CHAPTER 4

GENERATORS FOR HOMEOMORPHISMS ON G-SPACES

Observe that for a given compact Hausdorff space X the notion of a generator for (X,h) as defined in Definition 1.5 involves a homeomorphism h on the underlying space X while the notion of a generator for (X,G), G a discrete group acting on X, as defined in Definition 1.7 does not involve any homeomorphism on the underlying space X. Therefore, it is natural to ask whether the notion of a generator / weak generator involving a homeomorphism on an arbitrary compact Hausdorff G-space X, wherein G need not be discrete, can be meaningfully defined. In this chapter we attend to this problem and define such notions in this setting and terming them G-generators and weak G-generators carry out their study. We also define the notion of G-asymptotic points for homeomorphism on a metric G-space and study them relative to G-generators on a compact metric G-space.

Let H(X) throughout denote the collection of all homeomorphisms on the topological space X.

1. G-generators and weak G-generators.

Definition 4.1. Let X be a G-space and $h \in H(X)$. Then a finite cover $\mathcal{U}$ of X consisting of G-invariant open sets is called a G-*generator* (respectively *weak* G-*generator*) for (X,h) if for each bisequence $\{U_i\}_{i\in\mathbb{Z}}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$ (respectively $\bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$) contains at most one G-orbit.

Let us give an example of a G-generator.

Example 4.1. Consider the space $X = \{ 1/n, 1 - 1/n \mid n \in \mathbf{N} \}$ with the usual metric defined through the absolute value and the h in $H(X)$ which fixes 0 and 1 and sends t in $X - \{0,1\}$ to the point of X which is next to the right of t. Let $G \equiv \mathbf{Z}_2 = \{-1,1\}$ act on X with the action defined by $(-1)t = 1- t$, $1t = t$, $t \in X$. Then it is easily seen that the finite cover $\mathcal{U}$ of X consisting of G-invariant open sets U and $X - U$ where $U = \{ 1/2, 1/3, 2/3, 1/4, 3/4 \}$ is such that for any bisequence $\{U_i\}_{i\in\mathbb{Z}}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$ is either empty or is $\{0,1\}$. Thus, $\mathcal{U}$ is a G-generator for (X,h).

Note. Under the trivial action of G on X where X and G are as in Definition 4.1, a G-generator (respectively weak G-generator) for (X,h) is equivalent to a generator (respectively weak generator). But under a non-trivial action of G on X, a G-generator for (X,h) need not be a generator as can be seen in Example 4.1. In the following example we show that the converse is also not true, i.e., a generator need not be a G-generator.

Example 4.2. Let X, h and G be same as in Example 4.1. Choose any δ in $(0,1/6)$ and consider the open cover $\mathcal{U} = \{ B(x,\delta/2) \mid x \in X \}$ of X where $B(x,\delta/2)$ is the open ball centred at x with radius $\delta/2$.

Then any finite subcover $\mathcal{U}'$ of $\mathcal{U}$ is a generator but not a G-generator for (X,h) since the members of $\mathcal{U}$ are not G-invariant subsets of X.

Clearly a G-generator for (X,h) is also a weak G-generator for (X,h) while a weak G-generator need not be a G-generator. For the converse we have the following result.

Theorem 4.1. *Let* X *be a compact Hausdorff* G*-space with* G *compact and* h *in* H(X). *Then* (X,h) *has a* G*-generator whenever it has a weak* G*-generator.*

Proof. Let $\mathcal{U}$ be a weak G-generator for (X,h). For $x \in X$, choose U_x in $\mathcal{U}$ such that $x \in U_x$. Then $p(x) \in p(U_x)$ where $p : X \to X/G$ is the orbit map. Since an orbit map is known to be an open map [4,p.37], $p(U_x)$ is open in X/G. Also, X being regular Hausdorff and G being compact, it follows that X/G is a regular Hausdorff space and hence there exists an open set, say $A_{p(x)}$ of X/G, such that

$$p(x) \in A_{p(x)} \subseteq \mathrm{Cl}A_{p(x)} \subseteq p(U_x).$$

Since U_x is G-invariant, we obtain

$$x \in p^{-1}(A_{p(x)}) \subseteq p^{-1}(\mathrm{Cl}A_{p(x)}) \subseteq p^{-1}(p(U_x)) = U_x.$$

