(x_i),x_{i+1}),d_2(g_ih(y_i),y_{i+1})\}<\delta$$

Therefore $\{x_i:i\geq 0\}$ and $\{y_i:i\geq 0\}$ are δ_1-G pseudo orbit for f and δ_2-G pseudo orbit for h respectively. But each of f and h has the G–shadowing property, therefore $\{x_i:i\geq 0\}$ is $\frac{\varepsilon}{2}$–traced by a point of X, say, x and $\{y_i:i\geq 0\}$ is $\frac{\varepsilon}{2}$–traced by a point of Y, say, y. Hence for each $i\geq 0$, there exist $p_i,q_i\in G$ such that

$$d_1(f^i(x),p_ix_i)<\frac{\varepsilon}{2}\text{ and }d_2(h^i(y),q_iy_i)<\frac{\varepsilon}{2}$$

$$\Rightarrow \max\{d_1(f^i(x), p_ix_i), d_2(h^i(y), q_iy_i)\} < \frac{\varepsilon}{2}$$

$$\Rightarrow d((f^i(x), h^i(y), (p_ix, q_iy_i)\} < \frac{\varepsilon}{2}$$

Therefore θ is ε–traced by the point (x, y) of $X \times Y$ Since θ is an arbitrary $\delta - G \times G$pseudo orbit for $f \times h$, it follows that every $\delta - G \times G$pseudo orbit for $f \times h$ is ε–traced by a point of $X \times Y$. Therefore $f \times h$ has the $G \times G$–shadowing property.

For the converse part, one can easily verify that

(i) If $\{x_i : i \geq 0\}$ and $\{y_i : i \geq 0\}$ are $\delta - G$pseudo orbits for f and h respectively then $\theta = \{z_i = (x_i, y_i) : i \geq 0\}$ is a δ–$(G \times G)$ pseudo orbit for $f \times h$ in $X \times Y$.

(ii) If (x, y) ε-traces θ, then $\{x_i : i \geq 0\}$ is ε-traced by x and $\{y_i : i \geq 0\}$ is ε-traced by y.

Therefore each of f and h has the G–shadowing property.

In example 2.1.2(d) recall that the map has the $Z_2 \times Z_2$–shadowing property but does not have the Z_2–shadowing property. Thus if a group acts diagonally on the product of G–spaces then the product map need not have the G–shadowing property.

3. Characterization for the identity map to possess the G-shadowing property.

Recall the example 2.1.2(f) where the identity map has the G–shadowing. Observe that in that example the orbit space X/G of the space X is totally disconnected. This example helps us in finding the condition under which the identity map has the G-shadowing property. We first observe the following lemmas. The first lemma is proved in **[12]**.

Lemma 2.3.1. *Let X be a connected space, $a,b \in X$ and $\wp = \{U_\alpha : \alpha \in \Lambda\}$ be a family of open sets whose union is X. Then there is a simple chain with links form $\wp$ that connects a and b.*

Lemma 2.3.2. *Let $x,y \in X$, where X is a non–degenerate continuum. Then for a continuous map $f : X \to X$ and a $\delta > 0$, there exists a δ-pseudo orbit for f containing x,y in X.*

Proof. Let $\wp = \{U_1, U_2, ...U_n\}$ be a finite subcover of X with $diam U_i < \delta$, for each $i \in \{1,.....,n\}$. Since X is connected there is a simple chain which connects x and y. Take $x_0 = x$ and consider corresponding $f(x)$. Then by Lemma 3.3.1 there is a chain that connects x and $f(x)$. Take $x_1 \in U_k$, where U_k is that member of the chain between x and $f(x)$ for which $f(x) \in U_k$. Consider corresponding $f(x_1)$. Again there exists a chain connecting x_1 and $f(x_1)$. Continuing in this way we obtain a δ–pseudo orbit containing x and y. Since X is connected existence of such a δ–pseudo orbit is always guaranteed.

