CHAPTER - 1

A BRIEF ACCOUNT OF DEVELOPMENT OF INVERSE SERIES RELATIONS AND ASSOCIATED POLYNOMIALS

1.1 GENERALIZED HYPERGEOMETRIC SERIES AND ASSOCIATED POLYNOMIALS

One of the several ways in which the classical orthogonal polynomials and their various generalizations are introduced is through the generalized hypergeometric series ${}_rF_s$ which is defined as below.

$$(1.1.1)\qquad {}_rF_s\left[\begin{matrix} a_1,\ldots,a_r; \\ b_1,\ldots,b_s; \end{matrix}\; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_r)_n}{(b_1)_n\ldots(b_s)_n}\,\frac{x^n}{n!} \quad ,$$

where

$$(a)_n = \begin{cases} a(a+1)\ldots(a+n-1), & \text{if } n \text{ is a positive integer} \\ 1, & \text{if } n \text{ is zero} \\ \Gamma(a+n)/\Gamma(a), & \text{for arbitrary non zero 'a' and } n. \end{cases}$$

The above series converges under one of the following conditions.

(i) $|x| < \infty$, if $r \le s$ (ii) $|x|<1$, if $r = s+1$

(iii) $|x| = 1$, if $\operatorname{Re}\left(\sum_{j=1}^{s} b_j - \sum_{i=1}^{r} a_i\right) > 0$.

The ${}_rF_s$ function which is an elegant generalization of well known Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ is of prime importance in the theory of Special functions because most of the Special functions of mathematical physics, chemistry, astronomy, and statistics and, also those of electro-magnetic theory, statics, dynamics, fiber optics, vibration phenomena etc. are special cases of ${}_rF_s$.

When one of the numerator parameters assumes a negative integral value, the series representing this function becomes terminating in which case it represents a polynomial in its argument. The theory of various classical polynomials which are special cases of ${}_rF_s$, has been enriched by many eminent mathematicians like G.Szegö, H.Bateman, P.E.Bedient, L.Carlitz, R.P.Boas, W.N.Bailey, A.Erdélyi, R.C.Buck, E.D. Rainville, S.O.Rice, P.Humbert, R.L.Shively, sister M. Celine, D.Dikinson, H.W.Gould, R.Askey, W.A. Al-Salam, H.Exton, J.L. Burchnall, T.W. Chaundy, G.Gasper, H.M. Srivastava, T.S.Chihara, M.Rahman, M.E.H. Ismail, J.A. Wilson, A. Verma, R.P. Agarwal, C.M.Joshi, R.K. Saxena, N.K. Thakre and others. Several known polynomials which are expressible as special cases of ${}_rF_s$ are listed below.

(1.1.2) $$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1(-n;1+\alpha;x) = \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha)_n}{(1+\alpha)_k\ n!\ k!}\ x^k$$

(Laguerre polynomial),

(1.1.3) $$H_n(x) = (2x)^n\ {}_2F_0\left(\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}; \text{—}; -x^{-2}\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k\ n!(2x)^{n-2k}}{k!\ (n-2k)!}$$

(Hermite polynomial),

(1.1.4) $$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left(-n,\ 1+\alpha+\beta+n\ ;\ 1+\alpha\ ;\ \frac{1-x}{2}\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+n)_k(1+\alpha)_n}{(1+\alpha)_k\ n!\ k!\ 2^k}\ (1-x)^k$$

(Jacobi polynomial);

(1.1.5) $$P_n(x) = {}_2F_1\left(-n,\ n+1;\ 1;\ \frac{1-x}{2}\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k (1-x)^k}{k!\; k!\; 2^k} \,.$$

alternatively,

(1.1.6) $$P_n(x) = \frac{(1/2)_n (2x)^n}{n!}\; {}_2F_1\left(\frac{-n}{2}, \frac{-n}{2} + \frac{1}{2}; \frac{1}{2} - n \;;\; -x^2\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, (1/2)_{n-k}\, (2x)^{n-2k}}{k!\, (n-2k)!}$$

(Legendre polynomial),

(1.1.7) $$C_n^{\nu}(x) = \frac{(\nu)_n\, (2x)^n}{n!}\; {}_2F_1\left(\frac{-n}{2}, \frac{n}{2} + \frac{1}{2} \;;\; 1-\nu-n;\; -x^{-2}\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, (\nu)_{n-k}\, (2x)^{n-2k}}{k!\, (n-2k)!}$$

(Gegenbauer polynomial),

(1.1.8) $$Q_n(x;\alpha,\beta,N) = {}_3F_2\, (-n,\; 1+\alpha+\beta+n,\; -x \;;\; 1+\alpha, -N \;;\; 1)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k\, (1+\alpha+\beta+n)_k\, (-x)_k}{(1+\alpha)_k\, (-N)_k\, k!} \qquad (n=0,1,2,\ldots,N)$$

(Hahn polynomial (W.Hahn [1])),

(1.1.9) $$R_n(x(x+\gamma+\delta+1);\, \alpha,\beta,\gamma,\delta)$$

$$= {}_4F_3\left[\begin{matrix} -n,\; 1+\alpha+\beta+n,\; x+\gamma+\delta+1,\; -x;\; 1 \\ 1+\alpha,\;\; \beta+\delta+1,\; \gamma+1; \end{matrix}\right]$$

$$= \sum_{k=0}^{n} \frac{(-n)_k\, (1+\alpha+\beta+n)_k\, (x+\gamma+\delta+1)_k\, (-x)_k}{(1+\alpha)_k\, (\beta+\delta+1)_k\, (\gamma+1)_k\, k!}$$

(Racah polynomial (Askey and Wilson [1])), (1.1.10) $P_n(x^2) = (a+b)_n \, (a+c)_n \, (a+d)_n \, .$

$$\cdot {}_4F_3 \left[\begin{matrix} -n, \ a+b+c+d+n-1, \ a+ix, \ a-ix; \ 1 \\ a+b, \ a+c, \ a+d \ ; \end{matrix} \right]$$

$$= (a+b)_n \, (a+c)_n \, (a+d)_n \sum_{k=o}^{n} \frac{(-n)_k (a+b+c+d+n-1)_k \, (a+ix)_k (a-ix)_k}{(a+b)_k \, (a+c)_k \, (a+d)_k \, k!}$$

(Wilson polynomial (Askey and Wilson [1])).

Two worth mentioning generalizations of polynomials of Laguerre, Legendre, Gegenbauer, Jacobi etc. are the polynomials $g_n^c(x,r,s)$ and $f_n^c(x,y,r,m)$ studied by R. Panda [1], and J.P. Singhal and Savita Kumari [1], respectively. They are defined by the explicit forms as given below.

(1.1.11) $$g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk}}{(n-sk)!} \gamma_k \, x^k \, ,$$

and

(1.1.12) $$f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} \, y^k \, \gamma_{n-mk} \, x^{n-mk} .$$

When

$$\gamma_n = \left\{ \prod_{j=1}^{p} (a_j)_n \right\} \left\{ n! \prod_{i=1}^{q} (b_i)_n \right\}^{-1} ,$$

these polynomials admit the following hypergeometric representations.

