SUMMARY

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EXPANSIVE HOMEOMORPHISMS ON TOPOLOGICAL SPACES AND G-SPACES

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SUMMARY

Let X denote a topological space, H(X) denote the collection of all homeomorphisms on X, d denote the metric on X if X is considered a metric space and G denote a topological group acting continuously on X in case X is considered a G-space. Given a metric space X, an h in H(X) is called *expansive* if there exists a positive real number δ such that whenever $x, y \in X$, $x \neq y$, one can find an integer n satisfying $d(h^n(x),h^n(y)) > \delta$; δ is then called an expansive constant for h. Beginning with Utz's paper [12], extensive work has been done on such homeomorphisms, for example refer [5,6,10,11,13]. Also, the concept of expansiveness is defined and studied in various contexts by various authors [2,4,7,8,9,14]. However, the concept of expansive homeomorphisms is yet not defined and studied in the settings of general topological spaces and G-spaces. In the present thesis, we define the notion of expansiveness in these general settings and study the existence / non-existence of such homeomorphisms on different spaces, their properties, their different characterizations, extension problems and several other related problems. The material of the present thesis entitled 'Expansive homeomorphisms on topological spaces and G-spaces' is the outcome of researches carried out by the author mainly along these lines.

There are five chapters in the thesis. Chapter 1 aims at providing the introduction to the subject matter of the thesis through the recent developments in the field.

In Chapter 2, we formulate and study the notion of expansiveness of an h in H(X) for a general topological space X relative to a subset A of X×X. Observe that in the definition of an expansive homeomorphism on a metric space X as given in the beginning, $d(h^n(x),h^n(y)) > \delta$ may be reframed as $(h^n(x),h^n(y)) \notin d^{-1}[0,\delta]$, where $d^{-1}[0,\delta] \equiv A_\delta$, say, is a regular closed subset of X×X containing the diagonal. This motivates the following definition. Given $A \subset X\times X$, an h in H(X) is called *A-expansive* if for x, y in X, $x \neq y$, there exists an integer n such that $(h^n(x),h^n(y)) \notin A$. Since A-expansiveness depends on A, the notion provides a wider scope even on metric spaces in the sense that certain metric spaces which do not admit expansive homeomorphism do admit A-expansive homeomorphism. In fact, one knows [1] that any finite interval I or the unit circle S^1 with usual metric does not possess an expansive homeomorphism that is, an A_δ-expansive homeomorphism for some $\delta > 0$; however, we show that there are several non-trivial choices of A, $A \subset I\times I$ or $S^1\times S^1$, for which an A-expansive homeomorphism can be constructed on them. We study several properties, deal with extension problems and obtain different characterizations of such homeomorphisms. Some of the results obtained are stated below.

Theorem 1 generalizes Bryant's result [5] and shows that unlike expansiveness, A-expansiveness is a topological property.

Theorem 1. *Let* $g:X \to Y$ *be a homeomorphism from a topological space* X *to a topological space* Y. *Then an* h *in* H(X) *is A-expansive*, $A \subset X\times X$, *iff* $ghg^{-1} \in H(Y)$ *is B-expansive where* $B = (g\times g)(A)$.

Theorem 2. *Let* X *be a paracompact Hausdorff space,* $\mathcal{U}$ *be the uniformity on it consisting of all neighbourhoods of the diagonal in* X×X *and* h *in* H(X) *be such that* h^m, $m \neq 0$, *is uniformly continuous with respect to* $\mathcal{U}$. *Then* h *is* U*-expansive for some* U *in* $\mathcal{U}$ *implies* h^m, $m \neq 0$, *is* V*-expansive for some* $V \in \mathcal{U}$.

Theorem 3. *Let* X *be first countable, countably compact and Hausdorff and let* h *in* H(X) *be* A*-expansive where* A *is a neighbourhood of the diagonal in* X × X. *Then the set of fixed points of* h *is finite.*

The following Theorem 4 gives a characterization of an A-expansive homeomorphism using which a related extension problem is studied in Theorem 5. We need the following definitions. For h in H(X) (*i*) the set $O(x) = \{ h^n(x) \mid n \in \mathbb{Z} \}$ is called the h-*orbit* of x in X (*ii*) a set $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ is called a *basis* for (X,h) if $\bigcup_{\alpha\in\mathcal{A}} O(x_\alpha) = X$ and $\alpha \neq \beta$ implies $O(x_\alpha) \cap O(x_\beta) = \varphi$ (*iii*) h is said to A-*separate* h-orbits, $A \subset X \times X$, if given any basis $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ for (X,h), $x_\alpha \neq x_\beta$ implies the existence of an integer n such that $(h^n(x_\alpha), h^n(x_\beta)) \notin A$.