Clearly $\mathcal{V} = \{ p^{-1}(A_{p(x)}) \mid x \in X \}$ is an open cover of X and the fact that $p(y) = p(gy)$ for all g in G gives that $p^{-1}(A_{p(x)})$ is G-invariant for each x in X. Thus $\mathcal{V}$ is an open cover of X consisting of G-invariant subsets of X. We complete the proof by showing that any finite subcover $\mathcal{V}'$ of $\mathcal{V}$ is a G-generator for (X,h). Suppose $\{p^{-1}(A_i)\}_{i\in\mathbb{Z}}$ is any bisequence of members of $\mathcal{V}'$.

Then we have

$$\bigcap_{i=-\infty}^{\infty} h^{-i}(Clp^{-1}(A_i)) \subseteq \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}(ClA_i) \subseteq \bigcap_{i=-\infty}^{\infty} h^{-i}(U_i).$$

As $\mathcal{U}$ is a weak G-generator for (X,h), $\bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$ contains at most one G-orbit and therefore we get $\bigcap_{i=-\infty}^{\infty} h^{-i}[Clp^{-1}(A_i)]$ contains at most one G-orbit.

Recall that a homeomorphism h on a G-space X is called pseudoequivariant if h(G(x)) = G(h(x)) for all x in X, and in Chapter 3 we have noted that an equivariant map is pseudoequivariant but the converse is not true. The fact that an equivariant homeomorphism h on a G-space X induces a homeomorphism h_G on the orbit space X/G is shown, in the following lemma, to hold true even if the equivariancy condition is replaced by a weaker condition of pseudoequivariancy. This property of pseudoequivariancy is then used to prove in Theorem 4.3 that if (X,h) has a G-generator, where X is compact Hausdorff G-space, with G compact, then the orbit space X/G is metrizable.

Lemma 4.2. *Let* X *be a compact Hausdorff* G*-space with* G *compact and let* h *in* H(X) *be pseudoequivariant. Then the map* $h_G : X/G \to X/G$ *defined by* $h_G(G(x)) = G(h(x))$ *is a homeomorphism such that* $p \circ h = h_G \circ p$, *where* $p : X \to X/G$ *is the orbit map, i.e., the following diagram commutes :*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ p\downarrow & & \downarrow p \\ X/G & \xrightarrow{h_G} & X/G. \end{array}$$

Proof. First let us note that the map h_G is well defined. Let G(x) and G(y) ∈ X/G with G(x) = G(y). Then h(G(x)) = h(G(y)) and thus by the pseudoequivariancy of h we get G(h(x)) = G(h(y)) and hence $h_G(G(x)) = h_G(G(y))$. Next, suppose $h_G(G(x)) = h_G(G(y))$ for some G(x) and G(y) in X/G. Then we have G(h(x)) = G(h(y)) which by pseudoequivariancy of h implies h(G(x)) = h(G(y)). As h is injective, we get G(x) = G(y) which proves the injectivity of h_G. For the surjectivity of h_G, observe that if G(y) ∈ X/G, then $h_G(G(x)) = G(y)$ where h(x) = y. Also, for any x in X

$$p \circ h(x) = p(h(x)) = G(h(x))$$
$$= h_G(G(x)) = h_G(p(x)) = h_G \circ p(x).$$

Hence the above diagram is commutative. Next, let U be any open set of X/G. Then note that $(h_G)^{-1}(U) = ph^{-1}p^{-1}(U)$ and hence continuity of h implies the continuity of h_G as p is continuous as well as open. Finally, h_G is a closed map because X is given compact and Hausdorff and G being compact, X/G is also a compact Hausdorff space. Thus h_G is a homeomorphism.