Theorem 2.3.3. *Let X be a compact metric G–space, where G is compact. Then the identity map f on X has the G–shadowing property if and only if the orbit space X/G of X is totally disconnected.*

Proof. Suppose X/G is totally disconnected. Then clopen sets form a basis for topology of X. Since G is compact we can consider an invariant metric d on X compatible with topology of X. Let $\varepsilon > 0$ be given and let $\{U_1, U_2, \ldots\ldots, U_n\}$ be a finite subcover of X/G consisting of clopen sets such that $U_i \cap U_j = \phi$ for $i \neq j$ and $diam U_i < \varepsilon$, for each i in $\{1, 2, \ldots\ldots, n\}$. Set $V_i = \pi^{-1}(U_i)$ for each i,. Since U_i is a closed subset of X/G and π is a continuous map, $V_i = \pi^{-1}(U_i)$,is a closed subset of compact space X and hence compact. Also, $U_i \cap U_j = \phi \Rightarrow \pi^{-1}(U_i) \cap \pi^{-1}(U_j) = \phi \Rightarrow V_i \cap V_j = \phi$. Let $\delta_{ij} = d(V_i, V_j)$ for $i \neq j$. Then V_i, V_j being compact implies $\delta_{ij} > 0$, for $i \neq j$. Choose δ such that $0 < \delta < \min\{\delta_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$. In order to show that the identity map f has the $G-$ shadowing property we show that every $\delta - G$pseudo orbit for f is ε–traced by a point of X. Let $\theta = \{x_i : i \in \mathbf{Z}\}$ be a $\delta - G$pseudo orbit for f. Then for each $i \in \mathrm{Z}$, there exists a $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$ i.e.

$$d(g_i x_i, x_{i+1}) < \delta \qquad (*)$$

Note that if $x_i \in V_k$, then $x_{i+1} \in V_k$. For if $x_{i+1} \in V_j, j \neq k$, then V_k being G–invariant $g_i x_i \in V_k$ and $x_{i+1} \in V_j \Rightarrow d(g_i x_i, x_{i+1}) \geq d(V_k, V_j) = \delta_{kj} > \delta$ a contradiction to (*). Similarly, if $x_i \in V_k$, then $x_{i-1} \in V_k$. For, if $x_{i-1} \in V_j, j \neq k$, then V_j being G–invariant $g_{i-1} x_{i-1} \in V_j$ and $x_{i+1} \in V_j$ implies $d(g_{i-1} x_{i-1}, x_i) \geq d(V_k, V_j) = \delta_{kj} > \delta$ a contradiction to (*). Therefore for each $i \in \mathrm{Z}$, $x_i \in V_k$. This further implies $G(x_i) \in U_k$. But $diam U_k < \varepsilon$, therefore for any $G(x) \in U_k$ and for any $i \in Z$, $d_1(G(x), G(x_i)) < \varepsilon$. Since G is compact therefore for given $i \in Z$, there exists a $l_i, m_i \in G$ such that $d(l_i x, m_i x_i) < \varepsilon$. Thus for each $i \in Z$ there exists a $p_i \in G$ such that $d(f^i(x), p_i x_i) < \varepsilon$. Hence θ is ε–traced by the point x of X. Since θ is an arbitrary $\delta - G$ pseudo orbit for f, it follow that every $\delta - G$ pseudo orbit for f is ε–traced by a point of X. Hence f has the G–shadowing property.

Conversely, suppose the identity map f on X has the G–shadowing property. Since X/G is compact, it is sufficient to show that $\dim(X/G) = 0$. If possible suppose $\dim(X/G) \neq 0$. Since $\dim(X/G) \geq 1$ therefore there exists a closed connected subset F in X/G whose dimension is at least one. Since X/G is compact, F is a compact subset of X/G. Therefore there exists $G(a) \neq G(b) \in F$ such that $diamF = d_1(G(a), G(b)) = r$, say. Compactness of G implies there is $y_1 \in G(a)$ and $y_2 \in G(b)$ such that $r = d(y_1, y_2)$. Let $\varepsilon = \frac{r}{3}$. We obtain a contradiction by showing that for a given $\delta > 0$ there is a $\delta - G$ pseudo orbit for f which is not ε–traced by any point of X. By Lemma 3.3.2 there is a $\delta - G$ pseudo orbit $\{x_i : i \in \mathbf{Z}\}$ for f in X containing y_1 and y_2. Such a $\delta - G$ pseudo orbit can be obtained as follows: Since F is a compact connected subset of X/G by Lemma 3.3.2. there is a δ–pseudo orbit $\{G(x_i) : i \in \mathbf{Z}\}$ for $\hat{f}$ containing $G(a)$ and $G(b)$. This implies for each $i \in Z$, $d_1(\hat{f}(G(x_i)), G(x_{i+1})) < \delta$. Compactness of G implies for each $i \in Z$ there are $l_i, m_i \in G$ such that $d(l_i f(x_i), m_i x_{i+1}) < \delta$ which implies

