CHAPTER VII

Economic Design of np-Control Charts with Different Control Limits for Different Assignable Causes.

7.1 In this chapter we propose an np-control chart with two upper control limits defining three regions on the chart. The lower control limit of the chart is assumed to be zero. This chart is used to monitor a production process subject to two assignable causes. The advantage of this chart is that the chart not only detects the shift in the process but also suggests which corrective action is to be taken.

We develop the expected cost model for the proposed np-control chart with two upper control limits. The optimal values of the design variables are obtained for this cost model.

A comparision is made between the proposed two upper control limits chart and the "matched" single upper control limit chart from the cost point of view. It is observed that the former is an improvement over the latter.

The further improvement in the cost is achieved by using the curtailed sampling policy in place of the complete sampling policy in the expected cost model of the np-control chart with two upper control limits.

In Section 7.7 the use of curtailed sampling is discussed. Earlier sections are devoted to complete sampling.

7.2 Need for the Proposed Control Chart

7.2.1 The past research on the economic design of the control charts can be classified along two dimensions :

(i) Whether the process is subject to a single or the multiple assignable causes.

(ii) Whether the quality characteristic is measured on a continuous scale (variable control chart) or on a discrete scale (attribute control chart).

Duncan (1956) developed the economic design of Shewhart type $\bar{X}$-chart to control a process subject to a single assignable cause. Similar work on the control chart for attribute has been studied by Landy (1973), Chiu (1974) and Gibra (1978) for a process subject to a single assignable cause.

Lorenzen and Vance (1986) presented a unified approach for the economic design of the control charts for variables and attributes in the presence of a single assignable cause.

The economic models of variable control charts for a process subject to multiple assignable causes include those of Knappenberger and Grandage (1969) and Duncan (1971). The economic models of attribute control charts for a process subject to multiple assignable causes have been proposed by Montgomery, Heikes and Mance (1975), Chiu (1976), Gibra (1981) and Williams, Looney and Peters (1985, 1990).

7.2.2 A common characteristic in all the past research mentioned above is that a single response mode has been assumed even when there are multiple assignable causes shifting the process to different out-of-control states. Specifically, when the control chart indicates that the process is out of control, the same action is always taken, consisting of a perfect search for the assignable cause followed by an appropriate restoration procedure.

However, a single response mode is inadequate when there exist multiple assignable causes requiring different levels of restoration. Thereby a search for the assignable cause in effect is very expensive and time consuming. In such cases, it is desirable to use a control chart having different control limits to detect different assignable causes.

7.2.3 An introduction of three decision criteria done by two pairs of control limits has previously been proposed by Schmidt, Bannet and Case (1980) in their research work on acceptance sampling plans by variables. The two pairs of control limits divide the chart into three regions. An inspected lot may be (i) accepted or (ii) rejected and screened or (iii) rejected and scrapped according to the region of the chart in which the sample mean is plotted. As a matter of fact, rather from the historical point of view, we would like to mention that this concept is almost similar to the concept introduced by Pandey (1974) in his research work on acceptance sampling plans by attributes. Pandey (1974) introduced the three decision criteria done by two acceptance numbers. If x is the number of nonconforming units found in the sample of size n, then the inspected lot may be

(i) accepted if $0 \leq x \leq c_1$

or (ii) screened if $c_1 < x \leq c_2$

or (iii) rejected if $c_2 < x \leq n$.

Later on Tagaras and Lee (1988) used this concept in the construction of the economic design of control chart for sample means where the location of the sample point on the chart is not associated with the decision of acceptance / rejection of the lot but is associated with the determination of the appropriate corrective action. We propose to use this concept in designing the economic np-control chart under both uncurtailed and curtailed sampling.

Surprisingly, there is no mention of Pandey's (1974) work on three decision criteria by Schmidt et al. (1980) and by Tagaras and Lee (1988).

7.3 The Production Process and the Inspection Procedure.

The production process has an in-control state E_0 and two out-of-control states E_1 and E_2. The state E_1 represents a minor problem in the process caused by the assignable cause 1. The state E_2 represents a major problem in the process caused by the assignable cause 2. When the process is in the state E_i (i=0,1,2), the proportion of nonconforming units produced is p_i (i=0,1,2) such that $p_0 < p_1 < p_2$.

The production process starts in the in-control state. The time until the occurrences of the assignable causes 1 and 2 are assumed to be independently and exponentially distributed with means $1/\lambda_1$ and $1/\lambda_2$ hours of operating time respectively.

When the process is in the state E_1, the time until the occurrence of the assignable cause 2 is again exponential with mean $1/\lambda_{12}$ hours of operating time. When the process is in state E_2 it stays there until the external intervention and restoration are done to bring the process to the state E_0.

The process is observed by examining a sample of n units at the fixed intervals of h hours of operating time. Suppose R units are produced per hour of operating time. Then the number of units produced between two successive samples is $hR = k$. Let d be the number of nonconforming units found in the sample.

If $d < m_1$, then the process is declared to be in control. No action is taken and the production is continued.

If $m_1 \le d < m_2$, then the process is declared to be in state E_1 and the first level of action is taken. The first level of action corresponds to a minor adjustment in the process and is designed to restore the process from the state E_1 to the state E_0.

If $d \ge m_2$, then the process is declared to be in the state E_2 and the second level of action is taken. The second level of action calls for a major and more costly intervention. It is designed to restore the process from the state E_2 to the state E_0.

It is assumed that if the process is in the state E_2, then the first level of action will not be able to restore the process to E_0. However, the second level of action will always restore the process to E_0 regardless of whether the process has been in E_1 or E_2.

A typical np-control chart with two upper control limits is given in following figure 7.1.

