CHAPTER IV

POSITIVE EXPANSIVITY AND G–SHADOWING FOR MAPS ON G–SPACES

We recall that expansivity is another important dynamical property of maps on metric spaces. For homeomorphisms, it was defined by Utz **[45]** in 1950 and for continuous onto maps it was defined by Williams in **[49]**. Expansive maps have wide applications in topological dynamics, ergodic theory, continuum theory and symbolic dynamics **[47]**.

We also recall that notion of $G-$expansivity was defined and studied in detail for homeomorphism on metric $G-$spaces **[16]**. We introduce and study here the notion of $G-$expansivity for continuous maps on $G-$spaces. In Section 1, we define and give some interesting examples of positively expansive maps on $G-$spaces termed as positively $G-$expansive maps. In Section 2, we study properties of positively $G-$expansive map and provide necessary examples to strengthen hypothesis. We relate the positive $G-$expansivity of a map f on a metric $G-$space X with $G-$expansivity of the shift map σ on the inverse limit space X_f generated by f. In Section 3, observing that positive $G-$expansivity and $G-$shadowing property are independent concepts, we obtain a necessary and sufficient condition for a positively $G-$expansive map to possess $G-$shadowing property. In Section 4, we define and study the notion of non wandering points, chain recurrent points for maps on $G-$spaces and study the properties of sets of such points which we use in the Chapter 5 to obtain some applications of maps having $G-$shadowing property. Some of the results from this Chapter are accepted for publication in the Journal of Indian Mathematical Society.

1. Positively $G-$expansive maps : Definitions and examples.

In this section we define and give some examples of positively expansive maps on $G-$spaces termed as positively $G-$expansive maps. We begin with the following definition.

Definition 4.1.1. Let (X,d) be a metric $G-$space. A continuous onto map $f:X\to X$ is said to be *positively* $G-$*expansive*, if there exists a positive real number c such that for all $x,\ y$ in X with $G(x)\neq G(y)$, there exists a non-negative integer n such that

$$d(f^n(u),\ f^n(v))>c\text{, for all } u\in G(x) \text{ and } v\in G(y);$$

c is then called a $G-$*expansive constant* for f.

We first consider the following examples.

Examples 4.1.2 (a) Let $X=Z-\{0\}$ and let $G=Z_2$ act on X by the action $1x=x$ and $(-1)x=-x$, for all $x\in X$. Let d_1 be the usual metric on X and d_2 be the metric given by

$$d_2(m,n)=\left|\frac{1}{m}-\frac{1}{n}\right|,\ m,\ n\in X.$$

Let f be the identity map on X. Clearly f is positively Z_2-expansive with respect to metric d_1 with G-expansive constant $\delta, 0<\delta<1$. For a given $\varepsilon>0$, choose $n, m \in X$ such that $\frac{1}{n}<\frac{1}{m}<\frac{\varepsilon}{2}$. Then $d_2(n, m)=\left|\frac{1}{n}-\frac{1}{m}\right|<\varepsilon$ which gives $\left|f^k(\frac{1}{n})-f^k(\frac{1}{m})\right|<\varepsilon$ for all $k \geq 0$. Therefore for a given $\varepsilon>0$ there exists $n, m \in X$ with $G(n) \neq G(m)$ such that $d_2(n, m)<\varepsilon$. Hence f is not positively Z_2-expansive with respect to metric d_2.

4.1.2. (b) For each $n \in \mathrm{N}$, let X_n denote the $(m-1)$ sphere centered at origin and or radius $\frac{1}{n}$. Let $G=SO(m)$ act on $X=\bigcup_{n=1}^{\infty} X_n \cup \{0\}$ of R^{m} by the usual action of matrix multiplication, where 0 is the origin in R^{m}. Note that if $z \neq 0 \in X$ lies in X_n, then $G(z)=X_n$. Define $f: X \to X$ by

$$f(x)=\begin{cases} z, & \text{if } z=0 \text{ or } z \in X_1 \\ z', & \text{if } z \in X_n, n \neq 1, \text{where } z' \text{ is the point of intersection of the sphere } X_{n-1} \\ & \text{with the line joining } z \text{ and the origin} \end{cases}$$

Take δ such that $0<\delta<\frac{1}{6}$. For $z_1, z_2 \in X$ with $G(z_1) \neq G(z_2)$, there is an integer $n \geq 0$ such that $f^n(u) \in X_2$ and $f^n(v) \in X_3$ or $f^n(u) \in X_3$ and $f^n(v) \in X_2$. Therefore $d(f^n(u), f^n(v))>\frac{1}{6}>\delta$. Hence f is positively $SO(m)$-expansive with $SO(m)$-constant δ, $0<\delta<\frac{1}{6}$. Observe that f is not positively Z_2-expansive as the points of X_1 cannot be separated by f, f being the identity map on X_1. By similar arguments, f is not positively expansive on X_1.

4.1.2. (c) For each $n \in \mathrm{N}$, let X_n be the circle centered at origin and of radius n. For fix $k \in \mathrm{N}$ consider the subspace $X = \bigcup_{n=1}^{k} X_n \cup \{\bar{0}\}$ of R^2. Consider the usual action of $SO(2)$ on X. Then the identity map on X is not positively expansive but is positively $G-$expansive with $G-$expansive constant δ, $0 < \delta < 1$.

4.1.2. (d) Consider the space X and the homeomorphism h defined in Example 2.1.2 (C) Suppose $G = \{h^n : n \in \mathbf{Z}\}$ acts on X by usual action. Then the map $f : X \to X$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\} \\ 1 - \frac{1}{n-1}, & \text{if } x = 1 - \frac{1}{n},\ n \neq 1, 2 \\ \frac{1}{n-1}, & \text{if } x = \frac{1}{n},\ n \neq 1, 2 \\ -\left(1 - \frac{1}{n-1}\right), & \text{if } x = -\left(1 - \frac{1}{n}\right),\ n \neq 1, 2 \\ -\frac{1}{n-1}, & \text{if } x = -\frac{1}{n},\ n \neq 1, 2 \end{cases}$$

is positively expansive with expansive constant δ, $0 < \delta < \frac{1}{6}$.

Suppose f is positively $G-$expansive with $G-$expansive constant δ. Then for $G(\frac{1}{2}) \neq G(\frac{1}{3})$, there exists an integer $k \geq 0$ such that

$$d(f^k(u), f^k(v)) > \delta \text{ for all } u \in G(\tfrac{1}{2}) \text{ and } v \in G(\tfrac{1}{3}) \qquad (*)$$

Choose an integer $n > k$ such that $\frac{1}{2n-k}, \frac{1}{2n-k+1} < \frac{\delta}{2}$. For $u = \frac{1}{2n} \in G\left(\frac{1}{2}\right)$, $v = \frac{1}{2n+1} \in G\left(\frac{1}{3}\right)$, we have $f^k(u) = \frac{1}{2n-k}$ and $f^k(v) = \frac{1}{2n+1-k}$. Hence

$$d(f^k(u), f^k(v)) = d\left(\frac{1}{2n-k}, \frac{1}{2n+1-k}\right) < \frac{\delta}{2},$$

- a contradiction to (*). Therefore f is not positively G – expansive.

