

CHAPTER VI

Economic Design of Multivariate Control Charts

6.1 In this chapter the economic design of Hotelling T^2 -control chart is developed. The cost model developed is an adaptation of the cost model developed by us for np-control chart in chapter II. Hotelling T^2 -chart is a multivariate analog of \bar{x} -control chart. The economic design of \bar{x} -control chart is already developed in the chapter III using the cost model mentioned above. Hook-Jeeves search technique is used to find the optimal values of the sample size, the interval between samples and the critical region of T^2 -control chart. Numerical results are provided for several bivariate problems. When the out of control signal is given by Hotelling T^2 -control chart, the problem of immediate interest is which subset of p variates caused the signal. The solution of this problem is discussed in detail.

6.2 Multivariate Quality Control

There are many situations in which the simultaneous control of two or more related quality characteristics is necessary. For example, Suppose that a bearing has both an inner diameter (x_1) and an outer diameter (x_2) that together determine the quality of the product. Suppose that x_1 and x_2 have a bivariate normal distribution. As both quality characteristics are measurements, they could be controlled by applying the usual \bar{x} -chart to each

characteristic as illustrated in Figure 6.1 . The process is considered to be in control if both the sample means \bar{x}_1 and \bar{x}_2 fall within their respective control limits.

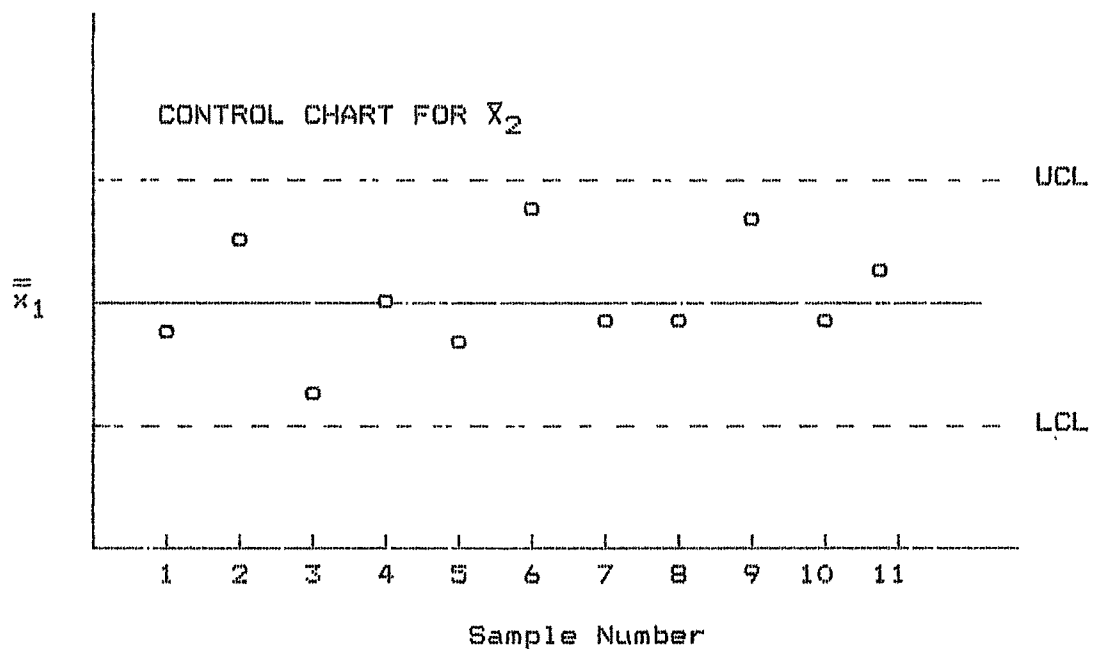
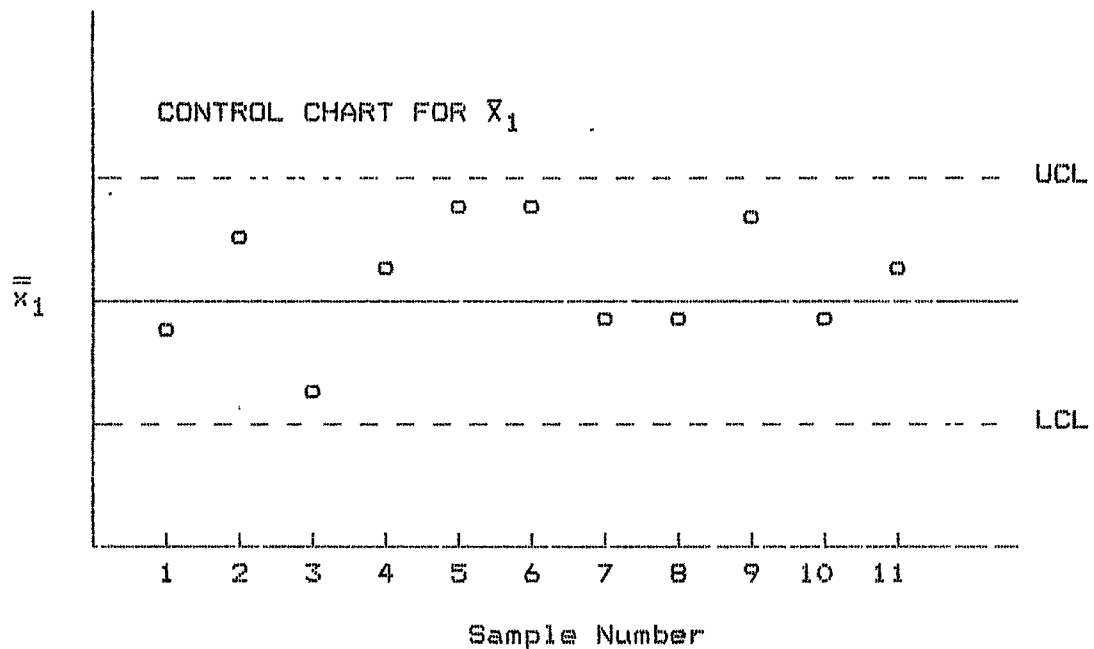


Figure 6.1

Controlling these two quality characteristics independently can be very misleading. Suppose the probability that either \bar{x}_1 or \bar{x}_2 exceeds 3-sigma control limits is 0.0027. However, the joint probability that the sample means for both variables exceed their control limits simultaneously when they are both in control is $(0.0027)(0.0027) = 0.00000729$ which is considerably smaller than 0.0027. Furthermore, the probability that both \bar{x}_1 and \bar{x}_2 will simultaneously plot inside the control limits when the process is really in control is $(0.9973)(0.9973) = 0.9946$ which is different from 0.9973. Therefore, the use of two independent \bar{x} -charts has distorted the simultaneous control of \bar{x}_1 and \bar{x}_2 , in that the probability of type I error and the probability of a point correctly plotting in control are not equal to their given levels for individual control charts.