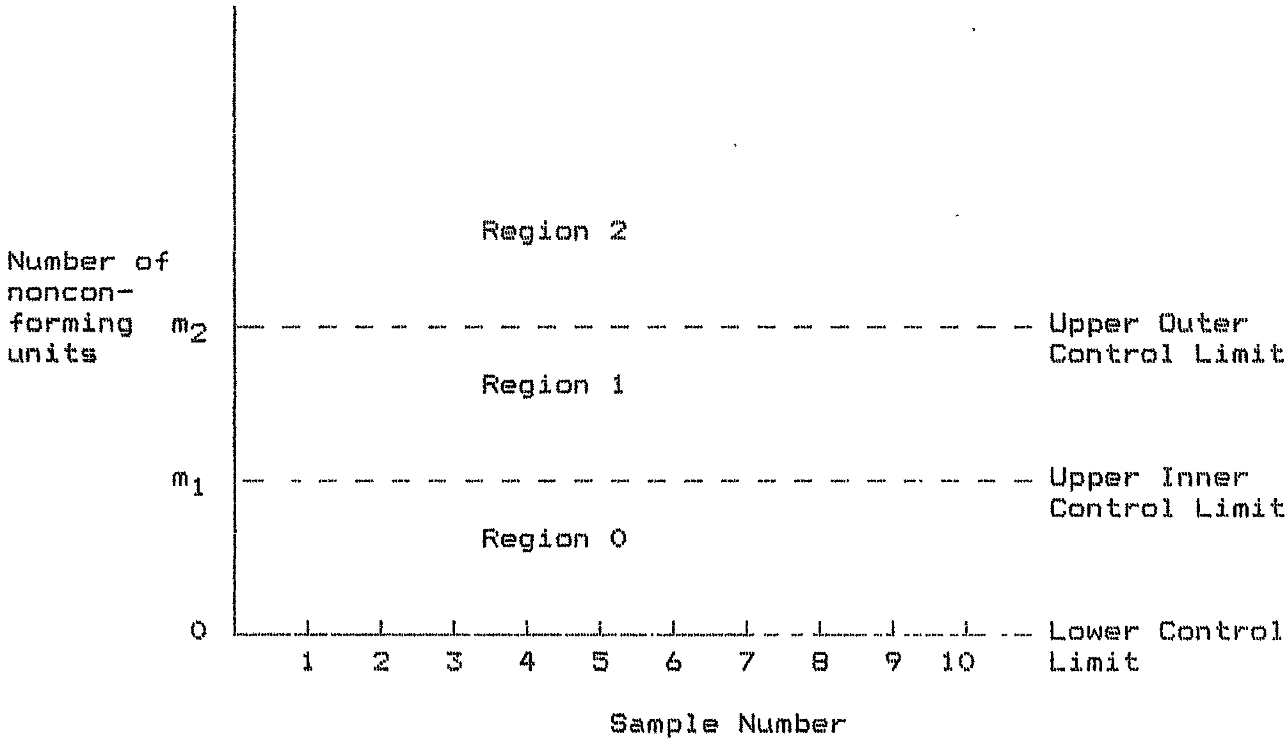

Figure 7.1

Let α_{ij} (i,j=0,1,2) be the probability that the jth level of action is taken when the process is in the state E_i. Level 0 denotes no action. Then

$$\alpha_{io} = \sum_{d=0}^{m_1-1} \binom{n}{d} p_i^d (1-p_i)^{n-d} \qquad \ldots(7.3.1a)$$

$$\alpha_{i1} = \sum_{d=m_1}^{m_2-1} \binom{n}{d} p_i^d (1-p_i)^{n-d} \qquad \ldots(7.3.1b)$$

$$\alpha_{i2} = \sum_{d=m_2}^{n} \binom{n}{d} p_i^d (1-p_i)^{n-d} \qquad \ldots(7.3.1c)$$

$$i=0,1,2$$

7.4 The Expected Cost Model

7.4.1 The cost model used is similar to the cost model developed by Tagaras and Lee (1988) for $\bar{X}$-chart with two pairs of control limits defining three regions on the chart.

A production cycle is defined as the time period from the beginning of the process in E_0 to the next beginning of the process in E_0 after detection and elimination of the assignable cause.

Let E(T) be the expected length of the production cycle. Let E(C) be the expected total cost incurred during the production cycle. Then the expected cost per time unit ECPTU is given by E(C)/E(T).

We compute ECPTU in the remaining part of this section.

7.4.2 Description of Certain Terms Required to develop the Expected Cost Model

$1/\lambda$ = the average operating time the process remains in state E_0 before shifting to E_1 or E_2 $(\lambda=\lambda_1+\lambda_2)$,

t = the expected sampling time per unit,

L_i = the expected time spent on ith level restoration (i=1,2),

T_0 = the time until the process shifts to an out-of-control state,

τ = the time between the shift of the process to out-of-control state and the first inspection thereafter,

T_i = the time between the first inspection after the process shifts to an out-of-control state and the end of the production cycle given that the first shift is to state E_i $(i=1,2)$,

b = the inspection cost per unit,

a = the cost of producing a nonconforming unit,

r_i = the cost of ith level restoration $(i=1,2)$,

C_O = the total cost incurred during T_O,

C_τ = the total cost incurred during τ,

C_i = the total cost incurred during T_i $(i=1,2)$,

$q_{ij}(u)$ = the probability that there are at least u inspections after the process shifts to an out of control state and before the end of the cycle and that the true state of the process is E_j at the time of uth inspection, given that the first shift of the process was to state E_i $(i=1,2;\ j=1,2;\ u=1,2,\ \ldots)$,

$A_i(u)$ = the expected number of nonconforming units produced between uth and (u+1)th inspection after the process shifts to an out-of-control state, given that the first shift of the process was to state E_i $(i=1,2)$,

π = the profit per time unit during the operation of the process in state E_O.

If the production is stopped during sampling and inspection the effective cost of sampling is

$$\theta = [nb + nt\pi] \qquad \ldots(7.4.1)$$

If the production continues during sampling and inspection, then set $t = 0$ in the above expression.

