Appendix A

The estimation method due to Pearson[4] solves the Euler-Lagrange equations (3.17) and (3.18) (with the boundary conditions in equations (3.19) and (3.20)) in an iterative manner. Let the equations (3.17) and (3.18) be represented respectively by the following general equations.

$$\bar{x}(i+1) = F\left[\bar{x}(i),\ \lambda(i),\ i\right] \tag{A.1}$$

$$\lambda(i-1) = G\left[\bar{x}(i),\ \lambda(i),\ i\right] \tag{A.2}$$

These are to be solved simultaneously satisfying the boundary conditions

$$\lambda(-1) = 0 \tag{A.3}$$

$$\lambda(N) = 0 \tag{A.4}$$

Let the estimate of $\bar{x}(i)$ obtained from N observations (called 1^{st} process) be denoted by $\bar{x}'(i)$ and that from $(N+1)$ observations (called 2^{nd} process) be denoted by $\bar{x}''(i)$. The corresponding trajectories for λ are indicated by $\lambda'(i)$ and $\lambda''(i)$. The trajectories $\bar{x}'(i)$, $\bar{x}''(i)$ and $\lambda'(i)$, $\lambda''(i)$ are shown in Figures A.1 and A.2 . Let the deviation between $\bar{x}'(i)$ and $\bar{x}''(i)$ and that between $\lambda'(i)$ and $\lambda''(i)$ be denoted by $\alpha(i)$ and $\beta(i)$ such that

$$\bar{x}''(i) = \bar{x}'(i) + \alpha(i) \tag{A.5}$$

and

$$\lambda''(i) = \lambda'(i) + \beta(i) \tag{A.6}$$

The boundary conditions are then given by

$$\lambda'(-1) = \lambda''(-1) = 0 \tag{A.7}$$

and

$$\lambda'(N) = \lambda''(N+1) = 0 \tag{A.8}$$

Now, following equation (A.1) and (A.2), the dynamic equations for the first and second processes are

$$\bar{x}'(i+1) = F\left[\bar{x}'(i),\ \lambda'(i),\ i\right] \tag{A.9}$$

$$\bar{x}''(i+1) = F\left[\bar{x}''(i),\ \lambda''(i),\ i\right] \tag{A.10}$$

$$\lambda'(i-1) = G\left[\bar{x}'(i),\ \lambda'(i),\ i\right] \tag{A.11}$$

$$\lambda''(i-1) = G\left[\bar{x}''(i),\ \lambda''(i),\ i\right] \tag{A.12}$$

Substituting for $\bar{x}''(i)$ and $\lambda''(i)$ from equations (A.5) and (A.6) in equation (A.10), one obtains

$$\bar{x}''(i+1) = F\left[\bar{x}'(i)+\alpha(i)\ ,\ \lambda'(i)+\beta(i)\ ,\ i\right] \tag{A.13}$$

By Taylor's expansion, this becomes

$$\bar{x}''(i+1) \approx F\left[\bar{x}'(i),\ \lambda'(i)\ ,\ i\right] + F_{\bar{x}'(i)}\,\alpha(i) + F_{\lambda'(i)}\,\beta(i)$$

$$\text{(neglecting other terms)} \tag{A.14}$$

Here, the sign "$\approx$" (approximately equal to) is replaced by the sign "$=$" (equal to) when the dynamic equation (A.1) is linear. Use of equations (A.9) and (A.5) in equation (A.14) gives

$$\alpha(i+1) = F_{\bar{x}'(i)}\,\alpha(i) + F_{\lambda'(i)}\,\beta(i) \tag{A.15}$$

Similarly, one can obtain

$$\beta(i-1) = G_{\bar{x}'(i)}\,\alpha(i) + G_{\lambda'(i)}\,\beta(i) \tag{A.16}$$

Let $\alpha(i+1)$ and $\beta(i)$ be related by the equation

$$\alpha(i+1) = P(i+1)\,\beta(i) \tag{A.17}$$

Therefore,

$$\alpha(i) = P(i)\,\beta(i-1) \tag{A.18}$$

Substituting for $\beta(i-1)$ from equation (A.16), this becomes

$$\alpha(i) = P(i)\,G_{\bar{x}'(i)}\,\alpha(i) + P(i)\,G_{\lambda'(i)}\,\beta(i) \tag{A.19}$$