This distortion in the control procedure increases as the number of quality characteristics increases. In general, if there are p statistically independent quality characteristics for a particular product and if an \bar{x} -chart with $P\{\text{Type I Error}\} = \alpha$ is monitored on each independently, then the true probability of type I error for the joint control procedure is

$$\alpha' = 1 - (1 - \alpha)^P \quad \dots(6.2.1)$$

and the probability that all p means will simultaneously plot inside their control limits when the process is in control is

$$P\{\text{all } p \text{ means plot in control}\} = (1 - \alpha)^P. \quad \dots(6.2.2)$$

Clearly the distortion in the joint control procedure can be severe even for moderate values of p .

Furthermore, if p quality characteristics are not independent, which would usually be the case if they relate to the same product, then equations (6.2.1) and (6.2.2) do not hold. In this situation we have no easy way even to measure the distortion in the joint control procedure.

Quality control problems in which the several related variables are of interest are known as multivariate quality control problems. The original work on multivariate quality control was done by Hotelling (1947). Multivariate quality control is important today, as automatic inspection procedures make it relatively easy to measure many parameters on each unit of the product. Also the installation of PC reduces the complexity of execution of multivariate control charts.

6.3 Hotelling T^2 -Control Chart

Suppose that the output of a process is described by p quality characteristics and that \underline{X} is a $(p \times 1)$ random vector whose j th element is the j th quality characteristic. Suppose that \underline{X} is distributed according to p variate normal so that

$$f(\underline{x}) = [(2\pi)^{p/2} |\Sigma|^{1/2}]^{-1} \exp[-(\underline{x} - \underline{\mu})' \Sigma^{-1}(\underline{x} - \underline{\mu})/2] \quad \dots(6.3.1)$$

We assume that the covariance matrix Σ is unknown but remains constant.

The control procedure for \underline{X} due to Hotelling (1947) is as follows.

From a random sample of size n say $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ we compute the sample mean vector and the sample covariance matrix as

$$\bar{\underline{X}} = \frac{1}{n} \sum_{i=1}^n \underline{X}_i \quad \dots(6.3.2)$$

$$S_{n-1}^{-1} = \frac{1}{n-1} \sum_{i=1}^n (\underline{X}_i - \bar{\underline{X}}) (\underline{X}_i - \bar{\underline{X}})' \quad \dots(6.3.3)$$

We then compute the test statistic

$$T^2 = n(\bar{\underline{X}} - \underline{\mu}_0)' S_{n-1}^{-1} (\bar{\underline{X}} - \underline{\mu}_0) \quad \dots(6.3.4)$$

where $\underline{\mu}_0$ denotes the value of $\underline{\mu}$ corresponding to the in-control state.

Then T^2 is distributed as Hotelling T^2 with p and $n-p$ degrees of freedom.

Define $T_{\alpha,p,n-p}^2$ as the upper 100α percentage point of Hotelling T^2 distribution with p and $n-p$ degrees of freedom such that

$$P[T^2 > T_{\alpha,p,n-p}^2] = \alpha \quad \dots(6.3.5)$$

The decision rule for T^2 -control chart is as follows.

If $T^2 \leq T_{\alpha,p,n-p}^2$, the process is in control.

If $T^2 > T_{\alpha,p,n-p}^2$, the process is out of control.

Thus Hotelling T^2 -control chart has only the upper control limit $T_{\alpha,p,n-p}^2$.

The percentage points of Hotelling T^2 distribution can be found from the tables of the cumulative F distribution since the random variable F given by

$$F = \frac{n-p}{p(n-1)} T^2 \quad \dots(6.3.6)$$

has F distribution with p and $n-p$ degrees of freedom.

We, therefore, have

$$T_{\alpha,p,n-p}^2 = \frac{p(n-1)}{(n-p)} F_{\alpha,p,n-p} \quad \dots(6.3.7)$$

where $F_{\alpha,p,n-p}$ is defined by

$$P [F > F_{\alpha,p,n-p}] = \alpha$$

Samples of size n are taken periodically, the quantity T^2 is computed according to the expression (6.3.4) and T^2 is plotted as a time oriented sequence on T^2 -control chart. The matrix S_{n-1}^{-1} is computed only once from the preliminary sample of size n taken when the process is in control. This practice is suggested by Montgomery and Klatt (1972 b) and also by Heikes, Montgomery and Yeung (1974) while developing the economic design of T^2 -control charts. The justification for this is due to the assumption made that Σ remains constant during the control process.

The power associated with the Hotelling T^2 -control chart depends on the distribution of T^2 when $\underline{\mu} \neq \underline{\mu}_0$. It is shown in several text books such as Anderson (1958) that if $\underline{\mu} \neq \underline{\mu}_0$ then T^2 has noncentral T^2 distribution with p and $n-p$ degrees of freedom and the noncentrality parameter

$$\gamma = n(\underline{\mu} - \underline{\mu}_0)' \Sigma^{-1}(\underline{\mu} - \underline{\mu}_0) \quad \dots(6.3.8)$$

Also it may be shown that if $\underline{\mu} \neq \underline{\mu}_0$ then the random variable F defined by (6.3.6) has noncentral F distribution with p and $n-p$ degrees of freedom and the same noncentrality parameter as given by (6.3.8).

If the variance covariance matrix Σ were known with certainty then S_{n-1}^{-1} in (6.3.4) would be replaced by Σ^{-1} . The statistic T^2 would then be distributed as χ^2 with p degrees of

freedom. Under this situation one has to use χ^2 -control chart with upper control limit $\chi_{\alpha,p}^2$. Here $\chi_{\alpha,p}^2$ is upper 100α percentage point of χ^2 distribution with p degrees of freedom.