In the similar manner, if the production is stopped during the restoration then the effective cost of ith level of restoration is

$$R_i = [r_i + L_i\pi] \qquad (i = 1,2) \qquad \ldots(7.4.2)$$

7.4.3 The Expected Length of the Production Cycle

The expected length of the production cycle is

$$E(T) = E(T_o) + E(\tau) + (\lambda_1/\lambda)E(T_1) + (\lambda_2/\lambda)E(T_2) \qquad \ldots(7.4.3)$$

The expectations on the right side are obtained in that turn. The expected number of samples taken during the in-control state, as given by Duncan (1956) is

$$N(0) = \frac{\exp(-\lambda h)}{1-\exp(-\lambda h)} \qquad \ldots(7.4.4)$$

Also there is a possibility of false alarm for each sampling inspection. Hence the expected length of time the process remains in the in-control state is

$$E(T_o) = 1/\lambda + (\alpha_{o1}L_1 + \alpha_{o2}L_2 + nt)N(0) \qquad \ldots(7.4.5)$$

The density function of τ as given by Duncan (1956) is

$$f(\tau) = \frac{\lambda \exp[-\lambda(h-\tau)]}{1-\exp(-\lambda h)} \qquad \ldots(7.4.6)$$

We therefore have

$$E(\tau) = \frac{\lambda h - 1 + \exp(-\lambda h)}{\lambda\{1-\exp(-\lambda h)\}} \qquad \ldots(7.4.7)$$

The expected values $E(T_1)$ and $E(T_2)$ consist of three parts :

(i) operation time in out-of-control state since the first inspection after the shift of the process,

(ii) inspection time,

(iii) restoration time.

Let u be the number of inspections required to detect the shift after the process becomes out of control, (u=1, 2, ...).

If the process shifts from E_o to E_1 then

$$E(T_1) = \sum_{u=1}^{\infty} [\{(u-1)h + unt + L_1\} q_{11}(u)\alpha_{11}$$

$$+ \{(u-1)h + unt + L_2\} \{q_{12}(u)\alpha_{22} + q_{11}(u)\alpha_{12}$$

$$+ L_1 q_{12}(u)\alpha_{21}] \qquad \ldots(7.4.8)$$

The expression for $q_{11}(u)$ and $q_{12}(u)$ are derived by Tagaras and Lee (1988). These expressions are as follows.

$$q_{11}(1) = \begin{cases} \lambda h \exp(-\lambda h)/[1 - \exp(-\lambda h)] & \text{if } \lambda_{12} = \lambda \\ \dfrac{\lambda \exp(-\lambda h)}{\{1-\exp(-\lambda h)\}} * \dfrac{1-\exp[-(\lambda_{12}-\lambda)h]}{(\lambda_{12}-\lambda)} & \text{if } \lambda_{12} \neq \lambda \end{cases}$$

$$q_{12}(1) = 1 - q_{11}(1)$$

$$q_{11}(u) = q_{11}(1)\, [(1-\alpha_1)\exp(-\lambda_{12}h)]^{u-1} \qquad \ldots(7.4.9)$$

$$u = 2, 3, 4, \ldots$$

$$q_{12}(u) = (1-\alpha_2)^{u-1} q_{12}(1) + b_1 \sum_{j=1}^{u-1} (1-\alpha_2)^{j-1} b_2^{u-j-1} \qquad \ldots(7.4.10)$$

where $\alpha_2 = \alpha_{22}$, $\alpha_1 = \alpha_{11} + \alpha_{12}$

$b_1 = q_{11}(1)(1-\alpha_1)[1-\exp(-\lambda_{12}h)]$

$b_2 = (1-\alpha_1)\exp(-\lambda_{12}h)$ $\qquad u = 2, 3, \ldots.$

If the process shifts from E_o to E_2 directly the number of inspections required to detect the shift is a geometric random variable with parameter α_{22}. Hence the expected number of samples required to detect the shift is $1/\alpha_{22}$.

Hence,

$$E(T_2) = (1/\alpha_{22} - 1)h + nt/\alpha_{22} + L_2 + \alpha_{21}L_1/\alpha_{22} \qquad \ldots(7.4.11)$$

where,

$(1/\alpha_{22} - 1)h$ = operation time in out-of-control state,

nt/α_{22} = sampling time,

L_2 = restoration time due to true alarm,

$\alpha_{21}L_1/\alpha_{22}$ = restoration time due to false alarm.

One can now use (7.4.5), (7.4.7), (7.4.8) and (7.4.11) to find E(T) of (7.4.3).

7.4.4 The Expected Cost Incurred During the Production Cycle

The total cost incurred during the production cycle is

$$E(C) = E(C_0) + E(C_\tau) + (\lambda_1/\lambda)E(C_1) + (\lambda_2/\lambda)E(C_2) \qquad \ldots(7.4.12)$$

The total cost incurred during T_0 consists of the cost of sampling, the cost of restoration due to false alarms and the cost of producing nonconforming units.

We therefore have,

$$E(C_0) = (G + \alpha_{01}R_1 + \alpha_{02}R_2)\,[\exp(-\lambda h)/\{1 - \exp(-\lambda h)\}] + aRp_0/\lambda \qquad \ldots(7.4.13)$$

Let $\tau_j\big|i$ be the portion of τ during which the process remains in state E_j given that the first shift of the process is to state E_i (i,j = 1,2).

Tagaras and Lee (1988) have shown that

$$E(\tau_1|1) = [1 - q_{11}(1)]/\lambda_{12} \qquad \ldots(7.4.14)$$

$$E(\tau_2|1) = E(\tau) - E(\tau_1|1) \qquad \ldots(7.4.15)$$

The total cost incurred during τ consists of the cost of producing nonconforming units. We therefore have

$$E(C_\tau) = (\lambda_1/\lambda)\ [a\{E(\tau_1|1)Rp_1 + E(\tau_2|1)Rp_2\}] + (\lambda_2/\lambda)\ [aE(\tau)Rp_2] \qquad \ldots(7.4.16)$$

The total cost incurred during T_1 consists of the cost of sampling, the cost of effective and ineffective restoration, and the cost of producing nonconforming units.

We therefore have,

$$E(C_1) = \sum_{u=1}^{\infty} [(uG + R_1)q_{11}(u)\alpha_{11} + (uG + R_2)\{q_{12}(u)\alpha_{22} + q_{11}(u)\alpha_{12}\} + R_1q_{12}(u)\alpha_{21} + A_{1u}] \qquad \ldots(7.4.17)$$

To find A_{1u}, we define h_1 and h_2 as the time within the sampling interval h that the process stays in state E_1 and E_2 respectively, given that the process is in state E_1 at the beginning of the interval.