Theorem 4.3. *Let* X *be a compact Hausdorff* G-*space with* G *compact and let* h ∈ H(X) *be pseudoequivariant. If* (X,h) *has a* G-*generator, then*

(i) $(X/G, h_G)$ *has a generator; and*

(ii) X/G *is metrizable.*

Proof. Let $\mathcal{U} = \{ U_1, \ldots\ldots, U_n\}$ be a G-generator for (X,h). Let $\mathcal{V} = \{ p(U_1), \ldots\ldots, p(U_n)\}$. Then we complete the proof of (i) by showing that $\mathcal{V}$ is a generator for $(X/G, h_G)$.

The surjectivity and the openness of the map p implies that $\mathcal{V}$ is a finite open cover of X/G. Let $\{p(U_i)\}_{i\in\mathbb{Z}}$ be any bisequence of members of $\mathcal{V}$. Then in order to show that $\bigcap_{i=-\infty}^{\infty} (h_G)^{-i}[\mathrm{Cl}p(U_i)]$ has at most one point, assume on the contrary that

$$G(x),\ G(y) \in \bigcap_{i=-\infty}^{\infty} (h_G)^{-i}[\mathrm{Cl}p(U_i)] \text{ and } G(x) \neq G(y).$$

Now,

$$\begin{aligned} G(x),\ G(y) \in \bigcap_{i=-\infty}^{\infty} (h_G)^{-i}\mathrm{Cl}p(U_i) &\Rightarrow p(x),\ p(y) \in \bigcap_{i=-\infty}^{\infty} (h_G)^{-i}[\mathrm{Cl}p(U_i)] \\ &\Rightarrow x,\ y \in \bigcap_{i=-\infty}^{\infty} p^{-1}(h_G)^{-i}[\mathrm{Cl}p(U_i)] \\ &\Rightarrow x,\ y \in \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}[\mathrm{Cl}p(U_i)] \\ &\Rightarrow x,\ y \in \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}p(\mathrm{Cl}U_i), \end{aligned}$$

as p is continuous and closed. Since closure of a G-invariant set is G-invariant, it therefore follows that $\mathrm{Cl}U_i$ is G-invariant for all i. Also, from the last implication we get $x,\ y \in \bigcap_{i=-\infty}^{\infty} h^{-i}[\mathrm{Cl}U_i]$. Finally, as $\mathcal{U}$ is a G-generator for (X,h) we must have G(x) = G(y). Hence $\mathcal{V}$ is a generator for $(X/G,h_G)$.

The proof of (ii) now follows by the Keynes and Robertson result [29, Corollary 2.8] which says that if there exists a generator for a homeomorphism on a compact Hausdorff space, then the space is metrizable.

For the converse, we have the following result.

Theorem 4.4. *Let* X *be a compact Hausdorff* G*-space with* G *compact and let* h *in* H(X) *be pseudoequivariant. Then* $(X/G,h_G)$ *has a generator implies* (X,h) *has a* G*-generator.*

Proof. Let $\mathcal{U} = \{ U_1,\dots,U_n \}$ be a generator for $(X/G,h_G)$. Consider

$\mathcal{V} = \{ p^{-1}(U_1),\ldots,p^{-1}(U_n) \}$ where $p : X \to X/G$ is the orbit map. Then $\mathcal{V}$ is clearly a finite open cover of X by G-invariant sets. We complete the proof by showing that $\mathcal{V}$ is a G-generator for (X,h). Suppose $\{p^{-1}(U_i)\}_{i\in\mathbb{Z}}$ is any bisequence of members of $\mathcal{V}$. Let x, y be in the $\bigcap_{i=-\infty}^{\infty} h^{-i}[Clp^{-1}(U_i)]$. We need to show that G(x) = G(y). Now, since $Clp^{-1}(U_i)$ are G-invariant for all i, we have

$$\bigcap_{i=-\infty}^{\infty} h^{-i}[Clp^{-1}(U_i)] = \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}p[Clp^{-1}(U_i)]$$
$$= \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}[Clpp^{-1}(U_i)]$$

as p is a closed map. Therefore, surjectivity of p and the fact that $h_G \circ p = p \circ h$ implies

$$\bigcap_{i=-\infty}^{\infty} h^{-i}[Clp^{-1}(U_i)] = \bigcap_{i=-\infty}^{\infty} h^{-i}p^{-1}(ClU_i)$$
$$= \bigcap_{i=-\infty}^{\infty} p^{-1}(h_G)^{-i}(ClU_i)$$
$$= p^{-1}[\bigcap_{i=-\infty}^{\infty}(h_G)^{-i}(ClU_i)].$$