(1.1.13) $$f_{n,p,q}^{c,r,s}\left[(a_p); \ (b_q); \ x \right] = \frac{(c)_n}{n!} \, .$$

$$\cdot {}_{p+r}F_{q+r} \left[\begin{matrix} \Delta(s;-n), \ \Delta(r-s;c+n), \ (a_p); \ (-s)^s (r-s)^{r-s} x/r^r \\ \Delta(r;c), \ (b_q); \end{matrix} \right],$$

where $r > s \geq 1$, and $\Delta(m,\lambda)$ denotes the sequence of m parameters

$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m}$, and (a_p) denotes the sequence of p parameters $a_1, a_2, \ldots, a_p$.

$$(1.1.14)\quad f^{c}_{n,p,q}\,[(a_p);(b_i);\,x] = x^n \left\{\prod_{j=1}^{p} (a_j)_n\right\} \left\{n!\prod_{i=1}^{q} (b_i)_n\right\}^{-1}$$

$$\cdot\, {}_{mr+mq+m}F_{mp+mr-1}\left[\begin{matrix} \Delta(m;-n),\Delta(rm;1-c-rn),\Delta(m;1-b_q-m); \\ \Delta(rm-1;1-(-rn),\Delta(m;1-a_p-n); \end{matrix}\ \frac{(rm)^{rm}(-m)^{mq-mp+1}}{(rm-1)^{rm-1}}\,yx^{-m}\right],$$

where $\Delta(m;\lambda_p)$ stands for the set $\Delta(m;\lambda_1),\ldots,\Delta(m;\lambda_p)$.

1.2 INVERSE SERIES RELATIONS

Let $\{U_n\}$ and $\{V_n\}$ be two sequences which are so related that

$$(1.2.1)\quad \begin{cases} U_n = \sum_{k=0}^{N} A(n,k)\ V_k\,, \\ V_n = \sum_{k=0}^{N} B(n,k)\ U_k\,, \end{cases}$$

where N may be finite or infinite.

The pair (1.2.1) is known as a pair of inverse series relations, and each one of the series is called an inverse series of the other. Such inverse series relations are useful in the study of combinatorial identities in several ways (see Riordan [2]). Apart from this, such relations also occur in

Approximation theory, Distribution theory, Partition theory, Coding theory (see e.g. sloane [1]) and also, in Probability theory (Feller [1]).

The simplest type of pair of inverse relations is

$$(1.2.2)\quad \begin{cases} a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k, \\ b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k, \end{cases}$$

which is suggested by the well known expansion formulae

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad x^n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (x+1)^k.$$

It is not difficult to see that each of the defining relations (1.1.2) to (1.1.12) can be viewed as one of the relations of the pair of inverse relations of the type (1.2.1). Their corresponding inverse relations have been obtained by using varried techniques; such as generating function relation, summation formula, orthogonal property, difference and shift operators and, recurrence relations (see for instance Rainville [1], Riordan [2]).

Given below are the pairs of inverse relations of various polynomials defined by (1.1.2) to (1.1.8), (1.1.11) and (1.1.12).

$$(1.2.3)\quad \begin{cases} L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{(1+\alpha)_k (n-k)!\, k!}, \\ x^n = \sum_{k=0}^{n} \frac{(-1)^k n!\, (1+\alpha)_n}{(n-k)!\, (1+\alpha)_k} L_k^{(\alpha)}(x) \end{cases}$$

(pair of inverse relation of Laguerre polynomial);

$$(1.2.4)\quad \begin{cases} H_n(x) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k\, n!\, (2x)^{n-2k}}{k!\,(n-2k)!}, \\[2ex] x^n = \sum_{k=0}^{[n/2]} \dfrac{n!\, H_{n-2k}(x)}{2^n\, k!\,(n-2k)!} \end{cases}$$

(pair of inverse relation of Hermite polynomial),

$$(1.2.5)\quad \begin{cases} P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \dfrac{(-n)_k (1+\alpha+\beta+n)_k (1+\alpha)_n}{(1+\alpha)_k\, 2^k\, k!\, n!} (1-x)^k, \\[2ex] \dfrac{(1-x)^n}{2^n\,(1+\alpha)_n} = \sum_{k=0}^{n} \dfrac{(-n)_k (1+\alpha+\beta)_k (1+\alpha+\beta+2k)}{(1+\alpha+\beta)_{n+k+1}\,(1+\alpha)_k} P_k^{(\alpha,\beta)}(x) \end{cases}$$

(pair of inverse relation of Jacobi polynomial),

$$(1.2.6)\quad \begin{cases} P_n(x) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k\, (1/2)_{n-k}\, (2k)^{n-2k}}{k!\,(n-2k)!}, \\[2ex] (2x)^n = \sum_{k=0}^{[n/2]} \dfrac{(2n-4k+1)\, n!\, P_{n-2k}(x)}{(3/2)_{n-k}\, k!} \end{cases}$$

(pair of inverse relation of Legendre polynomial),

$$(1.2.7)\quad \begin{cases} C_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k\, (\nu)_{n-k}\, (2x)^{n-2k}}{k!\,(n-2k)!}, \\[2ex] (2x)^n = \sum_{k=0}^{[n/2]} \dfrac{(\nu+n-2k)\, n!}{(\nu)_{n-k+1}\, k!} C_{n-2k}^{\nu}(x) \end{cases}$$

(pair of inverse relation of Gegenbauer polynomial),

$$
(1.2.8)\quad \begin{cases} Q_n(x;\alpha,\beta,N) = \sum_{k=0}^{n} \dfrac{(-n)_k(1+\alpha+\beta+n)_k\ (-x)_k}{(1+\alpha)_k\ (-N)_k\ k!} \\[2ex] \dfrac{(-x)_n}{(1+\alpha)_n\ (-N)_n} = \sum_{k=0}^{n} \dfrac{(-n)_k(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}\ k!}\ Q_k(x;\alpha,\beta,N) \end{cases}
$$

(pair of inverse relation of Hahn polynomial (Gasper [1])),

$$
(1.2.9)\quad \begin{cases} g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \dfrac{(c+rk)_{n-sk}}{(n-sk)!}\ \gamma_k\ x^k\ , \\[2ex] \gamma_n\ x^n = \sum_{k=0}^{sn} \dfrac{(-1)^{sn-k}(c+(rk/s))\ (c)_{rn}}{(c)_{rn-sn+k+1}\ (sn-k)!}\ g_k^c(x,r,s) \end{cases}
$$

(pair of inverse relation of Panda's polynomial (Singhal and S.Kumari [4])).

$$
(1.2.10)\quad \begin{cases} f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k}\ y^k\ \gamma_{n-mk}\ x^{n-mk}\ , \\[2ex] \gamma_n\ x^n = \sum_{k=0}^{[n/m]} (-y)^k\ \dfrac{c+nr-mrk}{c+nr-k}\ \binom{-c-nr+k}{k}\ f_{n-mk}^c(x,y,r,m) \end{cases}
$$

(pair of inverse relation of Singhal and S.Kumari's polynomial (Singhal and S.Kumari [1])).

A systematic study of the inverse series relations was taken up for the first time in the midst of this century. In fact, it appears from the works of Gould ([1] to [6]) that initially, such relations were merely an out come of a study of binomial series transformations; but later on, an independent development took place, and as a result of that a number of inverse pairs were discovered and also studied at length, by Gould, Carlitz, Riordan and others. The main aspects however, of their study were to obtain combinatorial identities and/or to obtain inverse series relations of particular polynomials. A brief account of this is given below which is follwed by some recent relevant results due to Singhal and S.Kumari.

In 1956, in an attempt to generalize the Vandermonde's convolution identity

$$\sum_{j=0}^{k} \binom{r}{j} \binom{m}{k-j} = \binom{r+m}{j}$$

Gould [1,Eq.(7),p.85] proved that

$$(1.2.11) \qquad \sum_{k=0}^{\infty} A_k(a,b)\, Z^k = x^a ,$$

where

$$(1.2.12) \qquad A_k(a,b) = \frac{a}{a+bk} \binom{a+bk}{k}, \quad \text{and} \quad Z = (x-1)\, x^{-b}.$$

By making use of the result (1.2.11), he obtained a binomial series transformation as well as its inverse transformation (see [3,theorems 1 and 2]), whence he deduced that

$$(1.2.13) \qquad \begin{cases} F(n) = \sum_{k=o}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{n} f(k), \\ \\ f(n) = \sum_{k=0}^{n} (-1)^k \binom{a+bn}{n}^{-1} A_{n-k}(a+bk-k,b)\, F(k), \end{cases}$$

wherein $A_k(a,b)$ is same as defined in (1.2.12).