Then

$$E(h_1) = \int_0^h t\lambda_{12}\exp(-\lambda_{12}t)\ dt + \int_h^{\infty} h\lambda_{12}\exp(-\lambda_{12}t)\ dt$$

$$= [1 - \exp(-\lambda_{12}h)]/\lambda_{12} \qquad \ldots(7.4.18)$$

$$E(h_2) = h - E(h_1) \qquad \ldots(7.4.19)$$

We therefore have,

$$A_{1u} = a[q_{11}(u)(1 - \alpha_{11} - \alpha_{12})\ \{E(h_1)Rp_1 + E(h_2)Rp_2\} + q_{12}(u)(1-\alpha_{22})hRp_2] \qquad u = 1,2,3,\ldots\ldots \qquad \ldots(7.4.20)$$

The total cost incurred during T_2 consists of the cost of sampling, the cost due to effective and ineffective restoration and the cost of producing nonconforming units.

We therefore have,

$$E(C_2) = G/\alpha_{22} + \alpha_{21}R_1/\alpha_{22} + R_2 + a(1/\alpha_{22} - 1)hRp_2 \quad \ldots(7.4.21)$$

Thus one can evaluate E(C) given by (7.4.12) using (7.4.13), (7.4.16), (7.4.17) and (7.4.21). The economic design problem is solved by finding the optimal values for the design variables n, m_1, m_2 and k which minimize the ratio E(C)/E(T). The ratio E(C)/E(T) represents the expected cost per time unit incurred during the production cycle which is denoted by ECPTU.

7.5 Solution Method and Numerical Example

Hooke-Jeeves search procedure is used to find the optimal values of n, m_1, m_2, k that minimize the expected cost per unit time.

Let $\lambda_1 = 0.01$, $\lambda_2 = 0.004$, $\lambda_{12} = 0.008$.

Let b = \$ 1.0, a = \$ 10.0, r_1 = \$ 100, r_2 = \$ 150.

Let t = 0, $L_1 = 1.0$, $L_2 = 2.0$.

Let R = 100, π = \$ 500, $p_0 = 0.01$, $p_1 = 0.10$, $p_2 = 0.50$.

For this combination of cost coefficients and process parameters the search technique yielded the following optimal procedure.

n = 11, $m_1 = 2$, $m_2 = 4$, k = 215 with minimum ECPTU = \$ 33.7357.

The values of some intermediate terms calculated for this numerical example at the optimal stage are as given below.

$\alpha_{01} = 0.0052$	$\alpha_{21} = 0.1074$	$E(T_1) = 5.8496$	$E(C_T) = \$ 233.23$
$\alpha_{02} = 0.0000$	$\alpha_{22} = 0.8867$	$E(T_2) = 2.3958$	$E(C_1) = \$ 1189.79$
$\alpha_{11} = 0.2841$	$E(T_o) = 71.5982$	$E(T) = 77.5431$	$E(C_2) = \$ 1248.83$
$\alpha_{12} = 0.0185$	$E(\tau) = 1.0803$	$E(C_o) = \$ 1176.02$	$E(C) = \$ 2615.91$

7.6 Comparision with the Traditional Single upper control Limit Model

7.6.1 It has been shown by Duncan (1971) and Chiu (1976) that single assignable cause approximation performs almost as good as multiple assignable cause model if the multiple out-of-control states are adequately approximated by single out-of-control state.

In view of the above comment we construct a "matched" single out-of-control state model corresponding to two out-of-control states model developed in Section 7.4 of this chapter.

7.6.2 The "Matched" single out-of-control State Model

The parameters of single out-of-control state model are obtained by matching the parameters of two out-of-control states model. The parameters are matched in the following manner.

λ = rate of shift from the in-control state

$= \lambda_1 + \lambda_2$.

p_o = proportion of nonconforming units produced in the in-control state

$= p_o$.

p_1 = proportion of nonconforming units produced in the out-of-control state

$$= \sum_{i=1}^{2} \lambda_i p_i / \lambda .$$

L = restoration time

$$= \sum_{i=1}^{2} \lambda_i L_i / \lambda .$$

r = restoration cost

$$= \sum_{i=1}^{2} \lambda_i r_i / \lambda .$$

R^* = effective restoration cost

$$= r + L\pi .$$

Recall the definations of the following terms from the Section 7.4.2.

T_O, Υ, C_O, C_Υ, b, a, π, Θ

Redefine the following terms.

T_1 = time between the first inspection after the shift of the process to the out-of-control state and end of the cycle,

C_1 = the total cost incurred during T_1,

α_{ij} = the probability that the jth level of action is taken when the process is in the state E_i $(i,j = 0,1)$.

$$\alpha_{io} = \sum_{d=0}^{m-1} \binom{n}{d} p_i^d (1-p_i)^{n-d}$$

$$\alpha_{i1} = \sum_{d=m}^{n} \binom{n}{d} p_i^d (1-p_i)^{n-d}$$

$$i = 0,1 \qquad \ldots(7.6.1)$$

The expected length of the production cycle is

$$E(T) = E(T_o) + E(\tau) + E(T_1) \qquad \ldots(7.6.2)$$

where

$$E(T_o) = (1/\lambda + \alpha_{01}L + nt)[\exp(-\lambda h)/\{1 - \exp(-\lambda h)\}] \quad ...(7.6.3)$$

$$E(\tau) = \frac{\lambda h - 1 + \exp(-\lambda h)}{\lambda\{1 - \exp(-\lambda h)\}} \quad ...(7.6.4)$$

$$E(T_1) = (1/\alpha_{11} - 1)h + L + nt/\alpha_{11} \quad ...(7.6.5)$$

The expected cost incurred during the production cycle is

$$E(C) = E(C_o) + E(C_\tau) + E(C_1) \quad ...(7.6.6)$$

where

$$E(C_o) = (G + \alpha_{01}R^*)[\exp(-\lambda h)/\{1 - \exp(-\lambda h)\}] + aRp_o/\lambda \quad ...(7.6.7)$$

$$E(C_\tau) = aE(\tau)Rp_1 \quad ...(7.6.8)$$

$$E(C_1) = G/\alpha_{11} + R^* + a(1/\alpha_{11}-1)hRp_1 \quad ...(7.6.9)$$

The expected total cost per time unit incurred during the production cycle under the "matched" single out-of-control state model, therefore, is

$$ECPTU = E(C)/E(T) \quad ...(7.6.10)$$

where E(C) is given by (7.6.6) & E(T) is given by (7.6.2).