We feel that in practice the perfect knowledge of Σ is unusual and hence χ^2 -control chart can be applied rarely. Thus the application of χ^2 chart is restricted even though it is ideal. On the other hand T^2 -control chart is used widely. Hence we prefer to develop the economic design of T^2 -control chart rather than χ^2 -control chart.

The following practical difficulties would arise while developing the economic design of T^2 -control chart .

(1) In the computation of the power of T^2 -control chart the value of noncentrality parameter γ given by (6.3.8) can not be computed exactly as Σ is unknown. Hence one has to use the approximate noncentrality parameter

$$\hat{\gamma} = n(\underline{\mu} - \underline{\mu}_0)' V^{-1}(\underline{\mu} - \underline{\mu}_0) \quad \dots(6.3.9)$$

in place of the true noncentrality parameter γ . Here V is an assessment of Σ based on some sample of suitable size taken when the process is in control.

(2) In the computation of the proportions p_i ($i=0,1$) of nonconforming units also there is a difficulty due to unknown Σ . Hence the approximate values of p_0 and p_1 are obtained by using the assessment V in place of Σ . The derivation of the expressions for p_0 and p_1 is discussed in Section 6.4.4.

In the next Section 6.4 we develop the expected cost model for T^2 -control chart.

6.4 Development of Economic Model for T^2 -control Chart

6.4.1 Introduction

Montgomery and Klatt(1972 b) developed the economic design of T^2 -control chart. The cost model used by them is the single assignable cause version of Knappenberger and Grandage's (1969) cost model for \bar{x} chart. We have seen in chapter II that Knappenberger and Grandage's (1969) cost model involves too many unrealistic assumptions.

In Chapter II we have developed a single assignable cause model which does not involve any of the unrealistic assumptions of Knappenberger and Grandage's (1969) cost model. The improvements of our model over Knappenberger and Grandage's model are already explained in Section 2.2 of Chapter II and hence not explained here.

Using this cost model the economic design of np-control chart is developed in chapter II Section 2.3, and the economic design of control charts for variables are developed in chapter III.

In this chapter the economic design of T^2 -control chart is developed using the same cost model. The expected cost model for T^2 -control chart is developed in the remaining sections of this section.

6.4.2 The Production Process and the Control Procedure

The production process starts in an in-control state in which the process mean vector is $\underline{\mu} = \underline{\mu}_0$. There is only one out-of-control state in which the process mean vector is $\underline{\mu}_1 = \underline{\mu}_0 + \underline{\delta} \sigma$

where the $(p \times 1)$ vectors $\underline{\mu}_0$, $\underline{\delta}$ are known. The time until the process remains in the in-control-state before shifting to the out-of-control state is assumed to be exponential random variable with mean $1/\lambda$ operation hours. The process is not self correcting. Once the process is out of control it stays there until the shift is detected by the chart.

The sampling and the inspection procedure is as follows. A sample of size n is taken after the production of every k units. The sample mean vector \bar{X} is calculated and the quantity T^2 is computed according to the expression (6.3.4). If $T^2 \leq T_{\alpha,p,n-p}^2$, the process is declared to be in control and the production continues. If $T^2 > T_{\alpha,p,n-p}^2$ the process is declared to be out of control. The production may or may not be halted and a search for the assignable cause is undertaken. If the assignable cause exists then the process is corrected and brought to the in-control state.

We want to find the optimal values of the design variables n , k , $T_{\alpha,p,n-p}^2$ which minimize the expected cost per unit of controlling the process during the production cycle.

6.4.3 The Probability of Type-I Error and the Power of T^2 -Control Chart

The probability of type-I error, q_0 , is the probability of concluding that the process is out of control when $\underline{\mu} = \underline{\mu}_0$.

The expression for q_0 is given by

$$q_0 = P [T^2 > T_{\alpha, p, n-p}^2 \mid \underline{\mu} = \underline{\mu}_0] \quad \dots(6.4.1)$$

where T^2 has Hotelling T^2 distribution with p and $n-p$ degrees of freedom.

It may be noted that q_0 represents the probability of a false alarm.

The power of T^2 -control chart, q_1 , is the probability of concluding that the process is out of control when $\underline{\mu} = \underline{\mu}_0 + \underline{\delta\sigma}$.

The expression for q_1 is given by

$$q_1 = P [T^2 > T_{\alpha, p, n-p}^2 \mid \underline{\mu} = \underline{\mu}_0 + \underline{\delta\sigma}] \quad \dots(6.4.2)$$

where T^2 has noncentral T^2 distribution with p and $n-p$ degrees of freedom and noncentrality parameter γ given by the expression (6.3.8).

As described in Section 6.3, it is not possible to compute the exact value of γ because Σ is unknown. Hence one has to use the approximate noncentrality parameter $\hat{\gamma}$ given by (6.3.9) in place of γ in the computation of q_1 .

6.4.4 The Proportions of Nonconforming Units

Let p_0 be the proportion of nonconforming units when the process mean vector is $\underline{\mu}_0$. Let p_1 be the proportion of nonconforming units when the process mean vector is $\underline{\mu} = \underline{\mu}_0 + \underline{\delta\sigma}$.

Let $\underline{u}_1, \underline{u}_2$ be the given specification limit vectors. Assuming that a unit is nonconforming if its measurements fall outside the given specification limits $(\underline{u}_1, \underline{u}_2)$, the expressions for p_0 and p_1 are as follows.

$$p_0 = 1 - P(\underline{u}_1 \leq \underline{X} \leq \underline{u}_2 \mid \underline{\mu} = \underline{\mu}_0) \quad \dots(6.4.3)$$

$$p_1 = 1 - P(\underline{u}_1 \leq \underline{X} \leq \underline{u}_2 \mid \underline{\mu} = \underline{\mu}_0 + \underline{\delta\sigma}) \quad \dots(6.4.4)$$

Recalling that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ one can see that the evaluation of p_0 and p_1 requires the knowledge of Σ . In the absence of such knowledge, one has to obtain p_0 and p_1 approximately replacing Σ by its assessment V .