$d(g_i f(x_i), x_{i+1}) < \delta$, for some $g_i \in G$ and hence $\{x_i : i \in \mathbf{Z}\}$ is a $\delta-G$pseudo orbit for f. Now, $\{G(x_i) : i \in Z\}$ contains $G(a)$ and $G(b)$. Therefore for some $k, p \in Z$, $G(x_k) = G(a)$ and $G(x_p) = G(b)$. Also, $y_1 \in G(a)$ and $y_2 \in G(b)$ implies $g' y_1 = x_k$ and $g'' y_2 = x_p$, for some $g', g'' \in G$. We replace x_k by $g' y_1$ and x_p by $g'' y_2$ in $\{x_i : i \in \mathbf{Z}\}$ and continue to denote the new $\delta-G$pseudo orbit containing y_1 and y_2 by $\{x_i : i \in \mathbf{Z}\}$. Suppose $\{x_i : i \in \mathbf{Z}\}$ is $\varepsilon-$traced by the point x of X. Therefore for each $i \in Z$, there exists $p_i \in G$, such that

$$d(x, p_i x_i) = d(f^i(x), p_i x_i) < \varepsilon \qquad \textbf{(I)}$$

Since $\{x_i : i \in Z\}$ is a $\delta-G$pseudo orbit for f containing y_1 and y_2, there exists $k, n \in Z$ such that $x_k = y_1$ and $x_n = y_2$ Therefore by (I),

$$d(x, p_k x_k) < \varepsilon \text{ and } d(x, p_n x_n) < \varepsilon,$$

which implies

$$d_1(G(x), G(p_k x_k)) < \varepsilon \text{ and } d_1(G(x), G(p_n x_n)) < \varepsilon$$

and hence

$$d_1(G(a), G(b)) \leq d_1(G(a), G(x)) + d_1(G(x), G(b))$$
$$< \varepsilon + \varepsilon = \frac{2r}{3}$$

a contradiction. This proves $\dim(X/G) = 0$.

4. Characterization for a map to have the G–shadowing property.

We obtain condition under which G–shadowing property of f on metric G–space X implies the shadowing property of the induced map on X/G and vice versa. We first recall the definition of a covering map. Let X and Y be metric spaces. A continuous onto map $f:X \to Y$ is called a *covering map* , if for each $y \in Y$, there exists an open neighborhood V_y of y in Y such that $f^{-1}(V_y) = \bigcup_i U_i$ $(i \neq i' \Rightarrow U_i \cap U_{i'} = \phi$, where each U_i is open in X and $f_{|U_i} : U_i \to V_y$ is a homeomorphism.

Theorem 2.4.1. *Let X be a compact metric G–space and $f:X \to X$ be a pseudoequivariant map. Suppose the orbit map $\pi : X \to X/G$ is a covering map. Then f has the G–shadowing property if and only if the induced map $\hat{f} : X/G \to X/G$ has the shadowing property.*