The orthogonal series relation viz.

$$(1.2.14) \qquad \sum_{k=0}^{n} (-1)^k \; A_{n-k}(a+bk,b) \binom{a}{k} = \binom{0}{n}$$

supplied by the pair (1.2.13) was further used by Gould who, in 1962, proved a more general pair of inverse relations ([4, p.394]) which is given below.

$$(1.2.15)\quad \begin{cases} F(a) = \sum_{k=0}^{M} (-1)^k A_k(a,b)\, f(a+bk-k) \\ \text{if, and only if} \\ f(a) = \sum_{k=0}^{M} \binom{a}{k} F(a+bk-k), \end{cases}$$

where M=[a/(1-b)] is finite if 'a' is positivie and 'b' is zero or a negative integer, otherwise $M=\infty$.

This general pair possesses a number of particular cases, for example when b=2, one finds

$$(1.2.16)\quad \begin{cases} F(a) = \sum_{k=0}^{\infty} (-1)^k A_k(a,2)\, f(a+k), \\ f(a) = \sum_{k=0}^{\infty} \binom{a}{k} F(a+k) \end{cases}$$

(for other special cases refer to Gould [4,p.395]).

By making a slight modification in (1.2.13), Gould introduced (in 1964) yet another inversion pair [5,p.326] :

$$(1.2.17)\quad \begin{cases} G(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+n+bk}{n} f(k), \\ \text{if, and only if} \\ f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bn+k}{k}^{-1} \dfrac{a+bk+k+1}{a+bn+k+1}\, G(k), \end{cases}$$

and thereby showed that the Bessel polynomial

(1.2.18) $\binom{a+n}{n} Y_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{a+n+k}{n} \binom{a+k}{k} k! \, (-x/2)^k$,

the Legendre polynomial (cf.(1.2.5))

(1.2.19) $P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n} \left(\frac{1-x}{2}\right)^k$.

and the Chebyshev polynomial $U_n(x) = \sin(n+1)\theta/\sin\theta$, where $x = \cos\theta$, possess the inverse series relations given by

(1.2.20) $\binom{a+2n}{n} \binom{a+n}{n} n! \, (x/2)^n$

$$= \sum_{k=0}^{n} (-1)^k \frac{a+2k+1}{a+2n+1} \binom{a+2n+1}{n-k} \binom{a+k}{k} Y_k^{(a)}(x),$$

(1.2.21) $\binom{2n}{n} \left(\frac{1-x}{2}\right)^n = \sum_{k=0}^{n} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} P_k(x)$,

and

(1.2.22) $2^n(1-x)^n = \sum_{k=0}^{n} (-1)^k \frac{k+1}{n+1} \binom{2n+2}{n-k} U_k(x)$

respectively.

In 1965, he introduced a generalized Humbert polynomial

(1.2.23) $P_n(m,x,y,p,C) = \sum_{k=0}^{[n/m]} \binom{p-n+mk}{k} \binom{p}{n-mk} C^{p-n-k+mk} y^k (-mx)^{n-mk}$

and obtained its inverse series in the form (Gould [6]):

(1.2.24) $\binom{p}{n}(-mx)^n = \sum_{k=0}^{[n/m]} (-1)^k \binom{p-n+k}{k} \frac{p-n+mk}{p-n+k} C^{n-k-p} \, y^k \, .$

$$.P_{n-mk}(m,x,y,p,C),$$

by establishing a novel type of inversion pair

$$(1.2.25)\quad \begin{cases} F(n) = \sum_{k=o}^{[n/m]} A_k(p-n,m)\, f(n-mk) \\ \text{if, and only if} \\ f(n) = \sum_{k=0}^{[n/m]} (-1)^k A_k(p-n,1)\, F(n-mk), \end{cases}$$

wherein $A_k(a,b)$ is same as defined in (1.2.12).

Several important particular cases of this generalized Humbert polynomial (1.2.23) such as the polynomials of Humbert : $\Pi^{\nu}_{n,m}(x) = P_n(m,x,1,-\nu,1)$, Kinney : $P_n(m,x) = P_n(m,x,1,-1/m,1)$, Pincherle : $\mathfrak{p}_n(x) = P_n(3,x,1,-1/2,1)$, Gegenbauer : $C^{\nu}_n(x) = P_n(2,x,1,-\nu,1)$, Legendre: $P_n(x) = P_n(2,x,1,-1/2,1)$ etc. are listed below along with their inverse series representations which are immediate consequences of the inverse series (1.2.24).

$$(1.2.26)\quad \begin{cases} \Pi^{\nu}_{n,m}(x) = \sum_{k=0}^{[n/m]} \dfrac{(mx)^{n-mk}}{\Gamma(-\nu-n+mk-k+1)\; k!\; (n-mk)!} \\ \dfrac{(-mx)^n}{n!} = \sum_{k=0}^{[n/m]} (-1)^k \dfrac{-\nu-n+mk}{-\nu-n+k}\; \dfrac{\Gamma(-\nu-n+k+1)}{k!}\; \Pi^{\nu}_{n-mk,m}(x); \end{cases}$$

$$(1.2.27)\quad \begin{cases} P_n(m,x) = \sum_{k=0}^{[n/m]} \dfrac{(mx)^{n-mk}}{\Gamma(-(1/m)-n+mk-k+1)\; k!\; (n-mk)!}, \\ \dfrac{(-mx)^n}{n!} = \sum_{k=0}^{[n/m]} (-1)^k \dfrac{-(1/m)-n+mk}{-(1/m)-n+k}\; \dfrac{\Gamma(-(1/m)-n+k+1)}{k!}\; . \\ \qquad .P_{n-mk}(m,x); \end{cases}$$

$$(1.2.28)\quad \begin{cases} \mathfrak{p}_n(x) = \sum_{k=0}^{[n/3]} \dfrac{(3x)^{n-3k}}{\Gamma((1/2)-n+2k)\; k!\; (n-3k)!} \\ \dfrac{(3x)^n}{n!} = \sum_{k=0}^{[n/3]} (-1)^k \dfrac{-(1/2)-n+3k}{-(1/2)-n+k}\; \dfrac{\Gamma((1/2)-n+k)}{k!}\; \mathfrak{p}_{n-3k}(x); \end{cases}$$

$$(1.2.29)\quad \begin{cases} C_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \dfrac{(-2x)^{n-2k}}{\Gamma(-\nu-n+k+1)\ k!\ (n-2k)!} \\[2ex] \dfrac{(-2x)^n}{n!} = \sum_{k=0}^{[n/2]} (-1)^k \dfrac{-\nu-n+2k}{-\nu-n+k} \dfrac{\Gamma(-\nu-n+k+1)}{k!} C_{n-2k}^{\nu}(x); \end{cases}$$

$$(1.2.30)\quad \begin{cases} P_n(x) = \sum_{k=0}^{[n/2]} \dfrac{(-2x)^{n-2k}}{\Gamma((1/2)-n+k)\ k!\ (n-2k)!}, \\[2ex] \dfrac{(-2x)^n}{n!} = \sum_{k=0}^{[n/2]} (-1)^k \dfrac{-(1/2)-n+2k}{-(1/2)-n+k} \dfrac{\Gamma((1/2)-n+k)}{k!} P_{n-2k}(x). \end{cases}$$

In 1973, the inversion pairs (1.2.13) and (1.2.17) were further extended in (or unified to) an elegant form by Gould and Hsu [1], who proved that if $\{a_i\}$ and $\{b_i\}$ be two sequences of numbers such that

$$\prod_{i=1}^{n} (a_i+xb_i) = \psi(x,n) \neq 0$$

for all non-negative x and n, and $\psi(x,0) = 1$, then

$$(1.2.31)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k,n)\ g(k) \\[2ex] \text{if, and only if} \\[2ex] g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_{k+1}+k\ b_{k+1})\ \psi(n,k+1)^{-1}\ f(k). \end{cases}$$

In their work [1], Gould and Hsu however do not discuss the reducibilities of (1.2.31) to the inverse series relations of various particular polynomials, yet it can be shown that the inverse relations of the polynomials quoted in (1.1.2) to

(1.1.10) are obtainable from this general pair.