4.1.2. (e) Consider the subspace $X_1 = X \times \mathrm{R}$ of R^2, where $X = \left\{\pm\frac{1}{n}, \pm\left(1-\frac{1}{n}\right) | n \in N\right\}$. Let $G = \mathrm{R}$ act on X_1 by the action $g(x,y) = (x, y+g)$. Then the map $f_1 : X_1 \to X_1$ defined by $f_1(x,y) = (f(x), y)$, where f is the map defined on X as in Example 4.1.2 (d), is not positively expansive. For given $\delta > 0$, choose $y_1, y_2 \in \mathrm{R}$ with $|y_1 - y_2| < \delta$. Then for $(x_1, y_1), (x_2, y_2) \in X_1$ there exists no integer $k \geq 0$ such that

$$d\left(f_1^k(x, y_1), f_1^k(x, y_2)\right) > \delta.$$

In fact, for each k, $d\left(f_1^k(x, y_1), f_1^k(x, y_2)\right) = |y_1 - y_2| < \delta$. Further, note that f_1 is R-expansive with R-expansive constant δ, $0 < \delta < \frac{1}{6}$. On the other hand if $h_1 : X_1 \to X_1$ is defined by $h_1(x,y) = (h(x), y)$, where h is the map h defined on X as in Example 4.1.2. (d) and $G_2 = \{h_1^n \mid n \in Z\}$ acts on X_1 by the usual action, then f_1 is not positively G_2 – expansive follows in a similar manner as in Example 4.1.2. (d).

Remark 4.1.3.

(i) Under the trivial action of G on a space X , notion of positive G – expansivity coincides with the notion of positive expansivity for a continuous onto map $f : X \to X$.

(ii) Examples 4.1.2 (b), 4.1.2 (c), 4.1.2 (d), and 4.1.2 (e) show that under a non-trivial action of G on X both the concepts are independent.

(iii) Example 4.1.2 (a) shows that for non-compact spaces positive $G-$expansivity depends upon the metric considered on the space.

(iv) Examples 4.1.2 (b), 4.1.2 (c), 4.1.2 (d), 4.1.2 (e) show that the notion depends upon the choice of G in the sense that it may be positively $G-$expansive with respect to one group but need not be with respect to another group.

2. Properties of positively $G-$expansive maps.

In this section we study some properties of positively $G-$expansive maps and give necessary examples to strengthen the hypothesis.

Following result gives the relation between the positive $G-$expansivity of a map f with the positive expansivity of the induced map.

Theorem 4.2.1. *Let (X,d) be a metric $G-$space, where G is compact and d is invariant.. Then a pseudoequivariant map $f:X\to X$ is positively $G-$expansive if and only if the induced map $\hat{f}:X/G\to X/G$ is positively expansive, where X/G is considered as a metric space with metric d_1 induced by d.*

Proof. Suppose f is positively $G-$expansive with $G-$expansive constant c. Then for $x,y\in X$ with $G(x)\neq G(y)$ there exists an integer $n\geq 0$ such that

$$d(f^n(u),f^n(v))>c, \text{ for all } u\in G(x) \text{ and } v\in G(y).$$

We show that $\hat{f}$ is positively expansive with expansive constant α, $0<\alpha<c$. Let $G(x),G(y)\in X/G$ with $G(x)\neq G(y)$. Since f is positively $G-$expansive there exist an integer $n\geq 0$ such that

$$d(f^n(u),f^n(v))>c, \text{ for all } u\in G(x) \text{ and } v\in G(y).$$

Observe that for this n, $d_1(\hat{f}^n(G(x)),\hat{f}^n(G(y))>\alpha$ which proves $\hat{f}$ is positively expansive on X/G with expansive constant α.

Conversely, suppose $\hat{f}$ is positively expansive with expansive constant e. Then for $G(x)$, $G(y)\in X/G$ with $G(x)\neq G(y)$, there exists an integer $n\geq 0$ such that $d_1(\hat{f}^n(G(x)),\hat{f}^n(G(y))>e$. We show that f is positively $G-$expansive with $G-$expansive constant e. Let $x,y\in X$ with $G(x)\neq G(y)$. Since $\hat{f}$ is positively expansive, there exists an integer $n\geq 0$ such that $d_1(\hat{f}^n(G(x)),\hat{f}^n(G(y))>e$ which implies

$$Inf\{d(f^n(u),f^n(v)):u\in G(x),v\in G(y)\}>e$$

and hence

$$d(f^n(u),f^n(v))\geq \inf\{d(f^n(u),f^n(v))|u\in G(x),v\in G(y)\}>e,$$

for all $u\in G(x)$, $v\in G(y)$. Therefore f is positively $G-$expansive with $G-$expansive constant δ.

Corollary 4.2.2. *Let (X,d) be a compact metric $G-$space, where G is compact and d is an invariant metric. If $f:X\to X$ is a pseudoequivariant positively $G-$expansive homeomorphism then the orbit space X/G is a finite space.*

Proof. Since f is positively $G-$expansive homeomorphism on X, by Theorem 4.2.1. the induced map $\hat{f}$ is positively expansive homeomorphism on a compact metric space X/G. Therefore by Theorem 1.10, X/G is finite.

If X is a metric $G-$space and $f:X\to X$ is a continuous onto map such that f^n is positively $G-$expansive for some $n>1$ then clearly f is positively $G-$expansive. The following example shows that $f:X\to X$ is positively $G-$expansive need not imply f^n is positively $G-$expansive for all $n>1$.

Example 4.2.3. Let

$$X_1=\{(n,0)\mid n\in,\ \boldsymbol{Z}-\{0\}\}$$

$$X_2=\{(n,n)\mid n\in \boldsymbol{Z},\ n \text{ odd}\}\cup\left\{\left(n,\frac{1}{n}\right)\mid n\in \boldsymbol{Z}-\{0\},\ n \text{ even}\right\}$$

and $X=X_1\cup X_2$ with the usual metric of $\boldsymbol{R}^2$. Suppose $G=\boldsymbol{Z}_2$ act on X by the action $1\cdot x=x$ and $(-1)\cdot x=-x$, for all $x\in X$. Define $f:X\to X$ by

$$f(z)=\begin{cases}(n+1,0), & \text{if } x=(n,0),\ n\neq -1\\ (1,0), & \text{if } x=(-1,0)\\ \left(n+1,\dfrac{1}{n+1}\right), & \text{if } x=(n,n),\ n \text{ odd and } n\neq -1.\\ (1,1), & \text{if } x=(-1,1)\\ (n+1,n+1), & \text{if } x=\left(n,\dfrac{1}{n}\right),\ n \text{ even}\end{cases}$$

We show that f is positively $\boldsymbol{Z}_2-$expansive with $\boldsymbol{Z}_2-$expansive constant δ, $0<\delta<1$. Let $z_1,z_2\in X$ with $G(z_1)\neq G(z_2)$. Then there is an integer $k>1$

such that $d(f^k(u), f^k(v)) > 1 > \delta$ for all $u \in G(z_1)$ and $v \in G(z_2)$. Therefore f is positively Z_2 -expansive. On the other hand observe that

$$f^2\left(\left(n, \frac{1}{n}\right)\right) = \left(n+2, \frac{1}{n+2}\right) \text{ and } f^2(n,0) = (n+2,0)$$

and hence

$$f^{2k}\left(\left(n, \frac{1}{n}\right)\right) = \left(n+2k, \frac{1}{n+2k}\right) \text{ and } f^{2k}((n,0)) = (n+2k,0),$$

which implies

$$d\left(f^{2k}\left(n, \frac{1}{n}\right), f^{2k}(n,0)\right) \to 0 \text{ as } k \to \infty.$$

Thus there is $z_1 = \left(n, \frac{1}{n}\right)$, $z_2 = (n,0)$ in X such that $G(z_1) \neq G(z_2)$ and there is $u \in G(z_1)$ and $v \in G(z_2)$ for which $d\left((f^2)^k(u), (f^2)^k(v)\right) \to 0$ as $k \to \infty$. This proves that f^2 is not positively Z_2 -expansive though f is.