6.4.5 The Expected Cost Model

Since the basic structure of the model under study of this chapter is the same as the structure of single assignable cause model developed in the Section 2.3 of Chapter II, the total expected cost incurred during the production cycle for T^2 -control chart is given by

$$E(C) = (a_1 + a_2 n)N + a_{3,1}q_0\theta/(1-\theta) + a_{3,2} + a_{4,1}S + a_{4,2}(D-S) \quad \dots(6.4.5)$$

where the definitions and the meaning of the various constants involved are already explained in Chapter II.

The expressions for q_0 and q_1 required in the computation of $E(C)$ are derived in the Section 6.4.3 and are given by (6.4.1) and (6.4.2) respectively. The expressions for p_0 and p_1 are derived in the Section 6.4.4 and are given by (6.4.3) and (6.4.4) respectively.

The total expected cost per unit of the product, ECPU, is then given by

$$ECPU = \frac{E(C)}{Nk} \quad \dots(6.4.6)$$

In the next Section we give the method for finding the optimal values of n , k , $T_{\alpha,p,n-p}^2$ which minimize ECPU given by (6.4.6) for bivariate case.

6.5 Solution Method and Numerical Examples

(A) Solution Method

We consider the bivariate sample problem given by Montgomery and Klatt (1972 a). The systems parameters and the cost coefficients, reproduced from their example, are as follows.

$$\underline{\mu}_0 = [0,0],$$

$$\text{Assessment of } \Sigma = 'V = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix},$$

The specification limit vectors

$$\underline{u}_1 = [-3,-3], \underline{u}_2 = [3,3],$$

Three shifts to be investigated are (i) 2σ (ii) 2.5σ (iii) 3σ . The corresponding three values of $\underline{\mu}_1$ are (i) $\underline{\mu}_1' = [2,2]$ (ii) $\underline{\mu}_1' = [2.5,2.5]$, (iii) $\underline{\mu}_1' = [3,3]$, so that (i) $\underline{\delta}' = [2,2]$, (ii) $\underline{\delta}' = [2.5,2.5]$, (iii) $\underline{\delta}' = [3,3]$. The cost coefficients are $a_1 = \$ 1$, $a_2 = \$ 0.1$, $a_{3,1} = \$ 10$, $a_{3,2} = \$ 10$, $a_{4,1} = \$ 1$, $a_{4,2} = \$ 1.5$, $\lambda = 1$, $R = 10000$.

The noncentrality parameter, for the bivariate case, is computed using the following expression.

$$\begin{aligned} \gamma &= \frac{n}{1-\rho^2} \left[\left(\frac{\mu_{11}-\mu_{01}}{\sigma_1} \right)^2 - 2\rho \left(\frac{\mu_{11}-\mu_{01}}{\sigma_1} \right) \left(\frac{\mu_{12}-\mu_{02}}{\sigma_2} \right) + \left(\frac{\mu_{12}-\mu_{02}}{\sigma_2} \right)^2 \right] \\ &= \frac{n}{1-\rho^2} [\delta_1^2 - 2\rho\delta_1\delta_2 + \delta_2^2] \quad \dots(6.5.1) \end{aligned}$$

To evaluate the above expression (6.5.1) we have used $\rho = 0.5$ from the assessment V of Σ . This approximate evaluation of γ is used in the computation of q_1 .

It is clear from the expression (6.5.1) that the noncentrality parameter γ is independent of σ_1 and σ_2 and depends upon only the correlation coefficient ρ so far the elements of Σ are concerned.

We now explain the method for computing the proportions of nonconforming units p_0 and p_1 .

The value of p_0 and the three values of p_1 for the three shifts are calculated using Table 2 of Pearson and Hartely Vol II and the results 26.3.7, 26.3.8, 26.3.9 and 26.3.10 given by Abramowitz and Stegun (1972) on pp 936. These results are as listed below.

$$26.3.7 \quad L(h, k, \rho) = L(k, h, \rho)$$

$$26.3.8 \quad L(-h, k, \rho) + L(h, k, -\rho) = Q(k)$$

$$26.3.9 \quad L(-h, -k, \rho) - L(h, k, \rho) = P(k) - Q(h)$$

$$26.3.10 \quad 2[L(h, k, \rho) + L(h, k, -\rho) + P(h) - Q(k)]$$

$$= \int_{-h}^h \int_{-k}^k g(x, y, \rho) dx dy$$

where

$$L(h, k, \rho) = \int_h^{\infty} dx \int_k^{\infty} g(x, y, \rho) dy,$$

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-t^2/2) dt,$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-t^2/2) dt,$$

$$g(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp [-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)]$$

In the evaluation of p_0 and p_1 the assessment V is used in place of Σ .

In computation of q_0 and q_1 , F and noncentral F distributions are utilized.

Let $P(F | v_1, v_2)$ be the distribution function of F with v_1, v_2 degrees of freedom.

$$\text{Let } Q(F | v_1, v_2) = 1 - P(F | v_1, v_2)$$

Then

$$Q(F | v_1, v_2) = I_x\left(\frac{v_2}{2}, \frac{v_1}{2}\right) \quad \dots(6.5.2)$$

where $I_x\left(\frac{v_2}{2}, \frac{v_1}{2}\right)$ is the incomplete beta integral with

$$x = \frac{v_2}{v_2 + v_1 F}.$$

For bivariate case we have $v_1 = 2, v_2 = n-2$.

We therefore have

$$q_0 = Q(F \mid 2, n-2) = I_x\left(-\frac{n-2}{2}, 1\right) = x^{\frac{n-2}{2}} \dots (6.5.3)$$

Thus the computation of q_0 becomes easy in bivariate case.

Let $P(F \mid v_1, v_2, \gamma)$ be the distribution function of noncentral F with v_1, v_2 degrees of freedom and the noncentrality parameter γ .