Theorem 4. *Let* $h \in H(X)$ *and* $A \subset X \times X$. *Then* h *is* A*-expansive iff* (*i*) h A*-separates* h*-orbits and* (*ii*) *given* p *in* X *and* n *in* $\mathbb{Z}$ *such that* $h^n(p) \neq p$, *there exists* $r \in \mathbb{Z}$ *satisfying* $(h^r(p), h^{r-n}(p)) \notin A$.

Theorem 5. *Let* $Y \subset X$, $A \subset X \times X$ *and* h *in* H(Y) *be* A*-expansive. Then a homeomorphic extension* f *of* h *to* X *is* A*-expansive if* (*i*) $f|_{(X-Y)}$ *is* A*-expansive and there exists a basis* $\mathcal{B}$ *for* (Y,h) *such that* $(x,y) \notin A$ *for* x *in* $\mathcal{B}$ *and* $y \in X-Y$.

Theorems 4 and 5 reduce to Theorems 1 and 2 of [13] if X, Y are metric spaces and $A = A_\delta$ for some $\delta > 0$.

Another characterization of A-expansive homeomorphisms appears in the following result in terms of generators defined as follows.

If X is a paracompact Hausdorff space and $h \in H(X)$, then a locally finite open covering $\mathcal{U}$ of X is called a *generator* for (X,h) if for each bisequence $\{ U_i \mid i \in Z \}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$ contains at most one point.

Theorem 6. *Let* X *be a paracompact Hausdorff space and* $h \in H(X)$. *Then* (X,h) *has a generator iff* h *is* A-*expansive for some neighbourhood* A *of the diagonal in* $X \times X$.

In Chapter 3 we define and study expansive homeomorphisms on metric G-spaces. Obviously, every metric space is a metric G-space under the trivial action of G on X; and, then every expansive h in H(X) with expansive constant $\delta > 0$ clearly satisfies the following condition : For x,y in X with $G(x) \neq G(y)$, there exists an integer n such that $d(h^n(u),h^n(v)) > \delta$ for each u in G(x) and each v in G(y), wherein for a in X, $G(a) = \{ ga \mid g \in G \}$ is called the G-*orbit* of a in X. In view of this, we consider in this chapter metric G-spaces X in general and define the notion of a G-expansive homeomorphism on X. More precisely, an h in H(X) is called G-*expansive* with G-*expansive constsnt* $\delta > 0$ if for x,y in X with $G(x) \neq G(y)$, there exists an integer n such that $d(h^n(u),h^n(v)) > \delta$ for each u in G(x) and each v in G(y). Examples are provided to show that expansiveness neither implies nor is implied by G-expansiveness. This leads us to determine some conditions under which expansiveness implies G-expansiveness and vice-versa. For example, we prove the following Theorem 7.

Theorem 7. *Let* X *be a metric* G-*space and* h *be in* H(X).*If (i)* h *is pseudoequivariant i.e. it satisfies* $h(G(x)) = G(h(x))$, $x \in X$;*(ii)* G *is a subgroup of* ISO(X), *the group of isometries on* X; *(iii) for each pair of distinct* G-*orbits* G(x) *and* G(y) *in* X, *there exists a* g_o *in* G *satisfying*

(a) $d(g_o x, y) = d(G(x), G(y))$ *and*

(b) for all g_o *in* G *satisfying (a),*

$d(h^i(g_o x), h^i(y)) \leq d(h^i(gg_o x), h^i(y))$, $g \in G$ *where* $i \in \{-1, 1\}$,

then h *is* G-*expansive with* G-*expansive constant* δ *whenever it is expansive with expansive constant* δ.

Next result gives a characterization of a G-expansive homeomorphism using which a related extension problem is studied inthe following Theorem 9. We need the following definition. Given $\delta > 0$, an h in H(X) is said to G-δ *seperate* h-orbits if given any basis $\mathcal{B} = \{x_\alpha \in X \mid \alpha \in \mathcal{A}\}$ of (X,h), whenever $G(x_\alpha) \neq G(x_\beta)$, there existsan integer r satisfying $d(h^r(gx_\alpha), h^r(kx_\beta)) > \delta$, $g, k \in G$.

Theorem 8. *If* h *is a pseudoequivariant homeomorphism on a metric* G-*space* X, *then* h *is* G-*expansive with* G-*expansive constant* δ *iff*

(i) h-*orbits are* G-δ *seperated by* h;

(ii) for p *in* X *and integer* n *such that* $h^n(p) \notin G(p)$, *there exists an integer* r *satisfying* $d(h^{n+r}(gp), h^r(kp)) > \delta$, $g, k \in G$.