Our next result gives a sufficient condition under which $f : X \to X$ positively G-expansive implies f^n positively G-expansive for all $n > 1$.

Theorem 4.2.4. *Let X be a compact metric G-space and let $f : X \to X$ be a positively G-expansive map. Then f^n is positively G-expansive, for any integer $n > 0$.*

Proof. Choose a positive integer n and fix it. Let $e > 0$ be a G-expansive constant for f. Since $f^i, 0 < i \leq n$, is uniformly continuous and X is compact, there exists $\eta > 0$ such that $d(x,y) < \eta \Rightarrow d(f^i(x), f^i(y)) < e$ for all

$i,\ 0 < i \leq n$ or equivalently we have

$$d(f^i(x), f^i(y)) \geq e \Rightarrow d(x,y) \geq \eta \qquad \textbf{(I)}$$

For $x, y \in X$ with $G(x) \neq G(y)$, since f is positively $G-$expansive, there exists an integer $m \geq 0$ such that

$$d(f^m(u), f^m(v)) > e, \text{ for all } u \in G(x) \text{ and } v \in G(y).$$

Note that if m and n are integers, then there exists $j \in N \cup \{0\}$ and $p \in \{0,1,....,n-1\}$ satisfying $m = nj + p$ Thus we have

$$e < d(f^m(u), f^m(v)) = d(f^{nj+p}(u), f^{nj+p}(v))$$

and therefore by using **(I)** we obtain

$$d(f^{nj}(u), f^{nj}(v)) \geq \eta \Rightarrow d((f^n)^j(u), (f^n)^j(v)) \geq \eta.$$

Thus for $x, y \in X$ with $G(x) \neq G(y)$, there is an integer $j \geq 0$ such that

$$d((f^n)^j(u), (f^n)^j(v)) \geq \eta', \text{ for all } u \in G(x) \text{ and } v \in G(y),$$

where $0 < \eta' < \eta$. Therefore f^n is positively $G-$expansive with expansive constant η'.

Note. Example 4.2.3. justifies compactness in Theorem 4.2.4.

Following result deals with product of positively $G-$expansive maps.

Theorem 4.2.5. *Let X and Y be two metric $G-$spaces with metrics d and ρ respectively. Suppose $f: X \to X$ and $h: Y \to Y$ are positively $G-$expansive maps. If G acts diagonally on the product space $X \times Y$, then the product map $f \times h: X \times Y \to X \times Y$ defined by $(f, h)(x, y) = (f(x), h(y))$ is positively $G-$expansive.*

Proof. Let e_1, e_2 be $G-$expansive constants for f and h respectively and let $0 < \delta < \min\{e_1, e_2\}$. We consider the metric D on $X \times Y$ defined by

$$D((x_1, y_1),(x_2, y_2)) = \left[[d(x_1, x_2)]^2 + [\rho(y_1, y_2)]^2\right]^{1/2}.$$

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $G(x_1, y_1) \neq G(x_2, y_2)$. Then either $G(x_1) \neq G(x_2)$ or $G(y_1) \neq G(y_2)$. Suppose $G(x_1) \neq G(x_2)$. Then positive $G-$expansivity of f implies there exists an integer $n \geq 0$ such that

$$d(f^n(u_1), f^n(u_2)) > e_1 \text{ for all } u_1 \in G(x_1) \text{ and } u_2 \in G(x_2).$$

Therefore, for any $(u_1, v_1) \in G(x_1, y_1)$, $(u_2, v_2) \in G(x_2, y_2)$, we have

$$D((f \times h)^n(u_1, v_1), (f \times h)^n(u_2, v_2))$$

$$= \left[[d(f^n(u_1), f^n(u_2))]^2 + [\rho(h^n(v_1), h^n(v_2))]^2\right]^{1/2}$$

$$\geq d(f^n(u_1), f^n(u_2)) > e_1 > \delta.$$

Similarly, if $G(y_1) \neq G(y_2)$ then h being positively $G-$expansive, there exists an integer $k \geq 0$ such that $\rho(h^k(v_1), h^k(v_2)) > e_2$, for all $v_1 \in G(y_1)$ and $v_2 \in G(y_2)$. Hence for any $(u_1, v_1) \in G((x_1, y_1))$, $(u_2, v_2) \in G((x_2, y_2))$, $D((f \times h)^k(u_1, v_1), (f \times h)^k(u_2, v_2)) > e_2 > \delta$. This proves $f \times h$ is positively $G-$expansive with $G-$expansive constant δ.

Theorem 4.2.7. *Let X and Y be compact metric $G-$spaces, with metric d and ρ respectively and $f : X \to X$ be a positively $G-$expansive map. If $h : X \to Y$ is a pseudoequivariant homeomorphism, then $f_1 = hfh^{-1} : Y \to Y$ is a positively $G-$expansive map on Y.*

Proof. Let e be a $G-$expansive constant for f. Since h^{-1} is a uniformly continuous map, there exists a $\delta > 0$ such that for all $y_1, y_2 \in Y$

$$d(h^{-1}(y_1), h^{-1}(y_2)) \geq e \Rightarrow \rho(y_1, y_2) \geq \delta \qquad \text{(I)}$$

Suppose $x_1, x_2 \in X$ be such that $h^{-1}(y_1) = x_1$ and $h^{-1}(y_2) = x_2$. Then w e have

$$d(x_1, x_2) \geq e \Rightarrow \rho(h(x_1), h(x_2)) \geq \delta \qquad \text{(II)}$$

Note that if $y_1, y_2 \in Y$ be such that $G(y_1) \neq G(y_2)$, then pseudoequivariancy of h gives $G(h^{-1}(y_1)) \neq G(h^{-1}(y_2))$ or equivalently $G(x_1) \neq G(x_2)$. Since f is positively $G-$expansive, there exists an integer $n \geq 0$ such that

$$d(f^n(u), f^n(v)) > e, \text{ for all } u \in G(x_1) \text{ and } v \in G(x_2).$$

Therefore from (II) for all $u \in G(x_1),\ v \in G(x_2)$

$$\rho(hf^n(u), hf^n(v)) \geq \delta \text{ and hence } \rho(f_1^n(h(u)), f_1^n(h(v))) \geq \delta$$

Thus we have $\rho(f_1^n(u'), f_1^n(v')) \geq \delta$ for all $u' \in G(y_1)$ and $v' \in G(y_2)$. This proves f_1^n is positively $G-$expansive map with $G-$expansive constant δ.

We recall the definition of $G-$expansive homeomorphisms defined and studied in **[17]**. Let X be a metric $G-$space and $h : X \to X$ be a homeomorphism. Then h is called *$G-$expansive* if there exists a $\delta > 0$ such that for $x, y \in X$ with $G(x) \neq G(y)$ there exists an integer n satisfying $d(f^n(u), f^n(v)) > \delta$, for all $u \in G(x)$ and $v \in G(y)$; δ is then called a *$G-$expansive constant* for h. In the following result we relate the positive $G-$expansivity of a continuous onto map f with $G-$expansivity of the homeomorphism σ, the shift map on the inverse limit space X_f.