We have

$$P(F \mid v_1, v_2, \gamma) = \sum_{j=0}^{\infty} \frac{\exp(-\gamma/2)(\gamma/2)^j}{j!} I_{(1-x)}\left(\frac{v_1}{2}+j, \frac{v_2}{2}\right) \dots (6.5.4)$$

For bivariate case this expression becomes

$$P(F \mid 2, n-2, \gamma) = \sum_{j=0}^{\infty} \frac{\exp(-\gamma/2)(\gamma/2)^j}{j!} I_{(1-x)}\left(1+j, \frac{n-2}{2}\right) \dots (6.5.5)$$

The probability q_1 given by the expression (6.4.2) can be obtained by using (6.5.5).

A computer program on Fortran is developed for evaluation of ECPU for given values of $n, k, T_{\alpha, 2, n-2}^2$.

This program uses a subroutine for the computation of incomplete beta integrals $I_{(1-x)}\left(1+j, \frac{n-2}{2}\right)$ $j = 0, 1, 2, \dots$ required in computation of q_1 . This program computes q_0 and q_1 and ultimately ECPU. The values p_0 and p_1 are computed externally and supplied as input parameters of the program. This program is linked to Hooke-Jeeves technique to find the optimal values of $n, k, T_{\alpha, 2, n-2}^2$. The listing of the program is given at the end of this chapter.

(B) Numerical Findings

In the following Table 6.1 we give the optimal values of the design variables n , k , T^2 with minimum ECPU for the three shifts.

Table 6.1

Shift	P_0	P_1	q_0	q_1	n	k	T^2	ECPU
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
[2, 2]	0.0053	0.2548	0.0107	0.9898	10	370	19.00	0.0211
[2.5, 2.5]	0.0053	0.4538	0.0148	0.9938	8	370	21.50	0.0260
[3, 3]	0.0053	0.6667	0.0087	0.9982	8	190	27.00	0.0279

From this table we observe that the larger shift in the process mean leads to smaller sample size, slightly smaller (or same) sampling intervals and larger control limits. The same trend has been observed by Montgomery and Klatt (1972a) while studying the economic design of T^2 -control chart using Knappenberger and Grandage's (1969) model.

We now consider the effect of changing the sign of ρ on the three design variables.

$$\text{Let } \rho = \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 1.1 \end{bmatrix}.$$

The effect of this change in the sign of ρ is studied on the optimal design variables in case of shift [2, 2]. The results are given in the Table 6.2.

Table 6.2

Design Variables	$\rho = 0.5$	$\rho = -0.5$
(1)	(2)	(3)
n	10.00	6.00
k	370.00	370.00
T^2	19.00	32.00
ECPU	0.0211	0.0219

The column(2) gives the optimal design variables and minimum ECPU in case of shift [2, 2] when $\rho = 0.5$. The column (3) gives the optimal design variables and minimum ECPU derived in case of shift [2, 2] when $\rho = -0.5$.

Comparing the columns (2) and (3) one can see that negative value of ρ leads to smaller sample size, slightly smaller (or the same) sampling interval, and larger control limit.

This is the same effect observed while increasing the magnitude of the shift of positive ρ . This is not unexpected because increasing δ and negative ρ both increase the power of the test.

6.6 Determination of Out-of-Control Variables

6.6.1 The main advantage of T^2 -control chart is that the state of the production process is characterized by a single number. However, if a point falls outside the control limits, it is not immediately obvious which subset of r (rip) quality characteristics are out of control. The work done in this respect is cited in the next para.

Jackson (1959) and Jackson and Mudholkar (1979) have suggested principle components technique to deal with this problem. Montgomery and Wadsworth (1971) have suggested simultaneous confidence intervals to deal with this problem. Murphy (1987) has suggested discriminant analysis for selecting the out of control variables. Recently Hawkins (1991) has suggested the multivariate quality control based on regression adjusted variables.

The method given by Murphy (1987) using discriminant analysis is described in the next Section.

6.6.2 Determination of out of control variables when Σ is known

The method based on discriminant analysis given by Murphy (1987) is as follows.

The decision on whether the process is in control or not is based on χ^2 -control chart mentioned in the Section 6.3. Samples of size n are taken periodically and the quantity

$$\chi^2 = n (\bar{X} - \underline{\mu}_0)' \Sigma^{-1} (\bar{X} - \underline{\mu}_0) \quad \dots(6.6.1)$$

is computed.

The decision rule is as follows.

If $\chi^2 \leq \chi^2_{\alpha, p}$, the process is in control.

If $\chi^2 > \chi^2_{\alpha, p}$, the process is out of control.

Here $\chi^2_{\alpha, p}$ is the upper 100α percentage point of χ^2 distribution with p degrees of freedom so that

$$P [\chi^2_{\alpha} > \chi^2_{\alpha, p}] = \alpha \quad \dots(6.6.2)$$

When $\underline{\mu} = \underline{\mu}_0$, χ^2 is distributed as χ^2 with p degrees of freedom.

When $\underline{\mu} \neq \underline{\mu}_0$, χ^2 is distributed as noncentral χ^2 with p degrees of freedom and noncentrality parameter

$$\gamma = n (\underline{\mu} - \underline{\mu}_0)' \Sigma^{-1} (\underline{\mu} - \underline{\mu}_0) \quad \dots(6.6.3)$$

When the out of control signal is given by χ^2 -control chart, the question of immediate interest is which subset of p variables caused the signal. An effective approach is to partition \bar{X} , $\underline{\mu}_0$, Σ as follows

$$\bar{X} = \left[\begin{array}{c} \bar{X}^{(1)} \\ \bar{X}^{(2)} \end{array} \right]_{p-r} \quad \underline{\mu}_0 = \left[\begin{array}{c} \underline{\mu}_0^{(1)} \\ \underline{\mu}_0^{(2)} \end{array} \right]_{p-r}$$

$$\Sigma = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array} \right]_{\substack{r \quad p-r}}$$

where $\bar{X}^{(1)}$ is the mean of the subset of r variables which we suspect caused the signal, and $\bar{X}^{(2)}$ is the mean of the remaining $p-r$ variables. $\underline{\mu}_0$ and Σ are partitioned as \bar{X} .