Theorem 9. *Let* X *be a* G-*invariant subspace of a metric* G-*space* Y *and* h in H(X) *be pseudoequivariant* G-*expansive with* G-*expansive constant* δ. *Then a pseudoequivariant extension* f *of* h, $f \in H(Y)$, *is* G-*expansive with* G-*expansive constant* δ *if* f *is* G-*expansive on* Y-X *with* G-*expansive constant* δ *and there exists a basis* $\mathcal{B}$ *for* (X,h) *such that* $d(gx, Y-X) > \delta$, *for each* g *in* G *and* x *in* $\mathcal{B}$.

Theorems 8 and 9 reduce to Theorems 1 and 2 of [13] if the action of G is trivial.

In Chapter 4, we define the notion of a G-generator and a weak G-generator for a homeomorphism on a compact Hausdorff G-space. In fact, given a compact Hausdorff G-space X and an h in H(X), a finite cover $\mathcal{U}$ of X consisting of G-invariant open sets is called a *G-generator* (respectively *weak G-generator*) for (X,h) if for each bisequence $\{ U_i \mid i \in Z \}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(ClU_i)$ (respectively $\bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$) contains at most one G-orbit. Examples are provided to show that a G-generator need not be a generator and vice-versa. We prove that existence of a G-generator on X implies the metrizability of the orbit space X/G of X. We obtain a characterization of a G-expansive homeomorphism on a compact metric G-space in terms of a G-generator. We also define the notion of G-asymptotic points for a homeomorphism on a metric G-space and study them relative to G-generators.

In Chapter 5, we introduce the concept of GA-expansiveness of a homeomorphism on a topological G-space X where $A \subset X \times X$. Examples are provided to show that A-expansiveness neither implies nor is implied by GA-expansiveness. We study properties of such homeomorphisms, obtain their characterizations which in turn gives a sufficient condition for the homeomorphic extension of a GA-expansive homeomorphism on a subspace to be GA-expansive on the whole space.

Some results of Chapter 3, in its original form, are published in the 'Proceedings of the Symposium on Topology', Second Biennial conference of the Allahabad Mathematical Society, Allahabad 1990.

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AN INTRODUCTION TO G-EXPANSIVE SELF-HOMEOMORPHISMS

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1. Introduction

The notion of an expansive self-homeomorphism on a metric space introduced by Utz [7] in 1950, with the term unstable homeomorphism found its application in various topics of Mathematical researches including the fixed point theory and topological dynamics. Enriching the space with group action, we find analogue of this notion and put the results of Wine [11] in this setting.

2. Notations and Terminologies

A metric space X on which a group G acts is called a *metric G-space*. The orbit of a point $x \in X$ is denoted by $G(x)$.

Let $h : X \to X$ be a homeomorphism of a metric space X. Then the *h-orbit* of a point $x \in X$, denoted by $O(x)$ is the set $\{h^n(x) | n \in Z$, where Z denotes the set of integers$\}$.

A self-homeomorphism h of a metric space (X, d) is called *expansive* if there is a $\delta>0$ such that whenever $x, y\in X$ with $x\neq y$, there exists an integer satisfying $d(h^n(x), h^n(y))>\delta$; δ is called an *expansive constant* for h.

Consider the following examples :

(i) Let $X=\{1/n, 1-1/n|n\in N$, where N denotes the set of positive integers$\}$, considered as a subspace of the real line R. Then $h : X\to X$ defined by fixing 0, 1 and sending $x\in X$ next to the right of x is an expansive homeomorphism with any real number δ lying between 0 and 1/6 as an expansive constant. However, if we consider the action of Z_2, the additive group of integers modulo two on X defined by $0.t=t$ and $1.t=1-t$, $t\in X$, then for any two points $x, y\in X-\{\frac{1}{2}\}$ with distinct orbits, there is no integer n satisfying $|h^n(u)-h^n(v)|>\delta$ for all $u\in G(x)$ and $v\in G(y)$.

(ii) Consider the subspace $X'=XU\{-1/n, -1+1/n|n\in N\}$ of the real line R (X denotes the subspace of R considered in example (i)) and define $h' : X'\to X'$ by sending x to $h(x)$, if $x\in X$, sending x next to the left of x, if $x<0$ and keeping -1 fixed. Then h' continues to be expansive with any real nember δ lying between 0 and 1/6 as an expansive constant. Moreover,under the action of Z_2 on X' defined by $0.t=t$ and $1.t=-t$, $t\in X'$, for every pair of points $x, y\in X'$ having distinct orbits, there is an integer n satisfying $|h^n(u)-h^n(v)|>\delta$, where $u\in G(x)$ and $v\in G(y)$.