Theorem 4.2.8. *Let X be a compact metric $G-$space, with G compact, and $f:X\to X$ be a positively $G-$expansive equivariant map. Consider the inverse limit space X_f and suppose G acts diagonally on X_f. Then the shift map σ defined on X_f by $\sigma((x_i))=(f(x_i))$ is an $G-$expansive homeomorphism.*

Proof. Let e be a $G-$expansive constant for f and let $\tilde{x},\tilde{y}\in X_f$ with $G(\tilde{x})\neq G(\tilde{y})$. Suppose $\tilde{x}=(x_m)$ and $\tilde{y}=(y_m)$. Then $G(\tilde{x})\neq G(\tilde{y})$ implies that there exists $m\in Z$ such that $G(x_m)\neq G(y_m)$. Since f is positively $G-$expansive there exists an integer $k\geq 0$ such that $d(f^k(u),f^k(v))>\delta$ for all for all $u\in G(x_m)$ and $v\in G(y_m)$. Set $n=k-m$ and observe that $\tilde{d}(\sigma^n(\tilde{u}),\sigma^n(\tilde{v}))>\delta$ for all $\tilde{u}\in G(\tilde{x})$ and $\tilde{v}\in G(\tilde{y})$. Hence σ is positively $G-$expansive homeomorphism on X_f.

The following result gives a class of maps which are not positively G-expansive.

Theorem 4.2.9. *Let $f:X\to X$ be a pseudo equivariant minimal open map defined on a compact metric $G-$space X, where G is compact and action of G on X is non-transitive. Then f is not a positively $G-$expansive map.*

Proof. If possible, suppose f is a positively $G-$expansive map. Then by Lemma 4.2.1, $\hat{f}$ is positively expansive. Thus $\hat{f}$ is a positively expansive open map. Therefore by Theorem 1.13 $\hat{f}$ has a fixed point in X/G, say $G(x)$. Observe that $\hat{f}(G(x))=G(x)$ implies $f^n(x)\in G(x)$ for each $n\geq 0$ and hence $O_f(x) \subset G(x)$. Minimality of map f and compactness of G gives $X = G(x)$ But this implies G acts on X transitively – a contradiction. Therefore f is not positively G-expansive.

3. Positively G-expansive maps having G-shadowing property.

Observing through examples that positive G-expansivity and G-shadowing property are independent concepts, we obtain here a necessary and sufficient condition for a positively G-expansive map to possess G-shadowing property. Consider the following examples:

Example 4.3.1. (a) Consider $\mathbf{Z}_2$-space I and let f be a pseudoequivariant continuous onto map defined on I satisfying the hypothesis of Theorem 3.3.2. Then f has the $\mathbf{Z}_2$-shadowing property. Observe that f is not positively $\mathbf{Z}_2$-expansive. For if f is positively $\mathbf{Z}_2$-expansive then by Theorem 4.2.1 the induced map $\hat{f}$ will be positively expansive map on $I/\mathbf{Z}_2 \cong [0, \frac{1}{2}]$. But there exists no positively expansive map on interval. Therefore f is not positively $\mathbf{Z}_2$-expansive.

4.3.1. (b) Consider the space, group and the map f of the Example 2.1.2 (e). Recall that f defined by $f(z) = z^2 = e^{2i\theta}$, does not possess the G-shadowing property. We show that f is positively G-expansive. Observe that for $z = e^{i\theta}$, $G(e^{i\theta}) = \left\{e^{i\theta}, e^{i(\theta+\frac{\pi}{2})}, e^{i(\theta+\pi)}, e^{i(\theta+\frac{3\pi}{2})}\right\}$ and $f^k(z) = e^{i2^k\theta}$. Let δ

be such that $0<\delta<1$. Then for $z_1, z_2 \in S^1$ with $G(z_1) \neq G(z_2)$ there is an integer $k \geq 0$ such that

$$d(f^k(u), f^k(v)) > 1 > \delta \text{ for all } u \in G(z_1) \text{ and } v \in G(z_2).$$

The above examples justifies that the notion of positive G-expansivity and G-shadowing property for a continuous onto map on a metric G-space are independent. So we obtain a necessary and sufficient condition for a positively G-expansive map to possess the G-shadowing property. We first observe the following result.

Lemma 4.3.2. *Let (X,d) be a compact metric G-space with G-compact. Then for each $\varepsilon>0$ there exists $\eta>0$ and $\delta>0$ satisfying $U_\eta^d(gx) \subset gU_\varepsilon^d(x)$ and $gU_\delta^d(x) \subset U_\varepsilon^d(gx)$ for all g in G and all x in X. Here $U_\delta^d(x)$ denotes the δ-neighbourhood of x with respect to metric d.*

Proof. Since (X,d) is a metric G-space, with G-compact, there is an invariant metric ρ on X, i.e. there is an equivalent metric ρ on X satisfying

$$\rho(gx, gy) = \rho(x,y), \text{ for each } g \in G \text{ and for all } x, y \in X.$$

Let $\varepsilon>0$ be given. Since d and ρ are equivalent metrics on a compact space X, there exists $\delta>0$ such that for each $x \in X$, $U_\delta^\rho(x) \subset U_\varepsilon^d(x)$ which implies

$$gU_\delta^\rho(x) \subset gU_\varepsilon^d(x), \; g \in G \qquad \textbf{(I)}$$

But ρ is an invariant metric on X. Therefore for all $x \in X$ and $g \in G$

$$gU_\delta^\rho(x) = U_\delta^\rho(gx). \qquad \textbf{(II)}$$

Again, d and ρ being equivalent metrics on a compact space X we have $\eta > 0$ such that for all $x \in X$ and $g \in G$

$$U_\eta^d(gx) \subset U_\delta^\rho(gx). \qquad \textbf{(III)}$$

Therefore from (I), (II) and (III) for all $x \in X$ and $g \in G$

$$U_\eta^d(gx) \subset U_\delta^\rho(gx) = gU_\delta^\rho(x) \subset gU_\varepsilon^d(x),$$

Thus for given $\varepsilon > 0$ there exists an $\eta > 0$ such that for each $x \in X$ and each $g \in G$

$$U_\eta^d(gx) \subset gU_\varepsilon^d(x),.$$

Similarly there exists $\delta > 0$ such that for each $x \in X$ and each $g \in G$

$$gU_\delta^d(x) \subset U_\varepsilon^d(gx).$$

Theorem 4.3.3. *Let X be a compact metric G – space with G compact and let $f : X \to X$ be a positively G – expansive pseudoequivariant map. Then f has the G – shadowing property if and only if for every open set U of X and for each x in U, there exists a $\delta > 0$ and a $g \in G$ such that*

$$gU_\delta(f(x)) \subset f(U) \qquad (*)$$

where $U_\delta(x)$ denotes the δ – neighbourhood of x.