We compute the full squared distance as

$$\chi_p^2 = n(\bar{X} - \underline{\mu}_0)' \Sigma^{-1} (\bar{X} - \underline{\mu}_0) \quad \dots(6.6.4)$$

We compute the reduced squared distance corresponding to the subset of r variables as

$$\chi_r^2 = n(\bar{X}^{(1)} - \underline{\mu}_0^{(1)})' \Sigma_{11}^{-1} (\bar{X}^{(1)} - \underline{\mu}_0^{(1)}) \quad \dots(6.6.5)$$

Consider the difference

$$D = \chi_p^2 - \chi_r^2 \quad \dots(6.6.6)$$

If D is small we accept the hypothesis that the subset of r variable caused the signal, if D is large we reject the hypothesis.

This D test is similar to the one which is well known in discriminant analysis for dealing with the variable selection problem.

In discriminant analysis we define the true full squared distance between the populations with means $\underline{\mu}_0$ and $\underline{\mu}$ as

$$\Delta_p^2 = n (\underline{\mu} - \underline{\mu}_0)' \Sigma^{-1} (\underline{\mu} - \underline{\mu}_0) \quad \dots(6.6.7)$$

and the true reduced squared distance as

$$\Delta_r^2 = n (\underline{\mu}^{(1)} - \underline{\mu}_0^{(1)})' \Sigma_{11}^{-1} (\underline{\mu}^{(1)} - \underline{\mu}_0^{(1)}) \quad \dots(6.6.8)$$

Then to test $H_0 : \Delta_p^2 = \Delta_r^2$ is equivalent to test that subset of r variables discriminate as good as the full set of p variables.

Murphy (1987) has shown that when $H_0 : \Delta_p^2 = \Delta_r^2$ is true, D defined in (6.6.6) is distributed as χ^2 with $p-r$ degrees of freedom.

The above method given by Murphy (1987) is applicable if Σ is known. As we have seen in the Section 6.3, the knowledge of Σ is unusual, and hence the application of Murphy's (1987) method based on χ^2 -control chart is restricted. In the next Section we describe the method for determining the out of control variables when Σ is unknown. The appropriate F test used in this method is based on Rao's U statistic described by Kshirsagar (1972).

6.6.3 A Method for Determination of out of control Variables when Σ is unknown

The decision on whether the process is in control or not is

based on Hotelling T^2 -control chart described in the section 6.3. When out of control signal is given by Hotelling T^2 -control chart, we are interested in finding which subset of r variables of the given set of p variables caused the signal. The method for finding this is as follows.

Partition $\underline{\bar{X}}$, $\underline{\mu}_0$, Σ as done in the Section 6.6.2. The sample covariance matrix S_{n-1} is also partitioned as Σ .

$$S_{n-1} = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array} \right] \begin{array}{l} r \\ p-r \end{array}$$

We compute the full squared distance based on the sample of p variables as

$$T_p^2 = n(\underline{\bar{X}} - \underline{\mu}_0)' S_{n-1}^{-1} (\underline{\bar{X}} - \underline{\mu}_0) \quad \dots(6.6.9)$$

and the reduced squared distance based on the sample of r ($r < p$) variables as

$$T_r^2 = n(\underline{\bar{X}}^{(1)} - \underline{\mu}_0^{(1)})' S_{11}^{-1} (\underline{\bar{X}}^{(1)} - \underline{\mu}_0^{(1)}) \quad \dots(6.6.10)$$

We then find Rao's U statistic as

$$U = \frac{1 + T_r^2/(n-1)}{1 + T_p^2/(n-1)} \quad \dots(6.6.11)$$

Let Δ_p^2 and Δ_r^2 be the true full squared distance and the true reduced squared distance as defined by (6.6.7) and (6.6.8) respectively.

To test $H_0 : \Delta_p^2 = \Delta_r^2$ is equivalent to test that the subset of r variables discriminates as good as the full set of p variables.

Taking vector $\underline{u} = \sqrt{n} (\bar{\underline{X}} - \underline{\mu}_0)$ and noting that $\underline{u} \sim N_p(\sqrt{n}(\underline{\mu} - \underline{\mu}_0), \Sigma)$, and using Theorem 4 of Section 3 of Chapter 5 given by Kshirsagar (1972) it can be seen that when $H_0 : \Delta_p^2 = \Delta_r^2$ is true then

$$\begin{aligned} F_r(\bar{\underline{X}}^{(1)}) &= \frac{n-p}{p-r} \left[\frac{1}{U} - 1 \right] \\ &= \frac{n-p}{p-r} \frac{T_p^2 - T_r^2}{(n-1) + T_r^2} \end{aligned} \quad \dots(6.6.12)$$

is distributed as F with $p-r$ and $n-p$ degrees of freedom. This statment that the above statistic is distributed as F under H_0 is also in accordance with the Theorem 2.11 on page 52 of Seber (1984). If $F_r(\bar{\underline{X}}^{(1)})$ is small we accept the hypothesis that the subset of r variables caused the signal, if $F_r(\bar{\underline{X}}^{(1)})$ is large we reject the hypothesis. This conclusion is similar to the one derived by Murphy (1987) for the case where Σ is known.

Particular Case : $p = 2$

Using the above method we study the situation when there are only two characteristics under study.

Let $\underline{X} \sim N_2(\underline{\mu}, \Sigma)$.

Let $\bar{\underline{X}}, \underline{\mu}_0, \Sigma, S_{n-1}$ be partitioned as

$$\bar{\underline{X}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix} \quad \underline{\mu}_0 = \begin{bmatrix} \mu_{01} \\ \mu_{02} \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix}$$

$$\Sigma = \left[\begin{array}{c|c} \sigma_1^2 & \sigma_{12} \\ \hline \sigma_{21} & \sigma_2^2 \end{array} \right] \begin{array}{l} 1 \\ 1 \end{array} \quad S_{n-1} = \left[\begin{array}{c|c} s_1^2 & s_{12} \\ \hline s_{21} & s_2^2 \end{array} \right] \begin{array}{l} 1 \\ 1 \end{array}$$

where

$$s_1^2 = \frac{1}{n-1} \sum_{i=1}^n (x_{i1} - \bar{x}_1)^2$$

$$s_2^2 = \frac{1}{n-1} \sum_{i=1}^n (x_{i2} - \bar{x}_2)^2$$

$$s_{12} = \frac{1}{n-1} \sum_{i=1}^n (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2)$$