The minimum of this ECPTU is to be compared with the minimum ECPTU of two assignable cause-two control limits model developed in Section 7.4. This comparision enables us to find the cost penalty associated with using the "matched" single assignable cause model in place of the two assignable cause model proposed in section 7.4.

7.6.3 Comparision by a Numerical Illustration.

Since there are two distinct responses in the original two assignable cause model the only meaningful comparision is against that "matched" single assignable cause model which always uses a second level of restoration as its single response. If the first level of restoration is used the process would remain indefinitely in state E_2 after its transition to E_2.

For Comparision we consider the same numerical example discussed in Section 7.5. The cost coefficients and the systems parameters of the "matched" single assignable cause-single control limit model obtained by using the expressions derived in Section 7.6.2 are as follows :-

$\lambda = 0.014$, $R = 100$, $\pi = \$\ 500.0$, $t = 0$, $L = 2.0$, $p_0 = 0.01$,

$p_1 = 0.2143$, $b = \$\ 1.0$, $a = \$\ 10.0$, $r = \$\ 150.0$

For this combination of cost coefficients and systems parameter Hooke-Jeeves technique yielded the following optimal procedure.

$n = 9$, $m = 2$, $k = 208$ with minimum ECPTU = \$ 37.1683.

Comparing this approximate solution with the exact solution derived in Section 7.5, it is seen that the improvement in the cost due to two control limits - two responses chart over its single control limit - single response approximation is 3.4326.

The proposed two control limits - two responses chart is expected to provide significant improvement in the cost in those situations where the relative frequency of assignable cause 2 is low (i.e the ratio λ_1/λ_2 is high) and the cost of second level of restoration is relatively high as compared to the first level of restoration.

7.7 Use of Curtailed Sampling Policy in Two Control Limits - Two Responses Model.

7.7.1 It has been shown in Chapter IV that the use of curtailed sampling policy leads to smaller expected cost. This is true for Montgomery's (1975) model as well as for the cost model developed by us in Section 2.3 of Chapter II. It is also proved that the fully-curtailed sampling policy is better than the semi-curtailed as well as the uncurtailed sampling policies from the cost point of view. In this section we use the curtailed sampling policy in place of the complete sampling policy in the cost model developed by us in the Section 7.4 of this chapter.

The curtailed sampling policy proposed by us to match the uncurtailed sampling of this chapter is as follows :-

After production of every k units, inspect the units one by one till one of the following occurs (whichever is earlier)

(i) m_2 nonconforming units are observed,

(ii) $n-m_1+1$ conforming units are observed,

(iii) n units are examined.

If (i) happens then the process is declared to be out of control and a search for the major assignable cause is undertaken (this implies the second level of action described in Section 7.3),

if (ii) happens then the process is declared to be in control and the production continues (this is known as action of level zero as per Section 7.3), if (iii) happens then the process is declared to be out of control and a search for the minor assignable cause is undertaken (this implies the first level of action described in Section 7.3).

A sort of the remark on the stopping rule and the decision rule given above is desirable. If m_2th nonconforming unit is observed after the completion of the inspection of the nth unit then it will be taken as if m_2th nonconforming unit is observed. Similarly, if $(n-m_1+1)$th conforming unit is noted after the completion of the inspection of nth unit then it will be taken as if $(n-m_1+1)$th conforming unit is observed.

The probability function associated with the above curtailed sampling policy is as given below.

$$P(Y=y, X=x) = \begin{cases} \binom{y-1}{m_2-1} p^{m_2} (1-p)^{y-m_2} & y = m_2, m_2+1, \ldots, n \\ & x = m_2 \\ \binom{y-1}{n-m_1} (1-p)^{n-m_1+1} p^{y-(n-m_1+1)} & y = n-m_1+1, n-m_1+2, \ldots, n \\ & x = y-(n-m_1+1) \\ \binom{n}{x} p^{x} (1-p)^{n-x} & y = n \\ & x = m_1, m_1+1, \ldots, m_2-1. \end{cases} \qquad \ldots(7.7.1)$$

where

p = the probability of getting a nonconforming unit,

Y = the actual number of units inspected when the sampling terminates,

X = the number of nonconforming units observed when the sampling terminates.

7.7.2 The Average Sample Number (ASN)

Let $\bar{n}_i$ be the average sample number when the process is in state E_i (i = 0,1,2). The expression for $\bar{n}_i$ is derived by taking into consideration the marginal distribution of Y and taking its expectation when $p = p_i$ (i = 0,1,2).

The expression for $\bar{n}_i$ is

$$\bar{n}_i = \sum_{y=m_2}^{n} y \binom{y-1}{m_2-1} p_i^{m_2} (1-p_i)^{y-m_2} + \sum_{y=n-m_1+1}^{n} y \binom{y-1}{n-m_1} (1-p_i)^{n-m_1+1} p_i^{y-(n-m_1+1)} + n \sum_{x=m_1}^{m_2-1} \binom{n}{x} p_i^{x} (1-p_i)^{n-x} \quad \ldots(7.7.2)$$

We next show that

$$\bar{n}_i \leq n \qquad i = 0,1,2 \qquad \ldots(7.7.3)$$

It is easy to see that

$$\bar{n}_i \leq n \sum_{y=m_2}^{n} \binom{y-1}{m_2-1} p_i^{m_2} (1-p_i)^{y-m_2} + n \sum_{y=n-m_1+1}^{n} \binom{y-1}{n-m_1} (1-p_i)^{n-m_1+1} p_i^{y-(n-m_1+1)} + n \sum_{x=m_1}^{m_2-1} \binom{n}{x} p_i^{x} (1-p_i)^{n-x} \quad \ldots(7.7.4)$$

The relation between the distribution functions of the negative binomial distribution and the binomial distribution is as follows.