Proof. Suppose for every open set U of X and for each x in U, there exist a $\delta > 0$ and a g in G such that $gU_\delta(f(x)) \subset f(U)$. We show that f has the

$G-$shadowing property. Since f is a positively $G-$expansive pseudoequivariant map on X, by Theorem 4.3.1, the induced map $\hat{f}$ is positively expansive on X/G. We first show that $\hat{f}$ is an open map. Let U be an open subset of X/G and let $y \in \hat{f}(U)$. Then there exists $x \in X$ such that $z = \pi(x) \in U$ and $y = \hat{f}(z) = \hat{f}(\pi(x))$. Since π is continuous $U_1 = \pi^{-1}(U)$ is open in X and $\pi(x) \in U$ implies $x \in \pi^{-1}(U) = U_1$. By hypothesis there exists $\delta > 0$ and g in G such that $gf(x) \in gU_\delta(f(x)) \subseteq f(U_1)$ which implies $y \in \pi(V) \subseteq \hat{f}(\pi(U_1)) = \hat{f}(U)$, where $V = U_\delta(f(x))$ is open in X. Since π is an open map, $\pi(V)$ is open in X/G. Therefore $\hat{f}(U)$ is open in X/G. Thus $\hat{f}$ is an open map. Further, since $\hat{f}$ is a positively expansive open map on compact metric space X/G, therefore by Theorem 1.12, $\hat{f}$ has the shadowing property. Next we show that f has the $G-$shadowing property. Let $\varepsilon > 0$ be given. By Lemma 4.3.2 there exists an $\eta > 0$ such that for each x in X and $g \in G$, $gU_\eta(x) \subset U_\varepsilon(gx)$. Since $\hat{f}$ has the shadowing property, for $\eta > 0$ there is a $\delta > 0$ such that every $\delta-$pseudo orbit for $\hat{f}$ is $\eta-$traced by a point of X/G. In order to show that f has the $G-$shadowing property, we show that every $\delta - G$pseudo orbit for f is $\varepsilon-$traced by a point of X. Let $\{x_i \mid i \geq 0\}$ be a $\delta - G$pseudo orbit for f Then for each $i \geq 0$, there exists a $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$ which gives $d_1(G(f(x_i)), G(x_{i+1})) < \delta$ and hence $\{G(x_i) \mid i \geq 0\}$ is a $\delta-$pseudo orbit for $\hat{f}$. Thus $\{G(x_i) \mid i \geq 0\}$ is $\eta-$traced by a point of X/G , say $G(x)$, which implies for each $i \geq 0$

$d_1(G(f^i(x)), G(x_i)) < \eta$. Since G is compact there exists l_i, $m_i \in G$ such that $d(l_i f^i(x), m_i x_i) < \eta$ which implies

$$f^i(x) \in l_i^{-1} U_\eta(m_i x_i) \subset U_\epsilon(l_i^{-1} m_i x_i)$$

and hence $d(f^i(x), p_i x_i) < \varepsilon$ for $p_i = l_i^{-1} m_i \in G$. Therefore f has the $G-$shadowing property.

Conversely, suppose f has the $G-$shadowing property. Let U be an open subset of X and $x \in U$. Choose an $\varepsilon > 0$ such that $x \in U_\varepsilon(x) \subset U$. By Lemma 4.3.2, there exists an η, $0 < \eta \leq e$ such that $U_\eta(gy) \subset gU_\varepsilon(y)$ for all $y \in X$ and $g \in G$. Here e is a $G-$expansive constant for f. Since f has the $G-$shadowing property there exists δ, $0 < \delta \leq e$, such that every $\delta - G$ pseudo orbit for f is $\eta-$traced by a point of X. Let $z \in U_\delta(x)$. Then the sequence $\{y_i \mid i \geq 0\} = \{x, z, f^2(z), ...\}$ is a $\delta - G$ pseudo orbit for f and therefore is $\eta-$traced by a point of X, say y, which implies for each $i \geq 0$, there exists $p_i \in G$, such that

$$d(p_i y_i, f^i(y)) < \eta \qquad \text{(I)}$$

Hence for $i \geq 1$, there exists $p_i \in G$ such that $d(p_i f^{i-1}(z), f^i(y)) < \eta \leq e$. Replacing i by $i+1$ we obtain for all $i \geq 0$, $d(p_{i+1} f^i(z), f^{i+1}(y)) \leq e$. Using pseudoequivariancy and positive $G-$expansivity of f, we get $G(z) = G(f(y))$ and therefore $f(y) = gz$, for some $g \in G$.

Also from (I) we get $d(p_0 y_0, f^0(y)) < \eta$ which implies $y \in U_\eta(p_0 x) \subset p_0 U_\epsilon(x)$ and hence $gz = f(y) \in f(p_0 U_\epsilon(x))$. Using pseudoequivariancy of f we get

$z \in g^{-1}f(p_0U_\varepsilon(x)) = tf(U_\varepsilon(x))$ and therefore $U_\delta(f(x)) \subset tf(U)$ for some t in G and hence $t'U_\delta(f(x)) \subset f(U)$, where $t' = t^{-1} \in G$. Thus the required condition holds. Hence the proof.

Note. Recall example 4.3.1 (b) in which map f on S^1 is defined by $f(e^{i\theta}) = e^{2i\theta}$. f being an open map satisfies the condition (*) of Theorem 4.3.3. Also f is positively U_4-expansive but f does not have the U_4-shadowing property. Observe that f is not a pseudo equivariant map. In fact for $e^{\frac{\pi}{2}i} \in S^1$,

$$G(f(e^{\frac{i\pi}{2}})) = G(e^{i\pi}) = \left\{e^{i\pi}, e^{i(\pi+\frac{\pi}{2})}, e^{i(\pi+\pi)}, e^{i(\pi+\frac{3\pi}{2})}\right\} = \left\{e^{i\pi}, e^{\frac{i3\pi}{2}}, e^{2\pi i}, e^{\frac{i\pi}{2}}\right\}$$

where as

$$f(G(e^{\frac{i\pi}{2}})) = f\left(\left\{e^{i\pi}, e^{\frac{i3\pi}{2}}, e^{2\pi i}, e^{\frac{i\pi}{2}}\right\}\right) = \{e^{i\pi}, e^{0\pi}\}.$$

This proves $G(f(e^{\frac{i\pi}{2}})) \neq f(G(e^{\frac{i\pi}{2}}))$. Thus pseudoequivariancy is a necessary condition in Theorem 4.3.3.

4. G-non wandering points and G-chain recurrent points: Definitions, examples and properties.

In this section, we define and study the notion of non wandering points, chain recurrent points for maps on G-spaces and study the properties of sets of such points. We first define these terms.

Definition 4.4.1. Let X be a metric $G-$space and $f:X \to X$ be a continuous onto map.

(a) A point x in X is said to be $G-$*non wandering point* of f if for every neighbourhood U of x, there exists an integer $n > 0$ and a $g \in G$ such that $gf^n(U) \cap U \neq \varphi$. We shall denote the set of all $G-$non wandering points of f by $\Omega_G(f)$.

(b) For $x, y \in X$ and $\delta > 0$, x is said to be $\delta - G$ *related* to y (denoted by $x \overset{\delta}{\sim}_G y$) if there are finite $\delta - G$ pseudo orbits $\{x = x_0, x_1, \ldots, x_k = y\}$ and $\{y = y_0, y_1, \ldots, y_n = x\}$ for f. If for every $\delta > 0$, x is $\delta - G$ *related* to y, then x is said to be G related to y (denoted by $x \sim_G y$).

(c) A point x is said to be $G-$*chain recurrent point* of f if $x \sim_G x$. We shall denote the set of all $G-$chain recurrent points of f by $CR_G(f)$.

Remark 4.4.2. (i) If the action of G on X is trivial then $\Omega(f) = \Omega_G(f)$. Under the non-trivial action $\Omega(f) \subset \Omega_G(f)$ (refer 4.4.3 (i)).

(ii) If X is compact then $\Omega(f) \neq \varphi$ and $\Omega(f) \subset \Omega_G(f)$ implies $\Omega_G(f) \neq \varphi$.

(iii)'$x \overset{\delta}{\sim}_G y$' is an equivalence relation.