Rearranging the terms, this gives

$$\alpha(i) = \left[I - P(i)\,G_{\bar{x}'(i)}\right]^{-1} P(i)\,G_{\lambda'(i)}\,\beta(i) \tag{A.20}$$

Substituting for $\alpha(i)$ in equation (A.15),

$$\begin{aligned}\alpha(i+1) &= F_{\bar{x}'(i)}\left[I - P(i)\,G_{\bar{x}'(i)}\right]^{-1} P(i)\,G_{\lambda'(i)}\,\beta(i) + \\ &\quad F_{\lambda'(i)}\,\beta(i) \\ &= \left[F_{\bar{x}'(i)}\left\{I - P(i)\,G_{\bar{x}'(i)}\right\}^{-1} P(i)\,G_{\lambda'(i)} + F_{\lambda'(i)}\right]\beta(i)\end{aligned} \tag{A.21}$$

Comparing this w. r. t. equation (A.17)

$$P(i+1) = F_{\bar{x}'(i)}\left\{I - P(i)\,G_{\bar{x}'(i)}\right\}^{-1} P(i)\,G_{\lambda'(i)} \tag{A.22}$$

For $i = N$, equation (A.6) becomes

$$\lambda''(N) = \lambda'(N) + \beta(N) \tag{A.23}$$

Using equation (A.8), this gives

$$\lambda''(N) = \beta(N) \tag{A.24}$$

Also writing $i = N+1$ in equation (A.11), one obtains

$$\begin{aligned}\lambda''(N) &= G\left[\bar{x}''(N+1)\ ,\ \lambda''(N+1)\ ,\ (N+1)\right] \\ &= G\left[\bar{x}''(N+1)\ ,\ 0\ ,\ (N+1)\right]\end{aligned} \tag{A.25}$$

Comparison of equations (A.24) and (A.25) yields

$$\beta(N) = G\left[\bar{x}''(N+1)\ ,\ 0\ ,\ N+1\right] \tag{A.26}$$

Equation (A.5), for $i = N+1$, becomes

$$\bar{x}''(N+1) = \bar{x}'(N+1) + \alpha(N+1) \tag{A.27}$$

Using equation (A.9), this becomes

$$\bar{x}''(N+1) = F\left[\bar{x}'(N)\ ,\ \lambda'(N)\ ,\ N\right] + \alpha(N+1)$$

$$= F\left[\bar{x}'(N)\ ,\ 0\ ,\ N\right] + \alpha(N+1) \tag{A.28}$$

where $\alpha(N+1)$ is given by (from equation (A.17))

$$\alpha(N+1) = P(N+1)\,\beta(N) \tag{A.29}$$

where $\beta(N)$ is given by equation (A.26) and $P(N+1)$ is given by (from equation (A.22)),

$$P(N+1) = F_{\bar{x}'(N)}\left[I - P(N)\,G_{\bar{x}'(N)}\right]^{-1} P(N)\,G_{\lambda'(N)} \tag{A.30}$$

Thus equation (A.28) indicates that the estimate $\bar{x}''(N+1)$ from (N+1) measurements can be obtained by updating, by the amount $\alpha(N+1)$, the extrapolated value (i.e. $F\left[\bar{x}'(N),\ 0,\ N\right]$) of the estimate $\bar{x}'(N)$ obtained from N measurements. This is essentially a sequential estimation scheme.

The iteration equations for the solution of equations (3.17) and (3.18) satisfying the boundary conditions in equations (3.19) and (3.20) can be readily obtained by comparing these with equations (A.1) to (A.4). For convenience, equations (3.17) to (3.20) are written here again as follows.

$$\bar{x}(i+1) = f\left[\bar{x}(i)\ ,\ i\right] \tag{A.31}$$

$$\lambda(i-1) = f'_{\bar{x}(i)}\,\lambda(i) + 2\,H^{T}\,Q\left[y(i) - H\,\bar{x}(i)\right] \tag{A.32}$$

$$\lambda(-1) = 0 \tag{A.33}$$

$$\lambda(N) = 0 \tag{A.34}$$

In view of these equations, the equation (A.28) becomes

$$\bar{x}''(N+1) = f\left[\bar{x}'(N),\ N\right] + \alpha(N+1) \tag{A.35}$$

where $\alpha(N+1)$ is given by equation (A.29) wherein $\beta(N)$ is obtained by comparing equation (A.32) with equation (A.26) for $i = N+1$ and $\lambda''(N+1) = 0$ and is given by