Hence $x,\ y \in \bigcap_{i=-\infty}^{\infty} h^{-i}[Clp^{-1}(U_i)]$ implies

$$p(x),\ p(y) \in \bigcap_{i=-\infty}^{\infty}(h_G)^{-i}(ClU_i).$$

That is

$$G(x),\ G(y) \in \bigcap_{i=-\infty}^{\infty}(h_G)^{-i}(ClU_i).$$

Now using the fact that $\mathcal{U}$ is a generator for $(X/G,h_G)$ we conclude that G(x) = G(y). It follows that $\mathcal{V}$ is a G-generator for (X,h).

2. G-generators and G-expansiveness.

Theorem 1.9 due to Keynes and Robertson [29] gives a characterization of an expansive homeomorphisms on a compact metric space in terms of a generator. In the following we characterize a G-expansive homeomorphisms on a compact metric G-space in terms of a G-generator.

Recall that if X is a metric G-space with metric d and G compact, then X/G is also a metric space with metric ρ defined by $\rho(G(x),G(y)) = \text{Inf}\ \{ d(gx,ky) \mid g,\ k \in G \}$.

Theorem 4.5. *Let* X *be a compact metric* G*-spacce with* G *compact and let* $h \in H(X)$ *be pseudoequivariant. Then* h *is* G*-expansive iff* (X,h) *has a* G*-generator.*

Proof. Let h be G-expansive with G-expansive constant δ. Then any finite subcover $\mathcal{V}$ of the open cover of the form

$$\mathcal{U} = \{ p^{-1}(B(G(x),\delta/2)) \mid x \in X \}$$

of X where $B(G(x),\delta/2)$ denotes the open ball in X/G centred at G(x) with radius $\delta/2$ under the metric ρ will work as a G-generator for (X,h). In fact, members of $\mathcal{V}$ are clearly G-invariant subsets of X. Moreover, for any bisequence $\{p^{-i}(B_i)\}_{i\in\mathbb{Z}}$ of members of $\mathcal{V}$, if x, $y \in \bigcap_{i=-\infty}^{\infty} h^{-i}[\text{Cl}p^{-1}(B_i)]$, then G(x) = G(y). To show this assume $G(x) \neq G(y)$. Using the closedness of p and the commutativity $p\circ h = h_G \circ p$, we get

$$G(x),\ G(y) \in \bigcap_{i=-\infty}^{\infty} (h_G)^{-i}(\text{Cl}B_i)$$

which implies

$$\rho(h^i(G(x)),h^i(G(y))) \le \delta$$

for all i in $\mathbb{Z}$. Further, as G is compact so G(x) and G(y) are disjoint compact sets and hence for a given i in $\mathbb{Z}$ there exist g_i, k_i in G such that

$$d(h^i(g_i x),h^i(k_i y)) = \rho(h^i(G(x)),h^i(G(y))) \le \delta.$$

But this contradicts the fact that h is G-expansive with G-expansive constant δ.