(iv) If the action of G on X is transitive, then for any $f:X \to X$, $\Omega_G(f) = X$. Example 4.3.3 (a) shows that $\Omega_G(f) = X$ need not imply that action is transitive.

Example 4.4.3. (a) Recall the space, map and group of Example 2.1.2 (c). In this case $X/G = \{G(0),\ G(1),\ G(-1),\ G(\frac{1}{2}),\ G(\frac{1}{3})\}$. Since $-1, 0, 1$ are fixed points of f, $-1, 0, 1 \in \Omega_G(f)$. Suppose $x \in X - \{-1, 0, 1\}$. Then $x \in G(\frac{1}{2})$ or $x \in G(\frac{1}{3})$. Now $f^2(G(\frac{1}{2})) = G(\frac{1}{2})$ and $f^2(G(\frac{1}{3})) = G(\frac{1}{3})$. Therefore if U is an $\varepsilon-$ neighbourhood of x, then there is an integer $n = 2$, such that $gf^n(U) \cap U \neq \varphi$. Thus $x \in \Omega_G(f)$ and $\Omega_G(f) = X$. Since f is a left shift fixing $-1, 0, 1$, $\Omega(f) = CR(f) = \{-1, 0, 1\}$.

4.4.3. (b) Denote by $\Sigma_t = \{0,1\}^Z$, $t = 1, 2$, and let $X = \Sigma_1 \cup \Sigma_2$ where Σ_1, Σ_2 are disjoint copies having one point in common $\bar{0} = (....,0,\ 0,\ .....)$. For x in X, we denote $x^t = (....,x_0,x_1,....) \in \Sigma_t$, $t = 1, 2$. Define a metric d on X by

$$d(x,\ y) = \begin{cases} 0, & \text{if } x = y \\ 2^{-|l|}, & \text{if } x,\ y \in \Sigma_t \\ 1, & \text{otherwise} \end{cases},$$

where l is the smallest integer for which $x_l = y_l$. Let σ_k be the shift map on Σ_t, $t = 1, 2$. Suppose $G = Z_2$ act on X by the action $1x = x$ for each x in X and $(-1)x^1 = x^2$ and $(-1)x^2 = x^1$. Consider the map $f : X \to X$ defined by

$$f(x) = \begin{cases} \sigma_1(x), & \text{if } x \in \Sigma_1 \\ \sigma_2(x), & \text{if } x \in \Sigma_2 \end{cases}.$$

We show that every point of X is a $G-$non wandering point of f. Let U be the $\varepsilon-$neighbourhood of $x^1 = (....,x_0,x_1,.....)$ then there exists an integer $N > 0$ such that $\frac{1}{2^N} < \frac{\varepsilon}{4}$. Let $n = 2N + 1$. Consider the point z^1 of Σ_1 such that

$$z^1 = \begin{cases} x_i, \text{ if } -N \le i \le N \\ y_i, \text{ if } -3N-2 \le i \le -N \text{ and } N+1 \le i \le 3N+2, \\ \quad \text{where } y_i = x_j, (-N \le j \le N) \\ 1, \text{ otherwise} \end{cases}$$

i.e. consider that point z^1 of Σ_1 which has 3 consecutive blocks of length n, where each block of length n is same as a block of length n in x^1 (i.e. a block from $-N$ to N). Then $z^1 \in U$. Also, $f^n(z^1) = \sigma_1^n(z^1) \in U$ as z^1 has $3-$blocks of same length. Therefore $f^n(U) \cap U \neq \varphi$ and hence $x^1 \in \Omega_G(f)$. Similarly if $x^2 \in \Sigma_2$ then $x^2 \in \Omega_G(f)$. Thus $\Omega_G(f) = X$. Also, $\Omega_G(f) \subset CR_G(f)$ implies $CR_G(f) = X$.

We now observe some properties of sets of $G-$non wandering points and $G-$chain recurrent points of f.

Theorem 4.4.4. *Let X be a compact metric $G-$space and $f : X \to X$ be a continuous pseudoequivariant onto map. Then $\Omega_G(f)$ is a non-empty closed $G-$invariant subset of X.*

Proof. We first show that $\Omega_G(f)$ is a closed subset of X. Let x be a limit point of $\Omega_G(f)$. Then there is a sequence $\{x_n\}$ in $\Omega_G(f)$ such that $x_n \to x$ as $n \to \infty$. For a given $\varepsilon > 0$ let U be the $\varepsilon-$neighbourhood of x in X. Then there exists an integer $N > 0$ such that $x_n \in U$, for each $n \ge N$. Choose $n \ge N$ and fix it. Then for this n, $x_n \in U \cap \Omega_G(f)$ implies there exists an integer $k > 0$ and $g \in G$ such that $gf^k(U) \cap U \neq \varphi$ which implies $x \in \Omega_G(f)$. This proves $\Omega_G(f)$ is a closed subset of X. Next we show that $\Omega_G(f)$ is a $G-$invariant subset of X. Let U be a neighbourhood of gx. Then $g^{-1}U$ is a neighbourhood of x. Since $x \in \Omega_G(f)$ there is an integer $n > 0$ and $k \in G$ such that $kf^n(g^{-1}U) \cap g^{-1}U \neq \varphi$. Observe that f^n is pseudoequivariant for each $n > 0$ and which gives $gkf^n(U) \cap U \neq \varphi$ and hence $gx \in \Omega_G(f)$. Thus $\Omega_G(f)$ is a $G-$invariant subset.

The following result gives a condition under which $f(\Omega_G(f))$ is contained in $\Omega_G(f)$.

Theorem 4.4.5. *Let X be a compact metric $G-$space and $f : X \to X$ be a continuous onto pseudoequivariant map. Then $f(\Omega_G(f)) \subset \Omega_G(f)$. Moreover, if f is a homeomorphism then $f(\Omega_G(f)) = \Omega_G(f)$ and $\Omega_G(f) = \Omega_G(f^{-1})$.*

Proof. We first show that $f(\Omega_G(f)) \subset \Omega_G(f)$. Let $y \in f(\Omega_G(f))$. Then there exists an x in $\Omega_G(f)$ such that $f(x) = y$. Let U be a neighbourhood of y. Then $f^{-1}(U)$ is a neighbourhood of x. Since $x \in \Omega_G(f)$, there exists an integer $k > 0$ and $g, g' \in G$ such that $gf^k(f^{-1}(U)) \cap f^{-1}(U) \neq \varphi$ implies $f(gf^{k-1}(U) \cap f^{-1}(U)) \neq \varphi$ and hence $g'f^k(U) \cap U \neq \varphi$. This establishes $f(\Omega_G(f)) \subset \Omega_G(f)$. Further, let f be a homeomorphism. Then $\Omega_G(f) = \Omega_G(f^{-1})$. Next applying the result $f(\Omega_G(f)) \subset \Omega_G(f)$ to the map f^{-1} we obtain $f^{-1}(\Omega_G(f^{-1})) \subset \Omega_G(f^{-1})$. Since $\Omega_G(f^{-1}) = \Omega_G(f)$, we have

$f^{-1}(\Omega_G(f^{-1})) \subset \Omega_G(f)$ which implies $\Omega_G(f^{-1}) \subset f(\Omega_G(f))$ and hence $f(\Omega_G(f)) = \Omega_G(f)$.

The following example justifies the necessity of pseudoequivariancy in Theorem 4.4.5.