$$\sum_{y=k}^{n} \binom{y-1}{k-1} p^{k} (1-p)^{y-k} = \sum_{x=k}^{n} \binom{n}{x} p^{x} (1-p)^{n-x}$$

...(7.7.5)

Using this relation in the expression (7.7.4) we have

$$\bar{n}_i \leq n \sum_{x=m_2}^{n} \binom{n}{x} p_i^{x} (1-p_i)^{n-x}$$

$$+ n \sum_{x=n-m_1+1}^{n} \binom{n}{x} (1-p_i)^{x} p_i^{n-x}$$

$$+ n \sum_{x=m_1}^{m_2-1} \binom{n}{x} p_i^{x} (1-p_i)^{n-x}$$

...(7.7.6)

Then using the relation

$$\sum_{x=n-k}^{n} \binom{n}{x} (1-p)^{x} p^{n-x} = \sum_{x=o}^{k} \binom{n}{x} p^{x} (1-p)^{n-x}$$

...(7.7.7)

in the middle term of the expression (7.7.6) and adding we have

$$\bar{n}_i \leq n \sum_{x=o}^{n} \binom{n}{x} p_i^{x} (1-p_i)^{n-x}$$

...(7.7.8)

$$= n.1$$

Thus $\bar{n}_i \leq n$ $(i = 0,1,2)$ is established.

7.7.3 The Expression for α_{ij} under Curtailed Sampling

α_{ij} is the probability that jth level of action is taken when the process is in the state E_i. Level o denotes no action. The expression for α_{ij} under the curtailed sampling is as follows.

$$\alpha_{i0} = \sum_{y=n-m_1+1}^{n} \binom{y-1}{n-m_1} (1-p_i)^{n-m_1+1} p_i^{y-(n-m_1+1)}$$
...(7.7.9a)

$$\alpha_{i1} = \sum_{x=m_1}^{m_2-1} \binom{n}{x} p_i^x (1-p_i)^{n-x}$$
...(7.7.9b)

$$\alpha_{i2} = \sum_{y=m_2}^{n} \binom{y-1}{m_2-1} p_i^{m_2} (1-p_i)^{n-m_2}$$
...(7.7.9c)

Using the relation given by (7.7.5) and (7.7.7), it can be easily seen that the expressions for α_{ij} $(i,j = 0,1,2)$ remain the same under the curtailed sampling and the complete sampling policies (i.e. (7.7.9a) is the same as (7.3.1a) and (7.7.9c) is the same as (7.3.1c)).

7.7.4 The Effects of Curtailment on Expected Costs

We now study the effect of using the curtailed sampling policy on the expected cost model developed in Section 7.4 of this chapter.

The total expected cost, E(C), incurred during the production cycle consists of four components $E(C_i)$ $i = 0,1,2$ and $E(C_T)$ such that

$$E(C) = E(C_0) + E(C_T) + (\lambda_1/\lambda)E(C_1) + (\lambda_1/\lambda_2)E(C_2)$$
...(7.7.10)

The effective cost of sampling under complete sampling is

$$\theta = n(b + t\pi)$$ ...(7.7.11)

Therefore effective cost of sampling under the curtailed sampling is

$$\theta_i = \bar{n}_i(b + t\pi)$$ ...(7.7.12)

when the sample is taken in the state E_i $(i = 0,1,2)$.

If the production continues during inspection then set $t = 0$ in the above expressions (7.7.11) and (7.7.12).

The result $\bar{n}_i \leq n$ $(i = 0,1,2)$ implies that

$$G_i \leq G \quad (i = 0,1,2). \qquad \ldots(7.7.13)$$

When curtailed sampling is to be used, then in the expressions for $E(C_i)$ $(i = 0,1,2)$, the original cost of sampling G is to be replaced by the new cost of sampling G_i $(i = 0,1,2,)$ in the following manner.

(1) In $E(C_0)$ all the samples are taken in the in-control state E_0. Hence G is to be replaced by G_0 in the expression for $E(C_0)$ of uncurtailed sampling given by (7.4.13).

(2) In the expression for $E(C_1)$ of uncurtailed sampling given by (7.4.17), when the multiplier of G is $q_{11}(u)$, G is to be replaced by G_1. This is obvious from the definition of $q_{11}(u)$. When the multiplier of G is $q_{12}(u)$, let w $(w < u)$ be the number of samples taken in the state E_1 before the process shifts to state E_2. This means that the remaining $u-w$ samples are taken in the state E_2. Hence for the first w samples G is to be replaced by G_1 and for the remaining $u-w$ samples G is to be replaced by G_2. We would like to mention that the expression for w is not derived, yet it does not create any difficulty since

$$WG_1 + (u-w)G_2 \leq wG + (u-w)G = uG \qquad \ldots(7.7.14)$$

whatever may be the value of w.

(3) In $E(C_2)$ all the samples are taken in the state E_2. Hence G is to be replaced by G_2 in the expression for $E(C_2)$ of uncurtailed sampling given by (7.4.21).

(4) The component $E(C_T)$ does not contain G. Hence the question of replacement does not arise.

Since α_{ij} $(i,j = 0,1,2)$ attain the same values under the curtailed and the complete sampling policies all the other terms of $E(C_i)$ $(i = 0,1,2)$ and all the terms of $E(C_T)$ remain unaltered when we use curtailed sampling in place of complete sampling.

Since $G_i \le G$ $(i = 0,1,2)$, it is easy to see that

$$[E(C_i)]_{curt} \le [E(C_i)]_{uncurt} \qquad \text{...(7.7.15)}$$

$$i = 0,1,2.$$

$$[E(C_T)]_{curt} = [E(C_T)]_{uncurt} \qquad \text{...(7.7.16)}$$

We therefore have

$$[E(C)]_{curt} \le [E(C)]_{uncurt} \qquad \text{...(7.7.17)}$$

This means that the total expected cost incurred during the production cycle is smaller when curtailed sampling is used in place of complete sampling policy. These expected costs are calculated using the fixed design variables (n, m_1, m_2, k).

The result (7.7.17) shows that the numerator of the objective function ECPTU is smaller when curtailed sampling is used in place of complete sampling.