For instance, the inverse relations involving the Laguerre polynomial $L_n^{(\alpha)}(x)$ (see (1.1.2)) may be obtained from (1.2.31) by setting $a_i=1$, $b_i=0$ for all i. Similarly the Jacobi polynomial and its inverse series relation (1.2.5) follow from (1.2.31) when $a_i=\alpha+\beta+i$, and $b_i=1$ for all i. It is interesting to see that the inverse relations of the recently introduced orthogonal polynomials of Racah (1.1.19), and of Wilson given in (1.1.10) can also be deduced from (1.2.31) which are mentioned below.

(1.2.32) $$\frac{(-x)_n(x+\gamma+\delta+1)_n}{(1+\alpha)_n(1+\beta+\delta)_n(1+\gamma)_n} = \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)}{(\alpha+\beta+k+1)_{n+1}\ k!} \cdot R_k(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$$

(inverse relation of Racah polynomial);

(1.2.33) $$\frac{(a+ix)_n\ (a-ix)_n}{(a+b)_n(a+c)_n(a+d)_n} = \sum_{k=0}^{n} \frac{(-n)_k\ (a+b+c+d+k+\ell k-1)\ \ P_k(x^2)}{(a+b)_k(a+c)_k(a+d)_k(a+b+c+d+k-1)_{n+1}\ k!}$$

(inverse relation of Wilson polynomial).

Recently, having motivated by the desirability to deducing the inverse relations of the general classes of polynomials $\{f_n^c(x,y,r,m)\}$ and $\{g_n^c(x,r,s)\}$ defined in (1.1.12) and (1.1.11), Singhal and S.Kumari ([1],[4]) in their study of classical polynomials, proved two more general inversion pairs which are as stated below.

$$(1.2.34)\quad \begin{cases} F(n) = \displaystyle\sum_{k=0}^{[n/m]} y^k \binom{p-\lambda n+\lambda mk}{k} f(n-mk) \\ \text{if, and only if} \\ f(n) = \displaystyle\sum_{k=0}^{[n/m]} (-y)^k \binom{p-\lambda n+k}{k} \frac{p-\lambda n+\lambda mk}{p-\lambda n+k} F(n-mk), \end{cases}$$

where p and λ are arbitrary parameters; and

$$(1.2.35)\quad \begin{cases} F(n) = \displaystyle\sum_{k=0}^{[n/s]} \binom{p+qsk-sk}{n-sk} f(k) \\ \text{if, and only if} \\ f(n) = \displaystyle\sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} F(k) \\ \text{and} \\ \displaystyle\sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} F(k)=0,\ n\neq ms,\ m=1,2,3,\ldots \end{cases}$$

On making use of the inversion formulae (1.2.34) and (1.2.35) they deduced the inverse series relations of $f_n^c(x,y,r,m)$ and $g_n^c(x,r,s)$ respectively, which are as mentioned in (1.2.10) and (1.2.9). As shown by them, the special instances of the pair (1.2.10) (or (1.2.34)) are the inverse series relations of the generalized Humbert polynomial and those associated to it (see (1.2.26) to (1.2.30)). On the other hand, the particular cases of the relations in (1.2.35) include the inverse relations of the classical orthogonal polynomials like Hermite polynomial $H_n(x)$, Laguerre polynomial $L_n^{(\alpha)}(x)$, Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and its associated polynomials such as the Gegenbauer polynomial: $(\nu+(1/2))_n\, C_n^{\nu}(x) = (2\nu)_n\, P_n^{(\nu-(1/2),\nu-(1/2))}(x)$, the

Ultraspherical polynomial: $P_n^{(\alpha,\alpha)}(x)$, the Chebyshev polynomials: $(1/2)_n\ T_n(x) = n!\ P_n^{(-1/2,-1/2)}(x)$ (of first kind), and $(3/2)_n\ U_n(x) = (n+1)!\ P_n^{(1/2,1/2)}(x)$ (of second kind).

Further, in view of the fact that the general hypergeometric polynomial considered by R.N. Jain [1] viz.

$$(1.2.36)\quad F_n^{(c,k)}[(a_p);(b_q);x] = \frac{(c)_n}{n!}\ .\ {}_{p+k}F_{q+k}\left[\begin{matrix} -n,\ \Delta(k+1;c+n),(a_p);\ (k+1)^{k-1}x \\ \Delta(k;c),(b_q); \end{matrix}\right]$$

(k is a positive integer),

the Brafman polynomial (Brafman [1]) :

$$(1.2.37)\quad B_n^{s}\ [(a_p);(b_q);x] = {}_{p+s}F_q\ \left[\begin{matrix}\Delta(s;-n),a_1,\dots,a_p\ ;\ x \\ b_1,\dots,b_q\ ; \end{matrix}\right],$$

and the Rainville's polynomial (Rainville [1]):

$$(1.2.38)\quad f_n(x) = \sum_{k=0}^{n} \frac{(c)_n\ (c+n)_k\ (-n)_k}{(c/2)_k(c/2+(1/2))_k\ n!}\ \gamma_k\ \ x^k$$

are particular cases of the set $\{g_x^c(x,r,s)\}$ (Panda [1]); the inverse series relations of these polynomials can easily be obtained from (1.2.9) in the forms as given below (in the same order).

$$(1.2.39)\quad (k-1)^{(k-1)n}\ x^n = \sum_{j=0}^{n} \frac{(-n)_j(c)_{kn}(c+kj)(b_1)_n\dots(b_q)_n}{(c)_{kn-n+j+1}(a_1)_n\dots(a_p)_n}\ .\ F_j^{(c,k)}\ [(a_p);(b_q);x]$$

(with $\gamma_n = \dfrac{(-1)^n (a_1)_n \cdots (a_p)_n}{n!\ (b_1)_n \cdots (b_q)_n} (k+1)^{(k+1)n}$, s=1, and

r=k a positive integer),

$$(1.2.40)\quad x^n = \sum_{k=0}^{sn} \frac{n!(b_1)_n \cdots (b_q)_n}{(sn-k)!\ (a_1)_n \cdots (a_p)_n\ k!}\ B_k^s\ [(a_p);(b_q);x]$$

(with $\gamma_n = \dfrac{(c)_{sn}(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n\ n!} (-1)^{sn}$, and r=s), and

$$(1.2.41)\quad 4^n\ x^n = \sum_{k=0}^{n} \frac{(-1)^k\ (c+2k)\ (c)_{2n}}{\gamma_n\ (c)_{n+k+1}\ (n-k)!}\ f_k(x)$$

(with s=1, r=2, and x is replaced by −4x).

1.3 CLASSIFICATION OF INVERSION PAIRS

The earlier works of Gould ([1] to [4]) on inversion of series evoked a wave of interest that was reflected in the works of John Riordan who studied the inverse series relations at length by classifying them into several classes namely, the simplest type pairs, Gould classes, simpler Chebyshev classes, Chebyshev classes, simpler Legendre classes and, the Legendre-Chebyshev classes of inverse relations.