Example 4.4.6. Consider the unit circle S^1 of $\boldsymbol{R}^2$, with the usual metric. Let G be the multiplicative group of fourth roots of unity acting on S^1 by the usual action of complex multiplication. We denote the points of S^1 by its argument θ. Consider the map $f : S^1 \to S^1$ defined by $f(\theta) = \dfrac{\theta}{2}$. Then for each $\theta \in S^1$, $f^n(\theta) \to 0$ as $n \to \infty$. For $\dfrac{\pi}{2} \in S^1$, let U be a neighbourhood of $\dfrac{\pi}{2}$. Then $f^n(\dfrac{\pi}{2}) \to 0$. Observe that $G = \{0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}\}$. Since for $g = \dfrac{\pi}{2} \in G$, $g0 = \dfrac{\pi}{2} \in U$ Therefore $gf^n(U) \cap U \neq \varphi$, where n is such that $f^n(\dfrac{\pi}{2}) \in U'$, for a neighbourhood U' of θ. Thus $\dfrac{\pi}{2} \in \Omega_G(f)$. But $f(\dfrac{\pi}{2}) = \dfrac{\pi}{4} \notin \Omega_G(f)$. For U is an open neighbourhood of $\dfrac{\pi}{4}$ such that $0, \dfrac{\pi}{2} \notin U$ then for no $g \in G$ and $n \in N$, $gf^n(U) \cap U \neq \varphi$. In fact we can always choose a small neighbourhood V of 0 such that gV, which is a neighbourhood of either $\dfrac{\pi}{2}$ or π or $\dfrac{3\pi}{2}$ does not contain $\dfrac{\pi}{4}$. Thus $f(\Omega_G(f)) \not\subset \Omega_G(f)$.

Theorem 4.4.7. *Let X, Y be compact metric $G-$spaces and f, p be continuous onto self maps on X and Y respectively. Suppose $h:X \to Y$ is a pseudoequivariant homeomorphism such that $ph = hf$ then $h(\Omega_G(f)) = \Omega_G(p)$.*

Proof. Let $y \in h(\Omega_G(f))$ then there exists x in $\Omega_G(f)$ such that $h(x) = y$. If U is a neighbourhood of $y = h(x)$ then $h^{-1}(U)$ is a neighbourhood of x. Since $x \in \Omega_G(f)$, using condition $ph = hf$ we get $y \in \Omega_G(p)$. Hence $h(\Omega_G(f)) \subset \Omega_G(p)$. Arguing as above we get $(\Omega_G(p)) \subset h(\Omega_G(f))$ and hence $h(\Omega_G(f)) = \Omega_G(p)$.

We now observe certain properties of sets of $G-$chain recurrent points of f.

Theorem 4.4.8. Let X *be a compact metric $G-$space where G is compact and f be a pseudo equivariant continuous onto map on X Then $\Omega_G(f) \subset CR_G(f)$.*

Proof. Let $y \in \Omega_G(f)$ and let $\varepsilon > 0$ be given. We show that $y \overset{\varepsilon}{\sim}_G y$. By Lemma 4.3.2 there is an $\varepsilon > \eta > 0$ such that $gU_n(x) \subset U_{\varepsilon/2}(g(x))$ for each $g \in G$ and each x in X. Also uniform continuity of f implies there is a δ, $0 < \delta < \varepsilon$ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \eta$. Let $U = U_\delta(y)$. Since $y \in \Omega_G(f)$, there is an integer $k > 0$ and $g \in G$ such that $gf^k(U) \cap U \neq \varphi$. Let $x \in gf^k(U) \cap U$. Then there exists $t \in U$ such that $f^k(g't) = x$. Observe that

$\{y_i \mid 0 \le i \le k\} = \{y, f(t), f^2(t), ..., f^{k-1}(t), y\}$ is a finite $\varepsilon - G$ pseudo orbit from y to itself. Hence $\Omega_G(f) \subset CR_G(f)$.

Theorem 4.4.9. *Let X be a compact metric G-space where G is compact and $f: X \to X$ be a pseudoequivariant continuous onto map. Then $CR_G(f)$ is a non-empty closed G-invariant subset of X.*

Proof. Since $\Omega_G(f) \subset CR_G(f)$ and for compact space $\Omega_G(f) \neq \varphi$ therefore $CR_G(f) \neq \varphi$. Let x be a limit point of $CR_G(f)$ and let $\varepsilon > 0$ be given. Choose an η, $0 < \eta < \frac{\varepsilon}{2}$, such that for each $y \in G$ and $x \in X$, $gU_\eta(x) \subset U_{\frac{\varepsilon}{2}}(gx)$. Since f is uniformly continuous therefore corresponding to η there exists a δ, $0 < \delta < \frac{\eta}{2}$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \frac{\eta}{2}$. Because x is a limit point of $CR_G(f)$, there is a sequence $\{x_k\}$ in $CR_G(f)$ such that $x_k \to x$ as $k \to \infty$. Therefore there exists an integer $N > 0$ such that $d(x, x_k) < \frac{\delta}{2}$ for all $k \ge N$.

Choose $k \ge N$ then $x_k \in CR_G(f)$ implies $x_k \overset{\frac{\delta}{2}}{\sim}_G x_k$. Therefore there is a finite $\frac{\delta}{2} - G$ pseudo orbit from x_k to itself, say $\{x_k = y_0', y_1', ..., y_m' = x_k\}$. We show that there exists a finite $\varepsilon - G$ pseudo orbit $\{x = y_0, y_1, ..., y_m = x\}$ with $y_0 = y_m = x$, $y_i = y_i'$, $1 \le i \le m-1$. For each $i \in \{1, 2, ..., m-2\}$ there exists $g_i \in G$ satisfying $d(g_i f(y_i'), y_{i+1}') < \frac{\delta}{2} < \varepsilon$. If $i = 0$, then there is a $g_0 \in G$

satisfying $d(g_0 f(x_k), y_1') < \frac{\delta}{2} < \frac{\varepsilon}{2}$. Note that whenever $d(x_k, x) < \frac{\delta}{2}$ we have $d(g_0 f(x_k), g_0 f(x)) < \frac{\varepsilon}{2}$. Therefore

$$d(g_0 f(x), y_1) \leq d(g_0 f(x), g_0 f(x_k)) + d(g_0 f(x_k), y_1).$$

If $i = m-1$, there is $g_{m-1} \in G$ satisfying $d(g_{m-1} f(y_{m-1}'), x_k < \frac{\delta}{2} < \frac{\varepsilon}{2}$. Thus

$$d(g_{m-1} f(y_{m-1}'), x) \leq d(g_{m-1} f(y_{m-1}'), x_k) + d(x_k, x).$$

Hence $\{x = y_1, ..., y_m = x\}$ is an $\varepsilon - G$ pseudo orbit from x to itself. Therefore $x \overset{\varepsilon}{\sim}_G x$, for every $\varepsilon > 0$, which implies $x \in CR_G(f)$. This proves $CR_G(f)$ is a closed subset of X.

Next, we show that $CR_G(f)$ is a G-invariant subset of X. Let $\varepsilon > 0$ be given. Choose an η, $0 < \eta < \frac{\varepsilon}{2}$, such that for all $g \in G$ and $x \in X$, $gU_\eta(x) \subset U_{\frac{\varepsilon}{2}}(gx)$. Since $x \in CR_G(f)$, $x \overset{\eta}{\sim}_G x$ implies there exists a finite $\eta - G$ pseudo orbit $\{x = x_0', x_1', ..., x_m' = x\}$ from x to itself. We show that $\{gx = x_0, x_1, ..., x_m = gx\}$, where $gx = x_0 = x_m$, $x_i = x_i'$ for $1 \leq i \leq m-1$, is an $\varepsilon - G$ pseudo orbit from gx to itself.