$$\beta(N) = 2\ H'\ Q\ \left[y(N+1) - H\ \bar{x}''(N+1)\right] \tag{A.36}$$

Thus, the equation (A.35) becomes

$$\bar{x}''(N+1) = f\ \left[\bar{x}'(N),\ N\right] + 2\ P(N+1)\ H'\ Q\ \left[y(N+1) - H\ \bar{x}''(N+1)\right] \tag{A.37}$$

where $P(N+1)$ is given by

$$P(N+1) = f_{\bar{x}'(N)} \left\{I + P(N)\ H'\ Q\ H\right\}^{-1} P(N)\ f_{\bar{x}'(N)} \tag{A.38}$$

Since the term $\bar{x}''(N+1)$ appears on both sides of equation (A.37), rearranging the terms and removing the superscripts (which are now superfluous) with $\bar{x}$, one obtains

$$\bar{x}(N+1) = f\left[\bar{x}(N),\ N\right] + C(N+1)\ H'\ Q\ \left[y(N+1) - H\ f(\bar{x}(N),\ N)\right] \tag{A.39}$$

where

$$C(N+1) = \left[I + P(N+1)\ H'\ Q\ H\right]^{-1} P(N+1) \tag{A.40}$$

where

$$P(N+1) = f_{\bar{x}(N)}\ C(N)\ f'_{\bar{x}(N)} \tag{A.41}$$

The computational procedure is summarized as follows :

(a) Choose $C(N)$. ($N=0$ in thebbeginning)

(b) Compute the Jacobian $f_{\bar{x}(N)}$ for the process described by equation (A.31) and then obtain $P(N+1)$ using equation (A.41).

(c) Obtain $C(N+1)$ from equation (A.40).

(d) Compute $\bar{x}(N+1)$ using equation (A.41). The initial guess on $\bar{x}(0)$ in the beginning is quite arbitrary.

This sequence is repeated to modify the estimates by including new observations at the successive sampling instants. When the dynamic noise is to be considered, the equation (3.2) is modified as given by

$$x(i+1) = f[x(i), i] + w(i) \tag{A.42}$$

where $w(i)$ accounts for the dynamic noise. In this context, the performance index to be minimized is given by

$$I = \sum_{i=0}^{N} \{[y(i) - H\bar{x}(i)]' Q [y(i) - H\bar{x}(i)] + [\bar{x}(i+1) - f\{\bar{x}(i), i\}]' R [\bar{x}(i+1) - f\{\bar{x}(i), i\}]\} \tag{A.43}$$

The Euler-Lagrange equations to be satisfied for the minimization of I are

$$\bar{x}(i+1) = f[\bar{x}(i), i] + R^{-1}\lambda(i) \tag{A.44}$$

$$\lambda(i-1) = f'_{\bar{x}(i)}\lambda(i) + 2H'Q[y(i) - H\bar{x}(i)] \tag{A.45}$$

This when solved in the forgoing manner gives the same filter equation given by equations (A.39) and (A.40), wherein the term $P(N+1)$ is given by

$$P(N+1) = f_{\bar{x}(N)} C(N) f'_{\bar{x}(N)} + R^{-1}(N) \tag{A.46}$$

The equations (A.39), (A.40) and (A.41) are to be used while considering only the additive noise (including measurement noise) at the output. When both the dynamic and measurement noise are considered, the filter equations to be used are (A.39), (A.40) and (A.46).