All the classes are recorded in the following tables.

Table-1 : *Simplest Inverse Relations*

(1)	$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k$	$b_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} a_k$
(2)	$a_n = \sum_{k=n} \binom{k}{n} b_k$	$b_n = \sum_{k=n} (-1)^{n+k} \binom{k}{n} a_k$
(3)	$a_n = \sum_{k=0}^{n} \binom{p-k}{p-n} b_k$	$b_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{p-k}{p-n} a_k$
(4)	$a_n = \sum_{k=0}^{n} \binom{p+n}{p+k} b_k$	$b_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{p+n}{p+k} a_k$
(5)	$a_n = \sum_{k=n} \binom{p+k}{p+n} b_k$	$b_n = \sum_{k=n} (-1)^{n+k} \binom{p+k}{p+n} a_k$
(6)	$a_n = \sum_{k=1} \frac{n!}{k!} \binom{n-1}{k-1} b_k$	$b_n = \sum_{k=1} (-1)^{n+k} \frac{n!}{k!} \binom{n-1}{k-1} a_k$

(Riordan [2,Table2.1, p.49])

Table-2 : *Gould classes of inverse relations*

$$a_n = \Sigma\, A_{n,k}\, b_k\ ;\quad b_n = \Sigma\, (-1)^{n+k}\, B_{n,k}\, a_k$$

	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{p+qk-k}{n-k}$	$\frac{p+qk-k}{p+qn-k}\binom{p+qn-k}{n-k}$
(2)	$\frac{p+qn-n+1}{p+qk-n+1}\binom{p+qk-k}{n-k}$	$\binom{p+qn-k}{n-k}$
(3)	$\binom{p+qn-n}{k-n}$	$\frac{p+qn-n}{p+qk-n}\binom{p+qk-n}{k-n}$
(4)	$\frac{p+qk-k+1}{p+qn-k+1}\binom{p+qn-n}{k-n}$	$\binom{p+qk-n}{k-n}$

(Riordan [2,Table - 2.2, p.52])

Table-3 : *Simpler Chebyshev inverse relations*

(1)	$a_n = \Sigma \binom{n}{k} b_{n-2k}$	$b_n = \Sigma (-1)^k \frac{n}{n-k}\binom{n-k}{k} a_{n-2k}$
(2)	$a_n = \Sigma \frac{n-2k+1}{n-k+1}\binom{n}{k} b_{n-2k}$	$b_n = \Sigma (-1)^k \binom{n-k}{k} a_{n-2k}$
(3)	$a_n = \Sigma \binom{n+2k}{k} b_{n+2k}$	$b_n = \Sigma (-1)^k \frac{n+2k}{n+k}\binom{n+k}{k} a_{n+2k}$
(4)	$a_n = \Sigma \frac{n+1}{n+k+1}\binom{n+2k}{k} b_{n+2k}$	$b_n = \Sigma (-1)^k \binom{n+k}{k} a_{n+2k}$
(5)	$a_n = \Sigma \binom{n-k}{k} b_{n-k}$	$b_n = \Sigma (-1)^k \frac{n-k}{n+k}\binom{n+k}{k} a_{n-k}$
(6)	$a_n = \Sigma \frac{n+1}{n-k+1}\binom{n+1-k}{k} b_{n-k}$	$b_n = \Sigma (-1)^k \binom{n+k}{k} a_{n-k}$

(Riondan [2,Table-2.3,p.62])

Table-4 : *Chebyshev classes of inverse relations*

$$a_n = \Sigma\, A_{n,k}\, b_{n+ck} \quad ; \quad b_n = \Sigma\, (-1)^k\, B_{n,k}\, a_{n+ck}$$

	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{n}{k}$	$\frac{n}{n+ck+k}\binom{n+ck+k}{k}$
(2)	$\frac{n+ck+1}{n-k+1}\binom{n}{k}$	$\binom{n+ck+k}{k}$
(3)	$\binom{n+ck}{k}$	$\frac{n+ck}{n}\binom{n+k-1}{k}$
(4)	$\frac{n+1}{n+ck-k+1}\binom{n+ck}{k}$	$\binom{n+k}{k}$

(Riordan [2, Table-2.4, p-63])

Table-5 : *Simpler Legendre inverse relations*

(1)	$a_n = \Sigma \binom{p+n+k}{n-k} b_k$	$b_n = \Sigma (-1)^{n+k} \frac{p+2k+1}{p+n+k+1} \binom{p+2n}{n-k} a_k$
(2)	$a_n = \Sigma \binom{p+2n}{n-k} b_k$	$b_n = \Sigma (-1)^{n+k} \frac{p+2n}{p+n+k} \binom{p+n+k}{n-k} a_k$
(3)	$a_n = \Sigma \binom{p+n+k}{k-n} b_k$	$b_n = \Sigma (-1)^{n+k} \frac{p+2n+1}{p+n+k+1} \binom{p+2k}{k-n} a_k$
(4)	$a_n = \Sigma \binom{p+2k}{k-n} b_k$	$b_n = \Sigma (-1)^{n+k} \frac{p+2k}{p+n+k} \binom{p+n+k}{k-n} a_k$
(5)	$a_n = \Sigma \binom{p+2n}{k} b_{n-2k}$	$b_n = \Sigma (-1)^{k} \frac{p+2n}{p+2n-3k} \binom{p+2n-3k}{k} a_k$
(6)	$a_n = \Sigma \frac{p+2n-4k+1}{p+2n-k+1} \binom{p+2n}{k} . .b_{n-2k}$	$b_n = \Sigma (-1)^{k} \binom{p+2n-3k}{k} a_{n-2k}$

(Riordan [2, Table-2.5, p.68])

Table-6 : *Legendre-Chebyshev classes of inverse relations*

$$a_n = \Sigma A_{n,k} b_k \quad ; \quad b_n = \Sigma (-1)^{n+k} B_{n,k} a_k$$

	$A_{n,k}$	$B_{n,k}$
(1)	$\binom{p+cn}{n-k}$	$\frac{p+cn}{p+ck} \binom{p+n+ck-k-1}{n-k}$
(2)	$\binom{p+cn}{k-n}$	$\frac{p+cn}{p+ck} \binom{p+ck+k-n-1}{k-n}$
(3)	$\binom{p+ck}{n-k}$	$\frac{p+ck}{p+cn} \binom{p+cn+n-k-1}{n-k}$
(4)	$\binom{p+ck}{k-n}$	$\frac{p+ck}{p+cn} \binom{p+cn-n+k-1}{k-n}$
(5)	$\frac{p+ck+1}{p+cn-n+k+1} \binom{p+cn}{n-k}$	$\binom{p+n+ck-k}{n-k}$
(6)	$\frac{p+ck+1}{p+cn+n-k+1} \binom{p+cn}{k-n}$	$\binom{p+ck+k-n}{k-n}$
(7)	$\frac{p+cn+1}{p+ck-n+k+1} \binom{p+ck}{n-k}$	$\binom{p+cn+n-k}{n-k}$
(8)	$\frac{p+cn+1}{p+ck-k+n+1} \binom{p+ck}{k-n}$	$\binom{p+cn-n+k}{k-n}$

(Riordan [2, Table - 2.6, p.69])

1.4 BASIC HYPERGEOMETRIC SERIES AND ASSOCIATED POLYNOMIALS

Nearly thirty years after the Gauss's introduction of hypergeometric series, E. Heine ([1],[2]) introduced an interesting extension of this series in the form:

$$(1.4.1)\quad 1+\frac{(1-q^a)\ (1-q^b)}{(1-q^c)\ (1-q)}\ x+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})\ (1-q)(1-q^2)}\ x^2+\ldots$$

$$(c \neq 0,-1,-2,\ldots;\ |x|<1,\ |q|<1).$$

Prior to this introduction, he defined a 'basic analogue' of a number 'a' in the form

$$[a;q] = \frac{1-q^a}{1-q}\ ,$$

where the arbitary number q ($\neq 1$) is called the base.