Conversely, if $\mathcal{U} = \{ U_1, \ldots, U_n \}$ is a G-generator for (X,h), then h will be G-expansive with G-expansive constant δ where δ is a Lebesgue number of the open cover $\mathcal{V} = \{ p(U_1), \ldots, p(U_n) \}$ of X/G. For, if not, then there exist $x, y \in X$ with $G(x) \neq G(y)$ such that for each $i \in \mathbb{Z}$ one gets $g_i, k_i \in G$ satisfying

$$d(h^i(g_i x), h^i(k_i y)) \leq \delta.$$

Thus for each i in $\mathbb{Z}$, we obtain

$$\begin{aligned} \rho(h^i(G(x)), h^i(G(y))) &= \operatorname{Inf} \{ d(h^i(gx), h^i(ky)) \mid g, k \in G \} \\ &\leq d(h^i(g_i x), h^i(k_i y)) \\ &\leq \delta. \end{aligned}$$

Since δ is a Lebesgue number for $\mathcal{V}$, one obtains for each integer i in $\mathbb{Z}$, a $p(U_i)$ in $\mathcal{V}$ such that

$$h^i(G(x)),\ h^i(G(y)) \in p(U_i)$$

which implies

$$G(x),\ G(y) \in (h_G)^{-i}(p(U_i))$$

and hence

$$x,\ y \in \bigcap_{i=-\infty}^{\infty} p^{-1}(h_G)^{-i} p(U_i).$$

Finally, in view of $p \circ h = h_G \circ p$ we get

$$x,\ y \in \bigcap_{i=-\infty}^{\infty} h^{-i} p^{-1} p(U_i) = \bigcap_{i=-\infty}^{\infty} h^{-i}(U_i) \subseteq \bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$$

with distinct G-orbits. But this contradicts the fact that $\mathcal{U}$ is a G-generator for (X,h).

Note. To see the fact that the condition of pseudoequivariancy of h in Theorem 4.5 is necessary, recall Example 4.1 and first note that the h defined in this example is not pseudoequivariant. In fact take $x = 1/3$, then

$$G(h(x)) = \{1/2\} \text{ where as } h(G(x)) = \{ 1/2, 3/4 \}$$

and hence

$$G(h(x)) \neq h(G(x)).$$

Next, note that as observed in Example 3.1(a) of Chapter 3, h is not G-expansive. However, (X,h) has a G-generator.

Now we use our Theorem 4.3, 4.4 ,4.5 and Theorem 1.9 stated in Chapter 1 to obtain the following interesting result.

Theorem 4.6. *Let* X *be a compact metric* G*-space with* G *compact and let* h *in* H(X) *be pseudoequivariant. Then* h *is* G*-expansive on* X *iff* h_G *is expansive on* X/G.

Proof. Suppose h is G-expansive on X. Then by Theorem 4.5 (X,h) has a G-generator and therefore by Theorem 4.3 $(X/G,h_G)$ has a generator. Now apply Theorem 1.9 to conclude that h_G is expansive on X/G.

Conversely, if h_G is expansive on X/G, then Theorem 1.9 guarantees the existence of a generator for $(X/G,h_G)$, and therefore by Theorem 4.4, (X,h) has a G-generator. Finally, apply Theorem 4.5 to obtain that h is G-expansive.

3. G-generators and G-asymptotic points.

The notion of positively and negatively asymptotic points are defined (recall Definition 1.8 from Chapter 1) and studied in detail [5,6,8,13,36,37,41]. Theorem 1.13 of Chapter 1 obtained by Bryant and Walter [8] gives a necessary and sufficient condition for two points to be positively asymptotic under a homeomorphism on a compact metric space which has a generator. In this section, we consider a metric G-space, define the notion of positively and negatively G-asymptotic points for a homeomorphism on such a space, give examples and study their relation with G-generators.

Throughout in this section d denotes the metric of a metric space X.

Definition 4.2. Let X be a metric G-space and let $h \in H(X)$. Then x y $\in$ X are called *positively G-asymptotic* (respectively *negatively G-asymptotic*) *points* with respect to h if given $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ (respectively $n \leq N$) one has $d(h^n(gx),h^n(ky)) < \varepsilon$ for some g, k in G.

Note. Under the trivial action of a G on X the notion of positively (respectively negatively) G-asymptotic points coincides with the notion of positively (respectively negatively) asymptotic points. Under a non-trivial action of G on X, obviously positively (respectively negativly) asymptotic points with rspect to a homeomorphism on X are positively (respectively negatively) G-asymptotic points; in fact take g = k = the identity element of G. However, the fact that the converse need not be true can be seen from the following example.