A system having the transfer function given by equation (4.51) and also described by differential equations (4.18), (4.19) and (4.20) with $x_2(0) = 0.4$ and $x_3(0) = 2.0$ was simulated on the digital computer. The output was obtained with a sampling interval of 0.125 sec. considering the actual input measured on the turbo-alternator. The average order of magnitude for this input was around 50 . The output thus obtained was added with random numbers (generated by RRN) with their values within the bounds ± 1 . This measurement noise with the output was considered as 2 %. The estimated trajectories $\bar{x}_1(i)$, $\bar{x}_2(i)$ and $\bar{x}_3(i)$ from these input and output are obtained using equations (A.39), (A.40) and (A.41) and are shown in Figures A.3, A.4 and A.5 respectively. The estimated trajectories $\bar{x}_1(i)$, $\bar{x}_2(i)$ and $\bar{x}_3(i)$ are also shown without considering measurement noise, for comparasiön. It can be seen from the figures that the convergence of $\bar{x}_2(0)$ and $\bar{x}_3(0)$ towards their true values (0.4 and 2.0 respectively) is excellent without noise. However, they fail to converge while considering noise. The noise level of ± 1 , though 2 % with respect to the magnitude of input, is quite considerable as compared to the parameters to be estimated. Moreover, the estimate of the present state depends only on the correction, to the extrapolated value of the previous estimate, based on the current observation, the sequential estimator is very sensitive to random disturbances in the observations. And especially, when the variance of this random disturbances is large as compared to the parameters, the filter equations fail to converge as is evident from Figures A.4 and A.5.

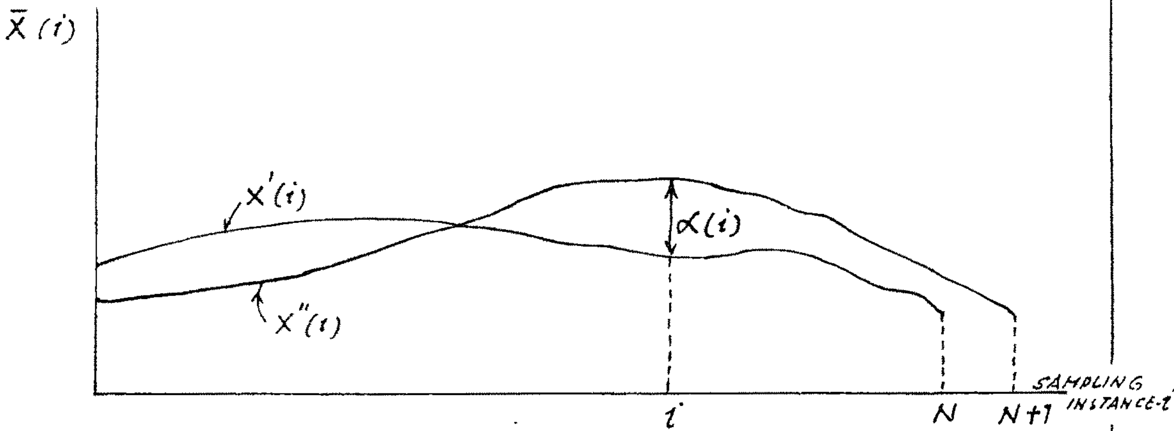


FIG A·1 COMPARISON OF ESTIMATED TRAJECTORIES $X'(i)$ AND $X''(i)$ FROM N AND $(N+1)$ OBSERVATIONS.

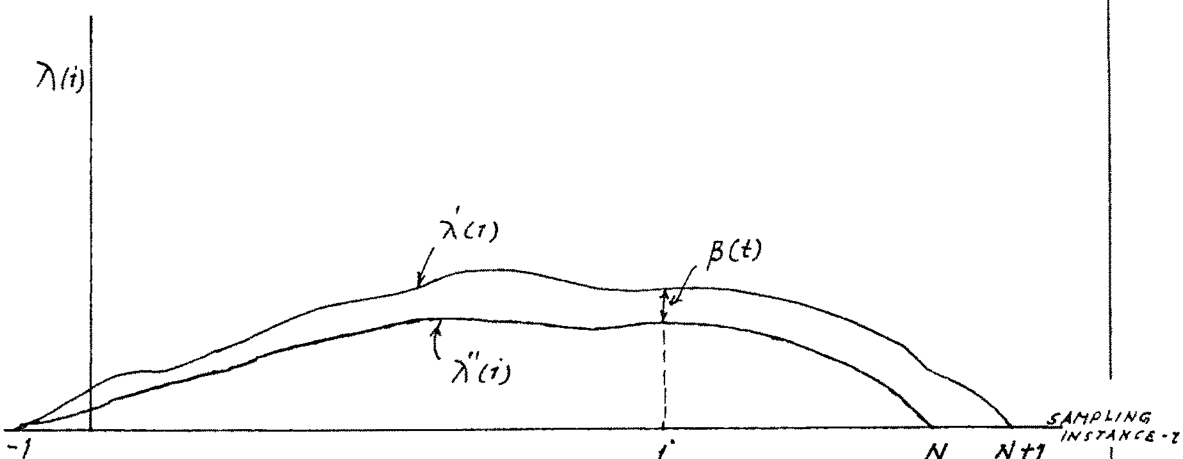


FIG. A·2 COMPARISON OF ESTIMATED TRAJECTORIES $\lambda'(i)$ AND $\lambda''(i)$ FROM N AND $(N+1)$ OBSERVATIONS.