If we assume that $t = 0$, the expected length of the production cycle E(T) given by (7.4.3) remains the same under both the sampling policies.

If $t > 0$ and assuming that it is small it can be seen that E(T) remains almost the same under both the sampling policies.

We therefore have

$$[ECPTU]_{curt} \le [ECPTU]_{uncurt} \qquad \text{...(7.7.18)}$$

This leads to a conclusion that curtailed sampling is no more expensive than the traditional complete sampling for the cost model developed in this chapter also.

```
C LISTING OF CHAPTER VII
      SUBROUTINE OBJ5(AKE,NSTAGE,SUM,ALDA1,ALDA2,ALDA12,NB,NA,NR1,
     1 NR2,T,TR1,TR2,NPIE,NU,PNOT,PONE,PTWO)
C  FILE NAME IS  B:SHY2
C PROGRAM ON ECONOMIC DESIGN OF NP-CHART WHEN THERE ARE TWO
C  CONROL LIMITS
      DIMENSION Q11(100),Q12(100),TERM(100),A1(100),A2(100),TERM5(100)
     1 ,Q22(100),AKE(5)
      WRITE(*,2) ALDA1,ALDA2, ALDA12
 2    FORMAT(1X,'ALDA1=',F10.4,'ALDA2=',F12.4,'ALDA12=',F12.4)
      WRITE(*,4) NB,NA,NR1,NR2
 4    FORMAT(1X,'NB=',I4,'NA=',I4,'NR1=',I4,'NR2=',I4)
      WRITE(*,6) T,TR1,TR2
 6    FORMAT(1X,'T=',F10.4,'TR1=',F10.4,'TR2=',F10.4)
      WRITE(*,8) NPIE ,NU
 8    FORMAT(1X,'NPIE=',I5,'NU=',I5)
      WRITE(*,10) PNOT,PONE ,PTWO
 10   FORMAT(1X,'PNOT=',F10.4,'PONE=',F10.4,'PTWO=',F10.4)
      N=AKE(1)
      M1=AKE(2)
      M2=AKE(3)
      K=AKE(4)
      WRITE(*,12) N,M1,M2,K
 12   FORMAT(1X,'N=',I5,'M1=',I5,'M2=',I5,'K=',I5)
      ALDA=ALDA1+ALDA2
      H=FLOAT(K)/NU
      WRITE(*,13) H
 13   FORMAT(1X,'H=',F10.6)
      PROB=PNOT
      MM=M1
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,PA,PIND)
      PR1=PR
      PROB=PNOT
      MM=M2
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,PA,PIND)
      PR2=PR
      ALPA01=PR1-PR2
      ALPA02=PR2
      WRITE(*,16) ALPA01,ALPA02
 16   FORMAT(1X,'ALPA01=',F10.6,'ALPA02=',F10.6)
      PROB=PONE
      MM=M1
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,RA,PIND)
      PR3=PR
      PROB=PONE
      MM=M2
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,PA,PIND)
      PR4=PR
      ALPA11=PR3-PR4
      ALPA12=PR4
```

```
      WRITE(*,17) ALPA11,ALPA12
17    FORMAT(1X,'ALPA11=',F10.6,'ALPA12=',F10.6)
      PROB=PTWO
      MM=M1
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,PA,PIND)
      PR5=PR
      MM=M2
      NN=N
      NT=NN-MM+1
      CALL BIN(PROB,MM,NT,PR,PA,PIND)
      PR6=PR
      ALPA21=PR5-PR6
      ALPA22=PR6
      WRITE(*,20) ALPA21,ALPA22
20    FORMAT(1X,'ALPA21=',F10.6,'ALPA22=',F10.6)
      POWER=ALDA*H
      PPOWER=-POWER
      EPOWER=EXP(PPOWER)
      TAW=(POWER-1+EPOWER)/(ALDA*(1-EPOWER))
      TO=1/ALDA+(ALPA01*TR1+ALPA02*TR2+N*T)*EPOWER/(1-EPOWER)
      WRITE(*,21) TAW,TO
21    FORMAT(1X,'TAW=',F10.6,'TO=',F10.6)
      ALPA1=ALPA11+ALPA12
      ALPA2=ALPA22
      D=ALDA12-ALDA
      GG=D*H
      G=-GG
      Q11(1)=(ALDA*EPOWER/(1-EPOWER))*((1-EXP(G))/(ALDA12-
     1 ALDA))
      F=ALDA12*H
      FF=-F
      DO 25 K=2,100
      Q11(K)=Q11(1)*((1-ALPA1)*EXP(FF))**(k-1)
25    CONTINUE
      Q12(1)=1-Q11(1)
      DO 30 K=2,100
      Q12(K)=Q12(K)*(1-ALPA2)**(K-1)+Q11(1)*(1-ALPA1)*(1-EXP(FF))
     1 *(((1-ALPA1)*EXP(FF))**(K-1)-(1-ALPA2)**(K-1))/
     1 ((1-ALPA1)*EXP(FF)-(1-ALPA2))
30    CONTINUE
      WRITE(*,31) (Q11(K),K=1,100)
      WRITE(*,31) (Q12(K),K=1,100)
31    FORMAT(1X,10F7.4)
      SUM=0
      DO 35  K=1,100
      TERM(K)=(((K-1)* H +K*N*T+TR1)*ALPA11*Q11(K)+((K-1) *H+K*N*T+
     1 TR2)*(ALPA12*Q11(K)+ALPA2*Q12(K))+TR2*ALPA21*Q12(K))
      SUM=SUM+TERM(K)
      IF(TERM(K).LE.0.00001) GO TO 40
35    CONTINUE
      T1=SUM
      WRITE(*,51)T1
      ISTOP=100
      GO TO 48