Example 4.3. Consider $X = \{ \pm 1/m, \pm(1 - 1/m) \mid m \in \mathbf{N} \}$ with usual metric and define $h : X \to X$ by

$$h(x) = x \text{ if } x \in \{-1,0,1\}$$
$$= \text{the point of } X \text{ which is immediately next to the right (left) of } x \text{ if } x > 0 \ (x < 0).$$

Obviously $h \in H(X)$. Let $G = \{-1,1\}$ act on X by $-1x = -x$ and $1x = x$ $x \in X$. Then the points $x = -1/8$ and $y = 1/4$ are seen to be positively G-asymptotic but are not positively asymptotic with respect to h [8].

The following result for G-generators is along the lines of Theorem 1.12 of Chapter 1 due to Bryant and Walter [8] concerning generators.

Theorem 4.7. *Let* X *be a compact metric* G*-space*, $h \in H(X)$ *and* $\mathcal{U}$ *be a* G*-generator for* (X,h). *Then for each non negative integer* n, *there exists* $\varepsilon > 0$ *such that for* x, y *in* X *with* $G(x) \neq G(y)$, $d(gx,ky) < \varepsilon$ *for some* g, k *in* G *implies the existence of* $A_{-n}, \ldots, A_0, \ldots, A_n$ *in* $\mathcal{U}$ *such that* $gx, ky \in \bigcap_{i=-n}^{n} h^{-i}(A_i)$. *Conversely, for each* $\varepsilon > 0$, *there is a positive integer* n *such that* $x, y \in \bigcap_{i=-n}^{n} h^{-i}(A_i)$ *with* $G(x) \neq G(y)$, *and* $A_{-n},\ldots,A_n \in \mathcal{U}$ *implies* $d(gx,ky) < \varepsilon$ *for some* g, k *in* G.

Proof. Since X is compact and $\mathcal{U}$ being a G-generator is an open cover of X, $\mathcal{U}$ will have a Lebesgue number, say η. Fix a non negative integer, say n. Obviously h^i, $|i| \leq n$, are uniformly continuous. Thus for above η, there exists an $\varepsilon > 0$ such that

$$d(x,y) < \varepsilon \text{ implies } d(h^i(x),h^i(y)) < \eta$$

for all i, $|i| \le n$. Now if for some g, k in G, $d(gx,ky) < \varepsilon$, then using the fact that η is a Lebesgue number for $\mathcal{U}$, for each i, $|i| \le n$, we will find an A_i in $\mathcal{U}$ such that $h^i(gx)$, $h^i(ky) \in A_i$ and hence gx, $ky \in \bigcap_{i=-n}^{n} h^{-i}A_i$.

Conversely, suppose $\varepsilon > 0$ is given. If the required result is not true, then for each positive integer j, there exists x_j, y_j in X with distinct G-orbits and $\{A_{j,i}\}_{-j\le i\le j}$ in $\mathcal{U}$ such that

$$x_j,\ y_j \in \bigcap_{i=-j}^{j} h^{-i}(A_{j,i}) \text{ and } d(gx_j,ky_j) \ge \varepsilon \qquad (*)$$

for each g, k in G. Since X is compact, sequences $\{x_j\}$ and $\{y_j\}$ will converge. Suppose they converge to x and y respectively. Then (*) implies $G(x) \ne G(y)$. In fact, if $G(x) = G(y)$ then $x = gy$ for some g in G and therefore $\operatorname{Lim} x_n = g \operatorname{Lim} y_n = \operatorname{Lim} gy_n$, i.e., for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n,gy_n) < \varepsilon$ for all $n > N$ but this is not possible in view of (*). Since $\mathcal{U}$ is a finite cover, infinitely many of $A_{j,o}$ are same, say equal to A_o and therefore for infinitely many j's, x_j, y_j belong to A_o. But this gives x, y $\in \mathrm{Cl}A_o$. Similarly, for each integer n, infinitely many of $A_{j,n}$ coincide; hence one gets A_n in $\mathcal{U}$ such that x, $y \in h^{-n}(\mathrm{Cl}A_n)$. Thus x, $y \in \bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}A_i)$. This contradicts the fact that $\mathcal{U}$ is a G-generator for (X,h).