From this it readily follows that as $q \to 1$, $[a;q] \to a$, and that the series in (1.4.1) approches to the Gauss hypergeometric series

$$1 + \frac{ab}{c}\ x + \frac{a(a+1)b(b+1)}{c(c+1)\ 1.2}\ x^2 + \ldots$$

$$(c \neq 0,-1,-2,\ldots;\ |x|<1).$$

Thus, Heine's series defines a basic analogue (or a q-analogue) of the Gauss series ; and for this reason the Heine's series is called a basic hypergeometric series (BHS) or a q-hypergeometric series.

Just as it happened with the Gauss series that it was known in other particular forms before its introduction, this q-series (1.4.1) was also known in special forms prior to its introduction. For example, the identity

$$1 + \sum_{n=1}^{\infty} (-1)^n \ \{\ q^{n(3n-1)/2} + q^{n(3n+1)/2}\ \} = \prod_{n=1}^{\infty} (1-q^n)$$

was given by Leonhard Euler in 1748 A.D.; also the triple product identity

$$\prod_{n=0}^{\infty} \{(1-xq^{n})\ (1-q^{n+1}\ x^{-1})\ (1-q^{n+1})\} = \sum_{n=-\infty}^{\infty} (-1)^{n} q^{n^2/2}\ x^{n} \quad ,$$

and the Theta Functions: $\theta_i(z,q)$, i= 1,2,3,4 were given by Carl Gustav Jacob Jacobi in 1829 A.D. But it was not until about sixteen years later that the field of BHS acquired an independent status when Heine (see [1],[2],[3]) introduced the q-series (1.4.1) and carried out a systematic study of it. During this study, he put forward a basic analogue of binomial theorem, basic transformation formulas, basic analogue of the Gauss's summation formula, and also discussed the q-contiguous functions relations (for detail, see Gasper and Rahman [1]). A q-Gamma function which he defined in the form

$$\Gamma_q(x) = \prod_{n=1}^{\infty} \frac{1-q^{n}}{1-q^{x+n-1}}$$

differs slightly from Thomae's definition [1]:

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^{n}}{1-q^{x+n-1}} .$$

Since then the field of BHS has developed notably in the hands of many eminent researchers among whom the names of F.H.Jackson, W.N.Bailey, D.B.Sears, L.J. Rogers, W. Hahn, L.J. Slater, L.Carlitz, H.Exton, R.P. Agarwal, G.E. Andrews, R.Askey, W.A. Al-Salam, H.M. Srivastava, A.Verma, M.E.H. Ismail, T.H. Koornwinder, J.A. Wilson, G.Gasper, S.C. Milne, M.Rahman, V.K. Jain are worth mentioning. It would not be out of place to say that quite a good number of formulae given by S. Ramanujan may be viewed as the special cases of the results involving BHS.

To denote the series in (1.4.1), Heine used the notation $\phi(a,b,c,q,x)$. However, the other notations:

$$ {}_2\phi_1(a,b;c;q,x), \qquad {}_2\phi_1 \left[\begin{matrix} a,b; & q, x \\ c; & \end{matrix} \right] $$

are often used in analysis, in terms of which the series (1.4.1) is representable in the form

$$ (1.4.2) \qquad {}_2\phi_1(a,b;c;q,x) = \sum_{n=0}^{\infty} \frac{(a;q)_n \, (b;q)_n}{(c;q)_n \, (q;q)_n} x^n , $$

where $(a;q)_n \equiv [a]_n$ is a basic factorial function defined as below.

$$ (a;q)_n = \begin{cases} (1-a)(1-aq) \dots (1-aq^{n-1}), & n=1,2,3,\dots \\ 1, & n=0 \\ [a]_\infty \,/\, [aq^n]_\infty , & n \text{ is arbitrary}, \end{cases} $$

in which $[a]_\infty \equiv (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, $0<q<1$.

A generalization of (1.4.2) which provides a basic analogue of (1.1.1) is ${}_r\phi_s$ function defined as (Askey and Wilson [1], also see Gasper and Rahman [1]):

$$ (1.4.3) \quad {}_r\phi_s \left[\begin{matrix} a_1,\dots,a_r; & q,x \\ b_1,\dots,b_s; & \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1]_n \dots [a_r]_n \, x^n}{[b_1]_n \dots [b_s]_n \, [q]_n} \cdot \left\{ (-1)^n \, q^{n(n-1)/2} \right\}^{s-r+1} . $$

The infinite basic series in (1.4.3) converges for all x if $0<|q|<1$ and $r \le s$. If $0<|q|<1$ and $r = s+1$ then it converges for $|x|<1$. The various specializations of this ${}_r\phi_s[x]$ function include the basic expenential functions defined by (cf. W.Hahn [2], Gasper and Rahman [1]) :

$$ e_q(x) = {}_1\phi_0(0;-;q,x) = \sum_{k=0}^{\infty} \frac{x^k}{[q]_k} = \frac{1}{[x]_\infty} \qquad (|x|<1) $$

and

$$E_q(x) = {}_0\phi_0(-;-;q,x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[q]_k} = [-x]_\infty ;$$

the basic sine and cosine functions viz.

$$\mathrm{Sin}_q x = \frac{x}{1-q}\ {}_0\phi_1(-;q^3;q^2,-x^2), \quad \sin_q x = \frac{x}{1-q}\ {}_2\phi_1(0,0;q^3;q^2;-x^2),$$

$$\mathrm{Cos}_q x = {}_0\phi_1(-;q;q^2,-qx^2), \quad \cos_q x = {}_2\phi_1(0,0;q;q^2,-x^2);$$

also, the Basic Gamma, and Beta functions given by

$$\Gamma_q(x) = [q]_\infty (1-q)^{1-x}\ {}_1\phi_0(0;-;q^x), \quad \text{and}$$

$$\beta_q(x,y) = (1-q)\ {}_1\phi_0\ [\ q^y;\ -;\ q,\ q^x\ ].$$

Moreover, the basic Bessel functions are also expressible in the particular ${}_r\phi_s$ functions by means of the following relations.

$$(1.4.4) \quad J_\nu^{(1)}(x;q) = \frac{[q^{\nu+1}]_\infty}{[q]_\infty}\ (x/2)^\nu\ {}_2\phi_1(0,0;q^{\nu+1};q,-x^2/4),$$

and

$$(1.4.5) \quad J_\nu^{(2)}(x;q) = \frac{[q^{\nu+1}]_\infty}{[q]_\infty}\ (x/2)^\nu\ {}_0\phi_1(-;q^{\nu+1};q,-x^2q^{\nu+1}/4),$$

(for further detail refer to Hahn ([1],[2]), Gasper and Rahman [1], and H.Exton [1]).

As the choice $a_i (\equiv q^{a_i}) = q^{-n}$, for atleast one i $(1 \le i \le r)$, $n=0,1,\ldots$, reduces the infinite series in (1.4.3) to a terminating series, the basic hypergeometric representations of various basic polynomials may be obtained by specializing the parameters in (1.4.3). As an illustration, putting $r=1, s=1$, $a_1 = q^{-n}$, $b_1 = \alpha q$ $(\equiv q^{\alpha+1})$, one gets by replacing x by $-xq^{n+\alpha+1}$, a basic Laguerre polynomial

$$(1.4.6) \quad L_n^{(\alpha)}(x;q) = \frac{[\alpha q]_n}{[q]_n} \; {}_1\Phi_1 \, (q^{-n};\alpha q;q,-xq^{\alpha+n+1}).$$

This polynomial was studied by D.S.Moak [1], and also, it was taken into account by Al-Salam and Verma [3] who constructed a pair of biorthogonal polynomials: $Z_n^{(\alpha)}(x;k|q)$ and $Y_n^{(\alpha)}(x;k|q)$ which are known as q-Konhauser polynomials. It is worth mentioning that the polynomial

$$(1.4.7) \quad Z_n^{(\alpha)}(x;k|q) = \frac{[\alpha q]_{kn}}{(q^k;q^k)_n} \sum_{j=0}^{n} \frac{(q^{-kn};\, q^k)_j}{(q^k;\, q^k)_j \, [\alpha q]_{jK}} \, . \, q^{kj(n+\alpha+1)+kj(kj-1)/2} \, x^{kj}$$

reduces to the polynomial (1.4.6) when k=1.