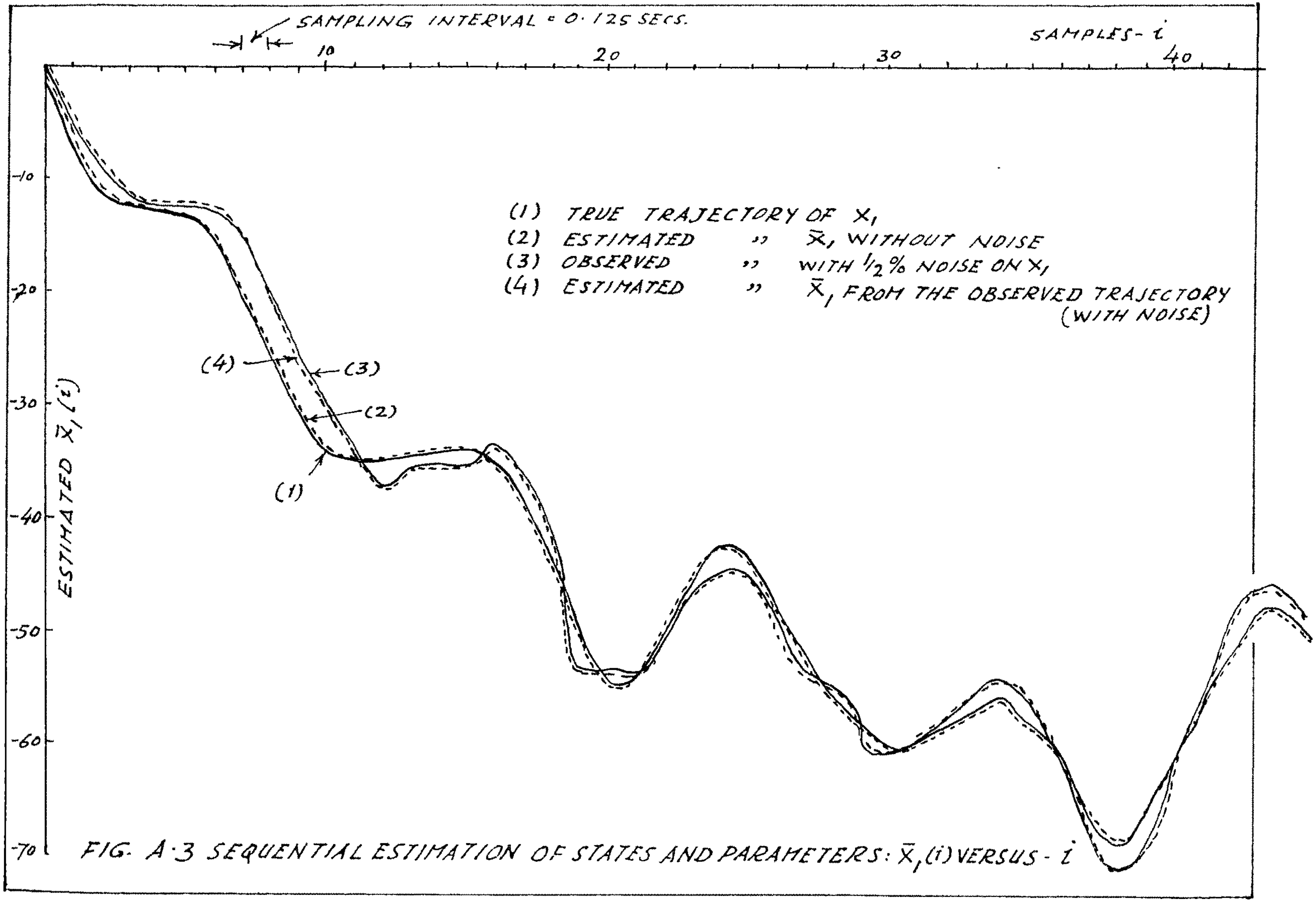


FIG. A·3 SEQUENTIAL ESTIMATION OF STATES AND PARAMETERS: $\bar{X}_1(i)$ VERSUS - i

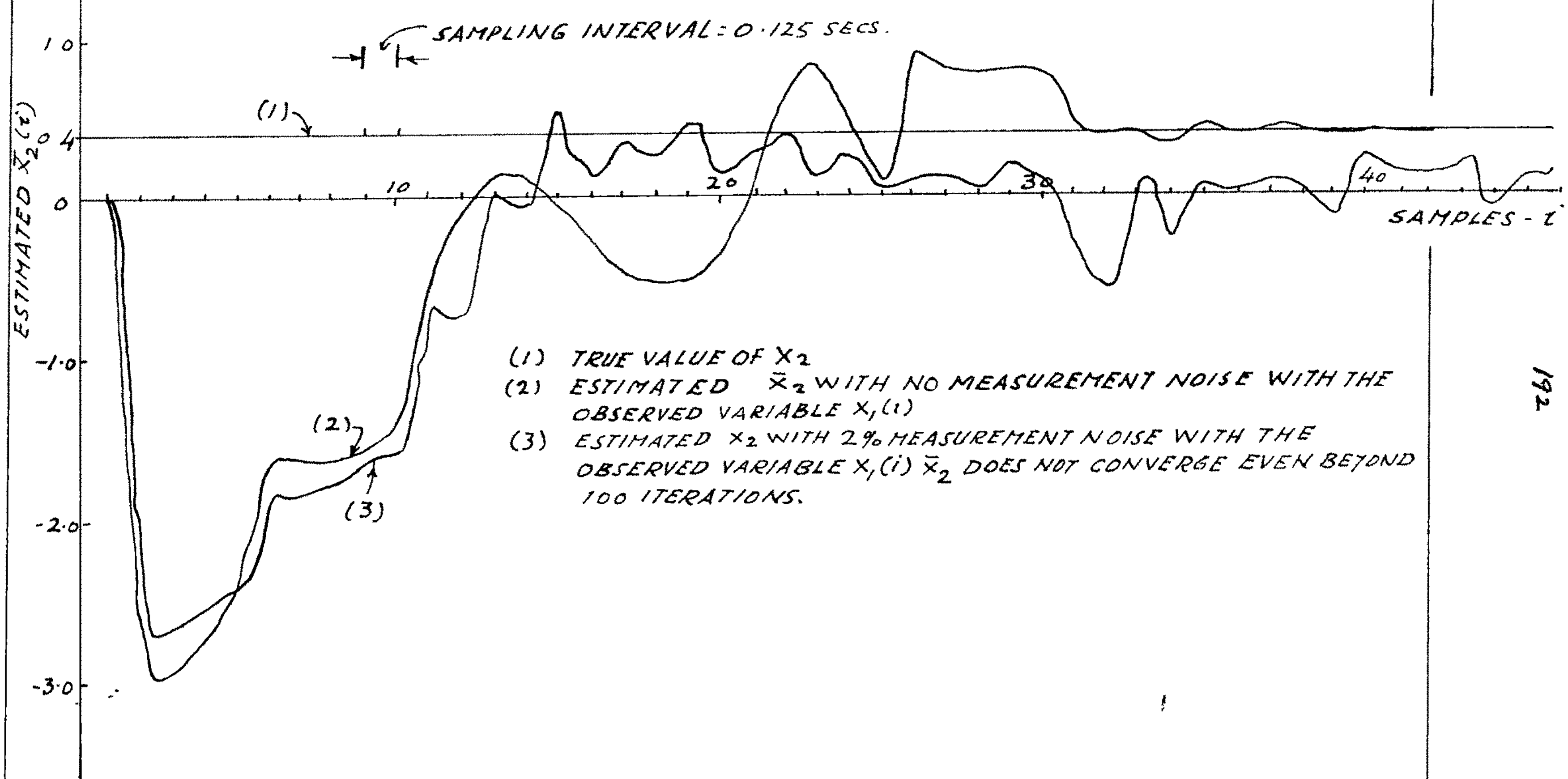
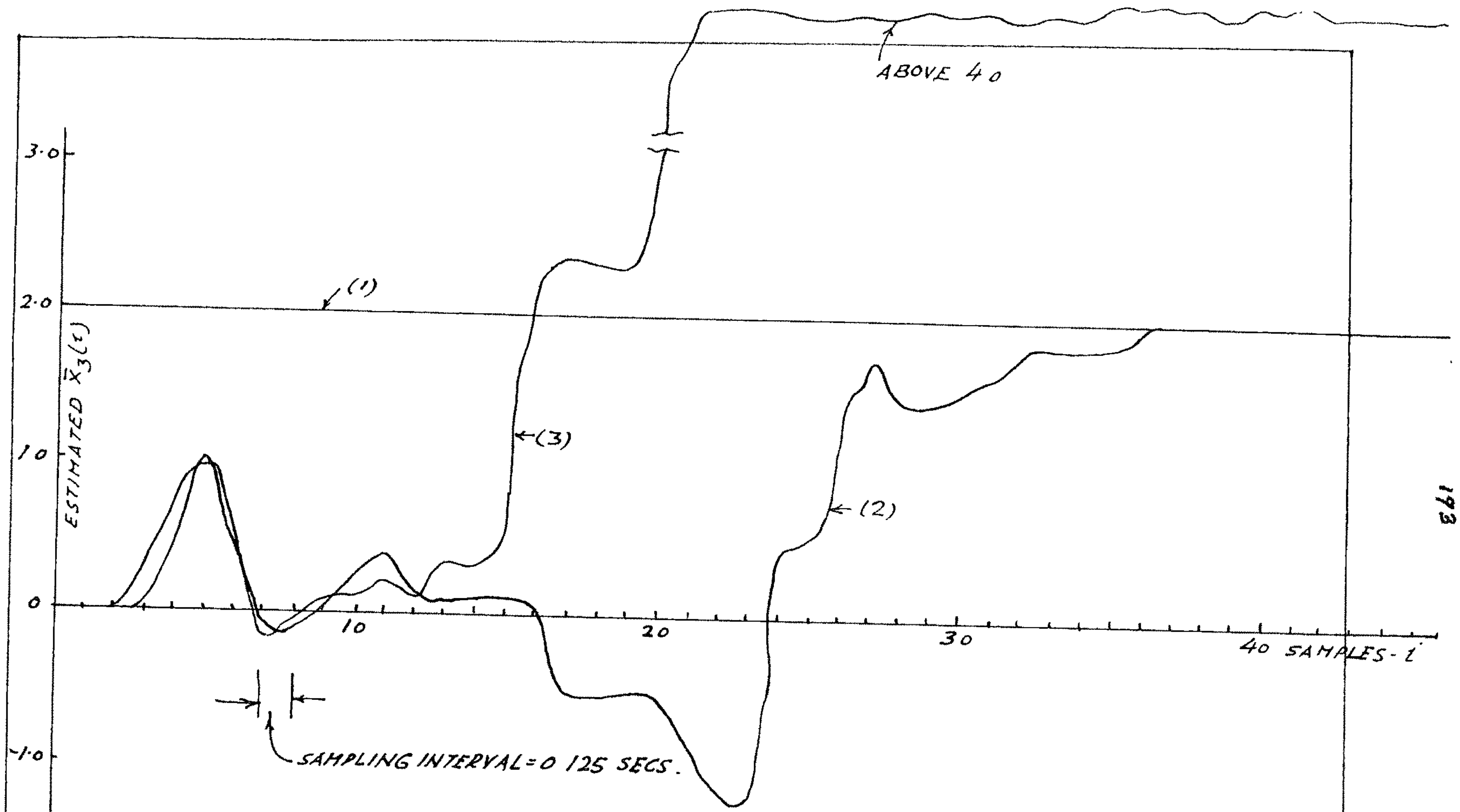


FIG. A·4 SEQUENTIAL ESTIMATION OF STATES & PARAMETERS $\bar{X}_2(i)$ VERSUS i



(1) TRUE VALUE OF X_3
(2) ESTIMATED $\bar{X}_3$ WITH NO MEASUREMENT NOISE WITH THE OBSERVED VARIABLE $X_1(i)$.
(3) " " " 2% " " " " " . $\bar{X}_3$ DOES NOT CONVERGE EVEN BEYOND 100 INTERATIONS.

FIG. A·5 SEQUENTIAL ESTIMATION OF STATES AND PARAMETERS · $\bar{X}_3(i)$ VERSUS i