This result helps us to obtain the following necessary and sufficient condition for two points to be positively G-asymptotic with respect to a homeomorphism on a compact metric G-space having a G-generator.

Theorem 4.8. *Let* X *be a compact metric* G*-space,* $h \in H(X)$ *be equivariant and* $\mathcal{U}$ *be a* G*-generator for* (X,h)*. Then* x, y *in* X *with distinct* G*-orbits are positively* G*-asymptotic with respect to* h *iff there exists an* N *in* **N** *such that for each* $i \geq N$*, there is an* A_i *in* $\mathcal{U}$ *with* $x, y \in \bigcap_{i=N}^{\infty} h^{-i}(A_i)$.

Proof. Suppose x, y in X with distinct G-orbits are positively G-asymptotic points. Then for a given $\varepsilon > 0$ there exists N in **N** such that

$$d(h^i(gx), h^i(ky)) < \varepsilon \text{ wherein } i \geq N$$

for some g, k in G. Take ε to be a Lebesgue number of $\mathcal{U}$. Then for each $i \geq N$, there exists A_i in $\mathcal{U}$ such that $h^i(gx), h^i(ky) \in A_i$ for some g, k in G and hence using equivariancy of h, we obtain

$$x, y \in \bigcap_{i=N}^{\infty} h^{-i}(A_i).$$

Conversely, suppose that there exists an integer N such that for each $i \geq N$, there exists an A_i in $\mathcal{U}$ such that $x, y \in \bigcap_{N}^{\infty} h^{-i}(A_i)$. Let $\varepsilon > 0$. Then by Theorem 4.7, obtain a positive integer n such that if $x, y \in \cap h^{-i}(A_i)$ with $G(x) \neq G(y)$ and $A_{-n}, \ldots, A_0, \ldots, A_n \in \mathcal{U}$, then $d(gx, ky) < \varepsilon$ for some g, k in G. Let $p \geq N + n$. Then

$$x, y \in \bigcap_{i=N}^{\infty} h^{-i}(A_i)$$

implies

$$x, y \in \bigcap_{p-n}^{p+n} h^{-i}(A_i).$$

Therefore

$$h^p(x), h^p(y) \in \bigcap_{p-n}^{p+n} h^{-(i-p)}(A_i)$$
$$= \bigcap_{j=-n}^{n} h^{-j}(A_{j+p}).$$

Also, $G(x) \neq G(y)$ implies $h^p(G(x)) \cap h^p(G(y)) = \varphi$ and from equivariancy of h we obtain $G(h^p(x)) \neq G(h^p(y))$ and hence for some

g, k in G

$$d(g.h^p(x),k.h^p(y)) < \varepsilon.$$

Now equivariancy of h gives

$$d(h^p(gx),h^p(ky)) < \varepsilon.$$

Thus given $\varepsilon > 0$, there exists N in $\mathbf{N}$ such that whenever $n \geq N$, for some g,k in G we have $d(h^n(gx),h^n(ky)) < \varepsilon$. This proves that x and y are positively G-asymptotic points with respect to h.

The following result concerning negatively G-asymptotic points can be proved in a similar manner.

Theorem 4.9. *Let* X *be a compact metric* G-*space,* h *in* $H(X)$ *be equivariant and* $\mathcal{U}$ *be a* G-*generator for* (X,h). *Then* x, y *in* X *with distinct* G-*orbits are negatively* G-*asymptotic with respect to* h *iff there exists an integer* N *such that for each* $i \leq N$, *there is an* A_i *in* $\mathcal{U}$ *with*

$$x,\ y \in \bigcap_{i=-\infty}^{N} h^{-i}(A_i).$$