Hence $\{x_0 = gx_1, x_2, ..., x_m = gx\}$ is a finite $\varepsilon - G$ pseudo orbit from gx to itself.

Therefore $gx \overset{\varepsilon}{\sim}_G gx$ for each $\varepsilon > 0$. This further implies that $gx \in CR_G(f)$. Thus $CR_G(f)$ is a G-invariant set.

Theorem 4.4.10. *Let X be a compact metric $G-$space where G is compact and $f:X\to X$ be a pseudoequivariant continuous onto map. Then $f(CR_G(f))\subset CR_G(f)$.*

Proof. Let $y\in f(CR_G(f))$ and $x\in CR_G(f)$ be such that $f(x)=y$. Let $\varepsilon>0$ be given. Uniform continuity of f implies that there exists $\delta>0$ such that $d(x,y)<\delta$ implies $d(f(x),f(y))<\varepsilon$. Since $x\in CR_G(f)$ there exists a finite $\delta-G$pseudo orbit $\{x=x_0,x_1,...,x_m=x\}$ from x to itself. Observe that $\{f(x)=f(x_0),f(x_1),...,f(x_m)=f(x)\}$ is a finite $\varepsilon-G$pseudo orbit for f from $f(x)$ to itself. Therefore $y=f(x)\in CR_G(f)$. Hence $f(CR_G(f))\subset CR_G(f)$.

Recall the Example 4.4.6 which says that $f(\Omega_G(f))$ may be a proper subset of $\Omega_G(f)$. In the following theorem we obtain the conditions under which $f(\Omega_G(f))=\Omega_G(f)$ and $\Omega_G(f)=CR_G(f)$.

Theorem 4.4.11. Let X *be a compact metric $G-$space where G is compact. Suppose a pseudoequivariant continuous onto map f defined on X has the $G-$shadowing property. Then*

(i) $f(\Omega_G(f))=\Omega_G(f)$.

(ii) $CR_G(f)=\Omega_G(f)$.

Proof. (i) In view of Theorem 4.4.5 it is sufficient to show that $\Omega_G(f)\subset f(\Omega_G(f))$. If possible, suppose $\Omega_G(f)\not\subset f(\Omega_G(f))$. This implies that there exists $x\in\Omega_G(f)$ such that $x\notin f(\Omega_G(f))$. Since $\Omega_G(f)$ is compact and f is continuous, $f(\Omega_G(f))$ is a compact subset of X and X being Hausdorff is closed in X. Therefore $X-\Omega_G(f)$ is open and hence there exists $\varepsilon>0$ such that $B_\varepsilon(x)\subset X-\Omega_G(f)$. By Lemma 4.3.2, there exists an η, $0<\eta<\varepsilon$, such that for each $g\in G$ and each $x\in X$, $U_\eta(gy)\subset gU_\varepsilon(y)$. Since f has the $G-$shadowing property, for $\eta>0$ there exists $\delta>0$ such that every $\delta-G$pseudo orbit for f is $\eta-$traced by a point of X. Also f is uniformly continuous. Therefore there is $\gamma>0$ such that $d(y,z)<\gamma\Rightarrow d(f(y),f(z))<\delta$. Let $U=U_\gamma(x)$. Since $x\in\Omega_G(f)$, there exists $k>0$ and $g\in G$ such that $gf^k(U)\cap U\neq\varphi$. Let $z\in gf^k(U)\cap U$, then there exists $t\in U$ such that $z=gf^k(t)$. Observe that $\{x_n\mid n\geq 0\}=\{x,f(t),f^2(t),...,f^{k-1}(t),x,....\}$ is a $\delta-G$pseudo orbit for f and hence is $\eta-$traced by a point of X, say y. This implies for each $\eta\geq 0$ there is $p_n\in G$ such that $d(p_nx_n,f^n(y))<\eta$. In particular, for $n=k_i$, $x_n=x$ and for each $i\geq 0$ there exists $p_{k_i}\in G$ satisfying $d(p_{k_i}x,f^{k_i}(y))<\eta$ which implies $f^{k_i}(y)\in U_\eta(p_{k_i}x)\subset p_{k_i}U_\varepsilon(x)$. Thus for each $i\geq 0$ there is $p_{k_i}\in G$ satisfying $f^{k_i}(y)\in p_{k_i}(U_\varepsilon(x))\subset G(U_\varepsilon(x))$, where $G(U_\varepsilon(x))=\bigcup_{g\in G}gU_\varepsilon(x)$. Therefore $\{f^{k_i}(y)\mid i\geq 0\}\subset G(U_\varepsilon(x))$. Since X is compact there is a subsequence of $\{f^{k_i}(y)\mid i\geq 0\}$ which is convergent. Suppose subsequence converges to y' in X. Since y' is a limit point a subsquence of $\{f^{k_i}(y)\mid i\geq 0\}$ and $\{f^{k_i}(y_i):i\geq 0\}\subset G(U_\varepsilon(x))$, there is $m\in G$ such that $f(my')\in U_\epsilon(x)$. Thus

$y' \in \omega(f)$, where $\omega(f)$ is the limit point set of f orbit of y and $\omega(f) \subset \Omega_G(f)$. Therefore $my' \in \Omega_G(f)$, $\Omega_G(f)$ being $G-$invariant subset of X, implies $f(my') \in f(\Omega_G(f))$ and also $f(my') \in U_\epsilon(x)$, which is a contradiction. Hence $f(\Omega_G(f)) = \Omega_G(f)$.

(ii) In view of Theorem 4.4.8 it is sufficient to show that $CR_G(f) \subset \Omega_G(f)$. Let $\varepsilon > 0$ be given. By Lemma 4.3.2 there exists $\eta > 0$ such that for all $y \in X$ and $g \in G$, $U_\eta(gy) \subset gU_\varepsilon(y)$. Since f has $G-$shadowing property there is a $\delta > 0$ such that every $\delta - G$ pseudo orbit for f is $\eta-$traced by a point of X. Let $x \in CR_G(f)$, $x \overset{\delta}{\sim}_G x$. Therefore there exists a finite $\delta - G$ pseudo orbit $\{x = x_0, x_1, ..., x_k = x\}$ for f. Since f has the $G-$shadowing property therefore there is a point y in X, $\eta-$tracing $\{x = x_0, x_1, ..., x_k = x\}$ which implies for each i, $0 \le i \le k$ there exists $p_i \in G$ satisfying $d(p_i x_i, f^i(y)) < \eta$. In particular, for $i = 0$ and $i = k$ there exists $p_0, p_k \in G$ such that $d(p_0 x, y) < \eta$ and $d(g_k x, f^k(y)) < \eta$. If $d(p_0 x, y) < \eta$ then $y \in U_\eta(p_0 x) \subset p_0 U_\varepsilon(x)$ which gives $f^k(y) \in p_0' f^k(U_\varepsilon(y))$. Also $d(g_k x, f^k(y)) < \eta$ implies

$$p_k U_\varepsilon(x) \cap p_0' f^k(U_\varepsilon(x)).$$

This implies that for the $\varepsilon-$neighbourhood U of x there exists an integer $k > 0$ and $p \in G$ such that $pf^k(U) \cap U \neq \varphi$. Therefore $x \in \Omega_G(f)$ and hence $CR_G(f) = \Omega_G(f)$.