40    T1=SUM
```

```
      WRITE(*,51) T1
      ISTOP=K
      WRITE(*,41)ISTOP
41    FORMAT(1X,'ISTOP=',I4)
51    FORMAT(1X,'T1=',F12.4)
48    T2=(1-ALPA2)*H/ALPA2+N*T/ALPA2+TR2+ALPA21*TR1/ALPA2
      WRITE(*,52)T2
52    FORMAT(1X,'T2=',F12.4)
      EXPT=T0+TAW+ALDA1*T1/ALDA+ALDA2*T2/ALDA
      WRITE(*,53) EXPT
53    FORMAT(1X,'EXPT=',F12.4)
      CR1=NR1+TR1*NPIE
      CR2=NR2+TR2*NPIE
      WRITE(*,55) CR1,CR2
55    FORMAT(1X,'CR1=',F12.4,'CR2=',F12.4)
      CS=N*(NB+T*NPIE)
      CO=(ALPA01*CR1+ALPA02*CR2+CS)*EPOWER/(1-EPOWER)+NA*NU*PNOT/ALDA
      WRITE(*,58)CS,CO
58    FORMAT(1X,'CS=',F12.4,'CO=',F12.4)
      TAW11=(1-Q11(1))/ALDA12
      TAW21=TAW-TAW11
      TAW22=TAW
      WRITE(*,60)TAW11,TAW21,TAW22
60    FORMAT(1X,'TAW11=',F10.6,'TAW21=',F10.6,'TAW22=',F10.6)
      CTAW=(ALDA1/ALDA)*(NA*(TAW11*NU*PONE+TAW21*NU*PTWO))+
     1 (ALDA2/ALDA)*(NA*TAW*NU*PTWO)
      WRITE(*,62) CTAW
62    FORMAT(1X,'CTAW=',F12.4)
      H1=(1-EXP(FF))/ALDA12
      H2=H-H1
      DO 65  K=1,100
      A1(K)=Q11(K)*(1-ALPA1)*(H1*NU*PONE+H2*NU*PTWO)*NA
     1 +Q12(K)*(1-ALPA2)*H*NU*PTWO*NA
65    CONTINUE
      DO 63 K=1,100
      Q22(K)=(1-ALPA2)**(K-1)
      A2(K)=Q22(K)*(1-ALPA2)*H*NU*PTWO
63    CONTINUE
      WRITE(*,31)(Q22(K),K=1,100)
      WRITE(*,70) (A1(K),K=1,100)
70    FORMAT(1X 7F10.6)
      WRITE(*,70) (A2(K),K=1,100)
      SUM5=0
      DO 75 K=1,100
      TERM5(K)= (K*CS+CR1)*ALPA11*Q11(K)+(K*CS+CR2)*(Q12(K)*ALPA2
     1 +Q11(K)*ALPA12)+CR1*Q12(K)*ALPA21+A1(K)
      SUM5=SUM5+TERM5(K)
      IF(TERM5(K).LE.0.00001) GO TO 90
75    CONTINUE
90    C1=SUM5
      C2=(CS+ALPA21*CR1+ALPA2*CR2+(1-ALPA2)*H*NU*PTWO)/ALPA2
      C=CO+CTAW+(ALDA1/ALDA)*C1+(ALDA2/ALDA)*C2
      ECPUT=C/EXPT
      WRITE(*,85) C1,C2,C
85    FORMAT(1X,'C1=',F12.4,'C2=',F12.4,'C3=',F12.4)
      WRITE(*,80) ECPUT
```

```
 80     FORMAT(1X,'ECPUT=',F12.6)
        SUM=ECPUT
        RETURN
        END


C   FILE NAME IS B:NSH2
C  PROGRAM FOR  ECPTU OF MATCHED SINGLE CONTROL LIMIT MODEL
       SUBROUTINE  OBJ1(RK,NSTAGE,SUM,A1,A2,A3,A4,RATE,ALEMDA,
     1 PNOT,PONE)
        DIMENSION RK(5)
    1   format(1x,4f10.4)
        write(*,1)a1,a2,a3,a4
  3     FORMAT(1X,2F10.4)
        WRITE(*,3)ALEMDA,RATE
        WRITE(*,3)PNOT,PONE
        SNOT=RK(1)
        SRNOT=RK(2)
        REJNOT=RK(3)
        WRITE(*,5)SNOT,SRNOT,REJNOT
  5     FORMAT(1X,3F10.4)
        POWER=ALEMDA*SRNOT/RATE
        PPOWER=-POWER
        THEETA=EXP(PPOWER)
        WRITE(*,7)THEETA
  7     FORMAT(1X,F10.6)
        MM=REJNOT
        NT=SNOT-REJNOT+1
        CALL BIN(PONE,MM,NT,CPR,CPL,PI)
        QONE=CPR
        WRITE(*,8)QONE
  8     FORMAT(1X,F10.6)
        MM=REJNOT
        NT=SNOT-REJNOT+1
        CALL BIN(PNOT,MM,NT,CPR,CPL,PI)
        QNOT=CPR
        WRITE(*,8)QNOT
        R=1/QONE+THEETA/(1-THEETA)
C  IR IS EXPECTED NO OF SAMPLES REQUIED TO DETECT SHIFT
        IR=R+0.5
        WRITE(*,9)IR
  9     FORMAT(1X,I3)
C   COMPUTATION  OF EXPECTED COSTS
        EC1=(A1+A2*SNOT)*IR
        EC2=A3*(QNOT*THEETA/(1-THEETA)+1)
        ADALTA=(1-(1+POWER)*THEETA)/(POWER*(1-THEETA))
        WRITE(*,35)ADALTA
  35    FORMAT(1X,'ADALTA=', F10.6)
        H=SRNOT/RATE
        TAW=ADALTA*H
        WRITE(*,36)TAW
  36    FORMAT(1X,'TAW=',F10.6)
        EC3=A4*(RATE*PNOT/ALEMDA+(H/QONE-TAW)*RATE*PONE)
        TC=EC1+EC2+EC3
        WRITE (*,30)TC,EC1,EC2,EC3
  30    FORMAT(1X,'TC=',F15.6,'EC1=',F10.6,'EC2=',F10.6,'EC3=',F10.6)
```

```
      BO=QNOT*THEETA/(1-THEETA)
      ET=1/ALEMDA+(H/QONE-TAW)+(BO+1)*2
      ECPT=TC/ET
      SUM=ECPT
      WRITE(*,40)ECPT
40    FORMAT(1X,'ECPT=',F10.6)
      RETURN
      END
```