For a given bivariate sample we compute

$$\begin{aligned} T_2^2(\underline{X}) &= n (\underline{X} - \underline{\mu}_0)' S_{n-1}^{-1} (\underline{X} - \underline{\mu}_0) \\ &= \frac{n}{s_1^2 s_2^2 - s_{12}^2} \left[s_2^2 (\bar{x}_1 - \mu_{01})^2 + s_1^2 (\bar{x}_2 - \mu_{02})^2 \right. \\ &\quad \left. - 2s_{12}(\bar{x}_1 - \mu_{01})(\bar{x}_2 - \mu_{02}) \right] \\ &\dots(6.6.13) \end{aligned}$$

If $T_2^2(\underline{X}) > T_{\alpha,2,n-2}^2$ the process is declared to be out of control. When the out of control signal is given by T^2 -control chart, we are interested in finding which of the two variables x_1 , x_2 or both caused the signal. We put $p = 2$ and $r = 1$ in the procedure described earlier. Then for this particular case we calculate

$$T_1^2(\bar{x}_1) = \frac{n(\bar{x}_1 - \mu_{01})^2}{s_1^2} \quad \dots(6.6.14)$$

$$T_1^2(\bar{x}_2) = \frac{n(\bar{x}_2 - \mu_{02})^2}{s_2^2} \quad \dots(6.6.15)$$

$$F_1(\bar{x}_1) = \frac{(n-2) [T_2^2(\bar{X}) - T_1^2(\bar{x}_1)]}{(n-1) + T_1^2(\bar{x}_1)} \quad \dots(6.6.16)$$

$$F_1(\bar{x}_2) = \frac{(n-2) [T_2^2(\bar{X}) - T_1^2(\bar{x}_2)]}{(n-1) + T_1^2(\bar{x}_2)} \quad \dots(6.6.17)$$

Let $F_{\alpha,1,n-2}$ be the upper 100α percentage point of F distribution with 1 and $n-2$ degrees of freedom.

Then one of the following four possibilities will occur.

- (A) $F_1(\bar{x}_1) \leq F_{\alpha,1,n-2}$, $F_1(\bar{x}_2) > F_{\alpha,1,n-2}$
- (B) $F_1(\bar{x}_1) > F_{\alpha,1,n-2}$, $F_1(\bar{x}_2) \leq F_{\alpha,1,n-2}$
- (C) $F_1(\bar{x}_1) \leq F_{\alpha,1,n-2}$, $F_1(\bar{x}_2) \leq F_{\alpha,1,n-2}$
- (D) $F_1(\bar{x}_1) > F_{\alpha,1,n-2}$, $F_1(\bar{x}_2) > F_{\alpha,1,n-2}$

It may be noted that the above $F_1(\bar{x}_1)$, $F_1(\bar{x}_2)$ statistics are calculated after receiving the out of control signal by the T^2 -control chart. Using this information we draw the following conclusions.

If (A) occurs, we conclude that x_1 alone caused the signal.

If (B) occurs, we conclude that x_2 alone caused the signal.

If either (C) or (D) occurs, we conclude that both x_1 and x_2 caused the signal.

C LISTING OF CHAPTER VI

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      SUBROUTINE MULT2(RK,NSTAGE,SUM,A1,A2,A3,A3P,A4,A4P,ALEMDA,
1    RATE,CPN,FNOT,FONE)
C    FILE NAME IS MCCA
C    PROGRAM FOR ECPU OF MULTIVARIATE TSQR CHART
      DIMENSION P(100),Q(100),R(100),RK(10),T(100)
      WRITE(*,2) A1,A2,A3,A3P,A4,A4P
2    FORMAT(1X,'A1=',F10.4,'A2=',F10.4,'A3=',F10.4,'A3P=',F10.4,
1    'A4=',F10.4,'A4P=',F10.4)
      WRITE(*,4) ALEMDA,RATE,CPN
4    FORMAT(1X,'ALEMDA=',F10.4,'RATE=',F10.4,'CPN=',F10.4)
      N=RK(1)
      K=RK(2)
      TSQR=RK(3)
      WRITE(*,6) N,K,TSQR
6    FORMAT(1X,'N=',I5,'K=',I5,'TSQR=',F10.4)
      WRITE(*,8) FNOT,FONE
8    FORMAT(1X,'FNOT=',F10.6,'FONE=',F10.6)
      CPN1=N*CPN
      POWER=ALEMDA*K/RATE
      PPOWER=-POWER
      THEETA=EXP(PPOWER)
      WRITE(*,9) THEETA
9    FORMAT(1X,'THEETA=',F10.6)
      F=(N-2)*TSQR/(2*(N-1))
      PNOT=(N-2)/((N-2)+2*F)
      QNOT=(PNOT)**((N-2)/2)
      WRITE(*,10) PNOT,QNOT
10   FORMAT(1X,'PNOT=',F10.6,'QNOT=',F10.6)
      DO 12 J=1,90
        NN=N-2
        MM=J
        X=2*F/(2*F+(N-2))
        CALL BETA(NN,MM,X,BI)
        P(J)=BI
        IF( P(J).LT.0.00001) GO TO 13
12   CONTINUE
13   ISTOP=J-1
      WRITE(*,20) (P(J),J=1,ISTOP)
20   FORMAT(1X,7F10.6)
      POW=CPN1/2
      PPOW=-POW
      R(1)=EXP(PPOW)*POW*P(1)
      IST=ISTOP-1
      DO 16 J=1,IST
16   R(J+1)=POW*P(J+1)*R(J)/(P(J)*(J+1))
      WRITE(*,20) (R(J),J=1,ISTOP)
      RNOT=EXP(PPOW)*(1-QNOT)
      TEM=RNOT
      DO 35 J=1,ISTOP
        TEM=TEM+R(J)
35   CONTINUE
      QONE=1-TEM
      write(*,45)qone
45   FORMAT(1X,'QONE=',F10.6)
      TNOS=THEETA/(1-THEETA)+1/QONE
      NOS=TNOS+0.5
      WRITE(*,50)NOS