Proof. Suppose $\hat{f}$ has the shadowing property. We show that f has the G–shadowing property. Let $\varepsilon > 0$ be given. Since π is a covering map and X is compact, there exists a $\delta > 0$ such that for $\pi(x) \in X/G$, $\pi^{-1}(U_\delta(\pi(x))) = \bigcup_{\alpha \in \Delta} U_\alpha$, where each U_α in X, $\alpha \in \Lambda$, $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \phi$ and $\pi_{|U_\alpha} : U_\alpha \to U_\delta(\pi(x))$ is a homeomorphism. For ε–neighborhood $U_\varepsilon(x)$ of x, consider U_α which contains x If $diam U_\alpha < \varepsilon$, we have $\pi^{-1}_{|U_\alpha}(U_\delta(\pi(x))) \subset U_\alpha \subset U_\varepsilon(x)$. If $diam U_\alpha \not< \varepsilon$, then choose $U'_\alpha \subset U_\alpha$ such that

$diamU'_\alpha < \varepsilon$ and $x \in U'_\alpha$, we have $\pi^{-1}_{|U'_\alpha}(U_\delta(\pi(x))) = U'_\alpha \subset U_\varepsilon(x)$. Since $\hat{f}$ has the shadowing property there is an $\eta > 0$ such that every η–pseudo orbit for $\hat{f}$ is δ–traced by a point of X/G. Uniform continuity of π implies there is $\gamma > 0$ such that $d(x,y) < \gamma \Rightarrow d_1(\pi(x), \pi(y)) < \eta$. In order to show that f has the G–shadowing property we show that every $\gamma - G$pseudo orbit for f is ε–traced by a point of X. Let $\{x_i : i \geq 0\}$ be a $\gamma - G$pseudo orbit for f. This implies for each $i \geq 0$, there is a $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \gamma$ which implies $d_1(\pi(f(x_i)), \pi(x_{i+1})) < \eta$ and hence we have $d_1(G(f(x_i)), G(x_{i+1})) < \eta$ which proves that $\{G(x_i) : i \geq 0\}$ is an η–pseudo orbit for $\hat{f}$. Since $\hat{f}$ has the shadowing property, $\{G(x_i) : i \geq 0\}$ is δ–traced by a point of X/G, say, $G(x)$ and hence $d_1(G(f^i(x)), G(x_i)) < \delta$, for each $i \geq 0$. But this gives

$$\pi(f^i(x)) \in U_\delta(\pi(x_i))$$

which implies

$$f^i(x) \in \pi^{-1}(U_\delta(\pi(x_i))) \subset U_\varepsilon(x)$$

and therefore $d(f^i(x), x_i) < \varepsilon$. This proves $\{x_i : i \geq 0\}$ is ε–traced by the point x of X. Hence f has the G–shadowing property.

Conversely, suppose f has the G–shadowing property. We show that $\hat{f}$ has the shadowing property. Let $\varepsilon > 0$ be given. Since π is uniformly continuous, there exists $\gamma > 0$ such that $d(x,y) < \gamma \Rightarrow d_1(\pi(x), \pi(y)) < \varepsilon$. Also, f has the G–shadowing property, therefore there is an $\eta > 0$ such that every $\eta - G$pseudo orbit for f is γ–traced by a point of X. Since π is a covering map on a compact space, there exists a $\delta > 0$ such that for each x in X we find an α_x satisfying $(\pi_{|U_{\alpha_x}})^{-1}(U_\delta(\pi(x))) \subset U_\eta(x)$. In order to show that $\hat{f}$ has the shadowing property we show that every δ–pseudo orbit for $\hat{f}$ is ε–traced by a point of X/G. Let $\{G(x_i): i \geq 0\}$ be a δ–pseudo orbit for $\hat{f}$. Then there exists $\alpha_{x_{i+1}}$ such that $x_{i+1} \in (\pi_{|U_{\alpha_{x_{i+1}}}})^{-1}(U_\delta(\pi(f(x_i)))) \subset U_\eta(f(x_i))$ which implies $\{x_i : i \geq 0\}$ is an η–Gpseudo orbit for f and therefore is γ–traced by some point x of X. Hence, for each $i \geq 0$, there is a $p_i \in G$ such that $d(p_i x_i, f^i(x)) < \gamma$. Further, using uniform continuity of the covering map π we get $d_1(G(f^i(x)), G(x_i)) < \varepsilon$. This proves $\{G(x_i): i \geq 0\}$ is ε–traced by $G(x)$. Hence $\hat{f}$ has the shadowing property