Amongst the other 'ordinary' polynomials, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is worth noting here for, it possesses two basic analogues. The one is given by (cf. Gasper and Rahman [1]):

$$(1.4.8) \quad p_n(x;\alpha,\beta;q) = {}_2\Phi_1 \, (q^{-n}, \alpha\beta q^{n+1}; \alpha q;\, q,\, xq),$$

which is known as 'little' q-Jacobi polynomial, and the other is given by

$$(1.4.9) \quad P_n(x;a,b,c;q) = {}_3\Phi_2 \, (q^{-n},\, abq^{n+1},\, x;\, aq,\, cq;\, q,\, q)$$

known as 'big' q-Jacobi polynomial.

The little q-Jacobi polynomial (1.4.8), with $\beta=0$ provides two more basic analogues of the Laguerre polynomial $L_n^{(\alpha)}(x)$; they are the Wall polynomial, and the stieltjes - Wigert polynomial as stated below (Gasper and Rahman [1,p.196]).

$$(1.4.10)\quad W_n(x;a,q) = (-1)^n\,[a]_n\, q^{n(n+1)/2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{q^{j(j-1)/2}}{[a]_j} \,.\, (-q^{-n}x)^j$$

and

$$(1.4.11)\quad S_n(x;p,q) = (-1)^n\, q^{-n(2n+1)/2} [p]_n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{q^{j^2}}{[p]_j} (-x\sqrt{q})^j.$$

Another q-polynomial with ${}_3\phi_2$ - representation besides the 'big' q-polynomial (1.4.9), is basic Hahn polynomial defined as

$$(1.4.12)\quad Q_n(x;\alpha,\beta,N|q) = {}_3\phi_2 \left[\begin{matrix} q^{-n},\ \alpha\beta q^{n+1},\ q^{-x};\ q,q \\ \alpha q,\ q^{-N}; \end{matrix} \right]$$

(cf. (1.1.8)).

Recently, in the study of general orthogonal q-polynomials, R. Askey and J.A. Wilson [1] considered the q-extensions of the Racah polynomial and Wilson polynomial (stated in (1.1.9) and (1.1.10)) in the forms

$$(1.4.13)\quad W_n(x;a,b,c,N|q) = {}_4\phi_3 \left[\begin{matrix} q^{-n},\ abq^{n+1},\ q^{-x},\ cq^{x-N};q,q \\ aq,\ q^{-N},bcq; \end{matrix} \right]$$

and

$$(1.4.14)\quad \frac{P_n(x;a,b,c,d|q)}{a^{-n}[ab]_n[ac]_n[ad]_n} = {}_4\phi_3 \left[\begin{matrix} q^{-n},\ abcdq^{n-1},ae^{i\theta},\ ae^{-i\theta};q\ ,q \\ ab,\ ac,\ ad; \end{matrix} \right]$$

which they call q-Racah polynomial, and Askey-Wilson polynomial respectively. It can be seen that these polynomials contain among the other polynomials, the q-Jacobi polynomials, q-Hahn polynomial, and the continuous q-Jacobi polynomial considered by M. Rahman [1] in the form :

$$P_n^{(\alpha,\beta)}(\cos\theta;q) = \frac{[\alpha q]_n[-\beta q]_n}{[q]_n[-q]_n}\ {}_4\phi_3\left[\begin{matrix} q^{-n},\ \alpha\beta q^{n+1}, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta};\ q, q \\ \alpha q,\ -\beta q,\ -q; \end{matrix}\right].$$

1.5 BASIC INVERSE RELATION

In the early sixties when the inverse series relations were being discovered by Gould ([3],[4],[5]), Riordan [2], and others (see e.g. Stanton and Sprott [1]), Carlitz [2] studied several inverse series relations and their basic analogues from the point of view of deriving the inverse relations involving certain polynomials. During his study, he was led to several more general inversion pairs. Out of these the following basic pairs are worth mentioning [2, p.196].

(1.5.1)
$$\begin{cases} U_n = \sum_{k=0}^{[n/2]} \left[{n \atop k} \right] V_{n-2k} \\ \text{if, and only if} \\ V_n = \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)/2} \dfrac{1-q^n}{1-q^{n-k}} \left[{n-k \atop k} \right] U_{n-2k}. \end{cases}$$

and

(1.5.2)
$$\begin{cases} U_n = \sum_{k=0}^{[n/2]} \left\{ \left[{n \atop k} \right] - \left[{n \atop k-1} \right] \right\} V_{n-2k} \\ \text{implies} \\ V_n = \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)/2} \left[{n-k \atop k} \right] U_{n-2k}. \end{cases}$$

It may be observed that the pair (1.5.1) provides a basic analogue of the simpler Chebyshev class No.(1) in Table-3, whereas the pair (1.5.2) provides a 'one sided' basic inverse relation of the class No.(2) of the same table.

In one of his other papers on q-inverse relations, Carlitz [3] proved a very general as well as useful result in the form of a basic analogue of the pair (1.2.31) due to Gould and Hsu. The result states that if $a_i+q^{-x}b_i\neq 0$, and $\psi(x,n,q)=\prod_{i=1}^{n}(a_i+q^{-x}b_i)$, then

$$(1.5.3)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \psi(k,n,q)\, g(k) \\ \text{if, and only if} \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \dfrac{(a_{k+1}+q^{-k}b_{k+1})}{\psi(n,k+1,q)} f(k). \end{cases}$$

With the aid of this pair, he obtained certain particular inverse series relations including the pair [3,p, 898] :

$$(1.5.4)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} a+n+k \\ n \end{bmatrix} g(k) \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \dfrac{1-q^{a+2k+1}}{1-q^{a+n+k+1}} \begin{bmatrix} a+n+k \\ k \end{bmatrix}^{-1} . \\ \qquad . f(k), \end{cases}$$

which provides a basic analogue of the pair in (1.2.17) when b=1. He also obtained a basic analogue of the pair (1.2.13) in the form:

$$(1.5.5)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \begin{bmatrix} a+k\lambda \\ n \end{bmatrix} g(k) \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \dfrac{1-q^{a+k\lambda-k}}{1-q^{a+nk-k}} \begin{bmatrix} a+n\lambda \\ k \end{bmatrix}^{-1} . \\ \qquad .f(k) \end{cases}$$

by means of a more general inverse relations [3, p.900]

$$(1.5.6)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-2n+1)/2} \left[{n \atop k} \right]_\lambda \psi(-k,n,q^\lambda)\, g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \left[{n \atop k} \right] (a_{k+1} + q^{k\lambda} b_{k+1}) \cdot \dfrac{f(k)}{\psi(-n,k+1,q^\lambda)}, \end{cases}$$

where

$$\left[{n \atop k} \right]_\lambda = \frac{(1-q^{n\lambda})(1-q^{(n-1)\lambda})\dots(1-q^{(n-k+1)\lambda})}{(1-q^{k\lambda})(1-q^{(k+1)\lambda})\dots(1-q^{2\lambda})(1-q^{\lambda})}, \quad \lambda \neq 0.$$