(iii) Consider the Euclidean 2-space R^2 and $h : R^2\to R^2$ defined by $h(x)=ax$, where $a>1$. Then h is an expansive homeomorphism with any positive real number δ as an expansive constant. Let R^2 be acted upon by the orthogonal group 0(2) by $T.x=T(x)$, $x\in R^2$. It can be seen that for every pair of points $x, y\in R^2$ with distinct orbits, an integer n making the n^{th}-iterates of two point $u\in G(x)$, $v\in G(y)$ at the shortest distance, to lie at a distance greater than δ satisfies the following :

$d(h^n(gx), h^n(ky))>\delta$, $g, k\in G$, where d is the Euclidean metric on R^2. We observe the same in the case of any integer $n>2$.

Motivated by the above examples we introduce the notion of G-expansive self-homeomorphism on a metric G-space.

A self-homeomorphism h on a metric G-space (X, d) is *G-expansive with expansive constant* $\delta>0$ if for $x, y\in X$ having distinct orbits there exists an integers n satisfying $d(h^n(gx), h^n(ky))>\delta$, $g, k\in G$.

Clearly a G-expansive self-homeomorphism is expansive. Example (i) shows that the converse is not true, in general. For the two notions to coincide we have the following result :

Let h be an expansive homeomorphism on a metric G-space X with expansive constant δ satisfying $h(G(x))=G(h(x))$, $x\in X$, where G is a subgroup of the group $ISO(X)$, of isometries of X. Suppose that each orbit is a closed set of X and for $x, y\in X$ with distinct orbits there is a $g_0\in G$ satisfying

(*a*) $$d(g_0x, y)=d(G(x), G(y))$$

and for $g'\in G$ satisfying (i) the following holds :

(b) $d(h^i(g'x), h^i(y))\leqslant d(h^i(gg'x), h^i(y))$, $g\in G$, *where* $i\in\{-1, 1\}$.

Then h is G-expansive with expansive constant δ.

Let h be a self-homeomorphism of a metric space (X, d). The set $\mathscr{B}=\{x_\alpha\in X|\ \alpha\in\mathscr{A}\}$, $\mathscr{A}$ is an index set} is called a *basis* of X with respect to (w.r.t.) h if $\bigcup_{\alpha\in\mathscr{A}} O(x_\alpha)=X$ and $O(x_\alpha)\cap O(x_\beta)=\phi$ for $\alpha\neq\beta$.

A self-homeomorphism h on a metric G-space (X, d), for a given $\delta>0$, is said to $G-\delta$ *separate orbits* if given any basis $\mathscr{B}=\{x_\alpha\in X|\alpha\in\mathscr{A}\}$ of X w.r.t. h, whenever $G(x_\alpha)\neq G(x_\beta)$ there exist integers m and M with $M-m\geqslant 2$ satisfying

$$d(h^i(gx_\alpha), h^i(kx_\beta))>\delta,\ g, k\in G,\ m<i<M.$$

We now state following two results, one regarding the characterization of G-expansiveness and the other for the extension of a G-expansive self-homeomorphism on an invariant subspace of a metric G-space to the whole space.

1. *Let (X, d) be a metric G-space and $h : X\to X$, a homeomorphism satisfying $h(G(x))=G(h(x))$, $x\in X$. Then h is G-expansive with expansive constant δ iff.*

(i) *orbits are $G-\delta$ separated by h,*

(ii) *for* $p \in X$ *and integer* n *such that* $h^n(p) \notin G(p)$ *there exists an integer* r *satisfying* $d(h^r(gp), h^{n+r}(kp)) > \delta$, $g, k \in G$.

2. *Let* X *be an invariant subspace of a metrtc* G*-space* (Y, d) *and* h *be a* G*-expansine self-homeomorphism on* X *with expansive constant* δ *satisfying* $h(G(x)) = G(h(x))$, $x \in X$. *A self-homeomorphism* f *on* Y *extending* h *and satisfying* $f(G(y)) = G(f(y))$, $y \in Y$, *is* G*-expansive with expansive constant* δ *if*

(a) f *is* G*-expansive on* $Y - X$ *with expansive constant* δ,

(b) *for any basis* $\mathscr{B}$ *of* X *w. r. t. h.*, $d(gx, Y - X) > \delta$, $g \in G$ *and* $x \in \mathscr{B}$.

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