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50  FORMAT(1X,'NOS=',I5)
    EC1=(A1+A2*N)*NOS
    BNOT=QNOT*THEETA/(1-THEETA)
    EC2=A3*BNOT+A3P
    TAW=(1-(1+POWER)*THEETA)/(1-THEETA)
    WRITE(*,55) TAW
55  FORMAT(1X,'TAW=',F10.4)
    H=K/RATE
    D=(RATE*FNOT/ALEMDA)+(H/QONE-TAW)*RATE*FONE
    S=THEETA*N*FNOT/(1-THEETA)+N*FONE/QONE
    WRITE(*,60) D,S
60  FORMAT(1X,'D=',F12.4,'S=',F12.4)
    EC3=A4*S+A4P*(D-S)
    EC=EC1+EC2+EC3
    ECPU=EC/(NOS*K)
    WRITE(*,65) EC1,EC2,EC3,EC,ECPU
65  FORMAT(1X,'EC1=',F12.4,'EC2=',F12.4,'EC3=',F12.4,'EC=',F12.4,
1    'ECPU=',F12.4)
    SUM=ECPU
    RETURN
    END

    SUBROUTINE MULT2(RK,NSTAGE,SUM,A1,A2,A3,A3P,A4,A4P,ALEMDA,
1  RATE,CPN,FNOT,FONE)
C  FILE NAME IS MCCB
C  PROGRAM FOR ECPU OF MULTIVARIATE TSQR CHART
    DIMENSION P(100),Q(100),R(100),RK(10),X(100),T(100)
    WRITE(*,2) A1,A2,A3,A3P,A4,A4P
2  FORMAT(1X,'A1=',F10.4,'A2=',F10.4,'A3=',F10.4,'A3P=',F10.4,
1  'A4=',F10.4,'A4P=',F10.4)
    WRITE(*,4) ALEMDA,RATE,CPN
4  FORMAT(1X,'ALEMDA=',F10.4,'RATE=',F10.4,'CPN=',F10.4)
    N=RK(1)
    K=RK(2)
    TSQR=RK(3)
    WRITE(*,6) N,K,TSQR
6  FORMAT(1X,'N=',I5,'K=',I5,'TSQR=',F10.4)
    WRITE(*,8) FNOT,FONE
8  FORMAT(1X,'FNOT=',F10.6,'FONE=',F10.6)
    CPN1=N*CPN
    POWER=ALEMDA*K/RATE
    PPOWER=-POWER
    THEETA=EXP(PPOWER)
    WRITE(*,9) THEETA
9  FORMAT(1X,'THEETA=',F10.6)
    F=(N-2)*TSQR/(2*(N-1))
    PNOT=(N-2)/((N-2)+2*F)
    QNOT=(PNOT)**((N-2)/2)
    WRITE(*,10) PNOT,QNOT
10  FORMAT(1X,'PNOT=',F10.6,'QNOT=',F10.6)
    DO 12 J=1,90
    MM=(N-2)/2
    NN=MM+J
    NT=NN-MM+1
    PROB=(N-2)/((N-2)+2*F)

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      CALL BIN(PROB,MM,NT,CPR,CPL,PI)
      Q(J)=CPR
      P(J)=1-Q(J)
      IF( P(J).LT.0.00001) GO TO 13
12    CONTINUE
13    ISTOP=J-1
      WRITE(*,20) (P(J),J=1,ISTOP)
20    FORMAT(1X,7F10.6)
      POW=CFN1/2
      PPOW=-POW
      R(1)=EXP(PPOW)*POW*P(1)
      IST=ISTOP-1
      DO 16 J=1,IST
16    R(J+1)=POW*P(J+1)*R(J)/(P(J)*(J+1))
      WRITE(*,20) (R(J),J=1,ISTOP)
      RNOT=EXP(PPOW)*(1-QNOT)
      TEM=RNOT
      DO 35 J=1,ISTOP
      TEM=TEM+R(J)
35    CONTINUE
      QONE=1-TEM
      write(*,45)qone
45    FORMAT(1X,'QONE=',F10.6)
      TNOS=THEETA/(1-THEETA)+1/QONE
      NOS=TNOS+0.5
      WRITE(*,50)NOS
50    FORMAT(1X,'NOS=',I5)
      EC1=(A1+A2*N)*NOS
      BNOT=QNOT*THEETA/(1-THEETA)
      EC2=A3*BNOT+A3P
      TAW=(1-(1+POWER)*THEETA)/(1-THEETA)
      WRITE(*,55) TAW
55    FORMAT(1X,'TAW=',F10.4)
      H=K/RATE
      D=(RATE*FNOT/ALEMDA)+(H/QONE-TAW)*RATE*FONE
      S=THEETA*N*FNOT/(1-THEETA)+N*FONE/QONE
      WRITE(*,60) D,S
60    FORMAT(1X,'D=',F12.4,'S=',F12.4)
      EC3=A4*S+A4P*(D-S)
      EC=EC1+EC2+EC3
      ECPU=EC/(NOS*K)
      WRITE(*,65)EC1,EC2,EC3,EC,ECPU
65    FORMAT(1X,'EC1=',F12.4,'EC2=',F12.4,'EC3=',F12.4,'EC=',F12.4,
1    'ECPU=',F12.4)
      SUM=ECPU
      RETURN
      END

```

```

      SUBROUTINE BETA(NN,MM,X,BI)
C   FILE NAME IS BTIC1
C   COMPUTATION OF INCOMPLETE BETA INTEGRAT
      DIMENSION TT(100),B(100)
      MM1=MM-1
      B(1)=2.0/NN
      DO 60 J=1,MM1

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        B(J+1)=(J/(J+NN/2.0))*B(J)
60      CONTINUE
        BAB=B(MM)
        RM=MM
        RN=NN
        TT(1)=(2*RM+RN)*X/(2*(RM+1))
        SUM=TT(1)
        DO 70 K=1,19
            RK=K
            COE=(2*RM+RN+2*RK)/(2*(RM+RK+1))
            TT(K+1)=COE*X*TT(K)
            SUM=SUM+TT(K+1)
70      CONTINUE
        CURLY=1+SUM
        RNN=NN/2.0
        TE=(X**RM)*((1-X)**RNN)/RM
        TEE=TE*CURLY
        BI=TEE/BAB
        RETURN
        END

```