The other consequences of the pair (1.5.3) although not discussed in [3], are worth mentioning here. They are the inverse series relations of the basic Jacobi polynomials (1.4.8) and (1.4.9), the q-Hahn polynomial (1.4.12), and the q-Racah and the Askey-Wilson polynomials mentioned in (1.4.13) and (1.4.14) respectively. As an illustration, replacing q by q^{-1} and, then setting $a_i=1$, $b_i= -q^{\alpha+\beta+i}$, and $g(n)= [\alpha\beta q]_n x^n / [\alpha q]_n$ in (1.5.3), one finds after a little simplification the following pair of inverse relation involving the little q-Jacobi polynomial.

$$(1.5.8)\quad \begin{cases} p_n(x;\alpha,\beta;q) = \sum_{k=0}^{n} \dfrac{[q^{-n}]_k\ [\alpha\beta q^{n+1}]_k}{[\alpha q]_k\ \ [q]_k}\, x^k q^k, \\ x^n = [\alpha q]_n \sum_{k=0}^{n} q^{nk} \dfrac{[q^{-n}]_k\ (1-\alpha\beta q^{2k+1})}{[\alpha q]_k\ [\alpha\beta q^{k+1}]_{n+1}}\, p_k(x;\alpha,\beta;q). \end{cases}$$

In a similar manner, the inverse relations of the other polynomials may be obtained in the forms as given below.

(1.5.9)
$$\begin{cases} Q_n(x;\alpha,\beta,N|q) = \sum_{k=0}^{n} \dfrac{[q^{-n}]_k\ [\alpha\beta q^{n+1}]_k\ [q^{-x}]_k}{[\alpha q]_k\ \ [q^{-N}]_k\ [q]_k}\ q^k, \\ [q^{-x}]_n = [q^{-N}]_n[\alpha q]_n \sum_{k=0}^{n} q^{nk}\ \dfrac{[q^{-n}]_k\ (1-\alpha\beta q^{2k+1})}{[\alpha\beta q^{k+1}]_{n+1}\ [q]_k} \\ \qquad .Q_k(x;\alpha,\beta,N|q). \end{cases}$$

(pair of inverse relation of basic Hahn polynomial);

(1.5.10)
$$\begin{cases} R_n(\mu(x);\alpha,\beta,\gamma,\delta;q) = \sum_{k=0}^{n} \dfrac{[q^{-n}]_k[\alpha\beta q^{n+1}]_k[q^{-x}]_k[\gamma\delta q^{x+1}]_k}{[\alpha q]_k\ \ [\beta\delta q]_k\ [\gamma q]_k\ [q]_k\ q^{-k}}, \\ \dfrac{[q^{-x}]_n\ [q^{x+1}\gamma\delta]_n}{[\alpha q]_n[\beta\delta q]_n[\gamma q]_n} = \sum_{k=0}^{n} q^{nk}\ \dfrac{[q^{-n}]_k\ (1-\alpha\beta q^{2k+1})}{[\alpha\beta q^{k+1}]_{n+1}\ [q]_k} \\ \qquad .\ R_k(\mu(x);\alpha,\beta,\gamma,\delta;q), \end{cases}$$

where $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$

(pair of inverse relation of basic Racah polynomial);

(1.5.11)
$$\begin{cases} \dfrac{P_n(\cos\theta;a,b,c,d|q)}{[ab]_n[ac]_n[ad]_n} = \sum_{k=0}^{n} \dfrac{[q^{-n}]_k[abcdq^{n-1}]_k[ae^{i\theta}]_k[ae^{-i\theta}]_k}{[ab]_k\ \ [ac]_k\ [ad]_k\ [q]_k\ q^{-k}}, \\ \dfrac{[ae^{i\theta}]_n\ [ae^{-i\theta}]_n}{[ab]_n\ [ac]_n\ [ad]_n} = \sum_{k=0}^{n} \dfrac{[q^{-n}]_k\ (1-abcdq^{2k-1})\ q^{nk}}{[abcdq^{k-1}]_{n+1}\ [ab]_k[ac]_k[ad]_k} \\ \qquad .P_k(\cos\theta;a,b,c,d|q). \end{cases}$$

(pair of inverse relation of Askey-Wilson polynomial).

On the other hand, the substitutions $a_i=1$, and $b_i=0$ in (1.5.3) lead to the inverse relations of the basic Laguerre polynomial: $L_n^{(\alpha)}(x;q)$ defined in (1.4.6), the Wall polynomial :

$W_n(x;a,q)$ given in (1.4.10), and the Stieltjes-Wigert polynomial: $S_n(x;p,q)$ cited in (1.4.11). In fact, with the aforementioned substitutions, the pair (1.5.3) assumes the simplest type of pair as given below.

$$(1.5.12)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k\, q^{k(k-1)/2} \left[\begin{matrix} n \\ k \end{matrix} \right] g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k\, q^{k(k-2n+1)/2} \left[\begin{matrix} n \\ k \end{matrix} \right] f(k), \end{cases}$$

whence the inverse relations of the above mentioned basic Laguerre polynomials are easily obtainable. In order to illustrate this, set

$$g(n) = q^{n(n+1)/2 + \alpha n} \frac{(1-q)^n\, x^n}{[\alpha q]_n}$$

in (1.5.12), then it follows from (1.4.6) that $[\alpha q]_n f(n)/[q]_n$ defines $L_n^{(\alpha)}(x;q)$ and thus, (1.5.12) yields the pair:

$$(1.5.13)\quad \begin{cases} L_n^{(\alpha)}(x;q) = \sum_{k=0}^{n} q^{k(k+1)/2} \dfrac{[q^{-n}]_k\, [\alpha q]_n\, (1-q)^k\, (q^{\alpha+n}x)^k}{[\alpha q]_k\, [q]_k\, [q]_n} \\ \dfrac{x^n}{[q]_n} = q^{-\alpha n - n(n+1)/2} \sum_{k=0}^{n} \dfrac{[q^{-n}]_k\, [\alpha q]_n}{[\alpha q]_k}\, q^k\, L_k^{(\alpha)}(x;q). \end{cases}$$

Likewise, with $f(k) = x^k/[a]_k[q]_k$, the above pair (1.5.12) gives

$$(-1)^n\, [a]_n [q]_n\, q^{n(n+1)/2}\, g(n) = W_n(x;a,q)$$

and consequently, one finds the pair of relations:

$$(1.5.14)\quad \begin{cases} W_n(x;a,q) = (-1)^n [a]_n q^{n(n+1)/2} \sum_{k=0}^{n} (-1)^k q^{k(k-2n-1)/2} \cdot \\ \qquad \cdot \left[{n \atop k} \right] \dfrac{x^k}{[a]_k} , \\ x^n = [a]_n \sum_{k=0}^{n} q^{-k} \left[{n \atop k} \right] \dfrac{W_k(x;a,q)}{[q]_k} . \end{cases}$$

and similarly, putting $g(k) = q^{k+(k^2/2)}x^k$ in (1.5.12) and, comparing it with (1.4.11), one arrives at the inverse relations involing Stieltjes-Wigert polynomial as mentioned below.

$$(1.5.15)\quad \begin{cases} S_n(x;p,q) = (-1)^n q^{-n(2n+1)/2} [p]_n \sum_{k=0}^{n} (-1)^k q^{k^2+k/2} . \\ \qquad \cdot \left[{n \atop k} \right] \dfrac{x^k}{[p]_k} \\ q^{n+n^2/2} x^n = [p]_n \sum_{k=0}^{n} q^{k-nk+3k^2/2} \left[{n \atop k} \right] \dfrac{S_k(x;p,q)}{[p]_k} . \end{cases}$$