CHAPTER - 3

INVERSE SERIES RELATION II

3.1 INTRODUCTION

As mentioned in chapter-1, Carlitz [3] gave a basic inversion pair in the form :

$$(3.1.1)\quad \begin{cases} f(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-2n+1)/2} \left[{n \atop k} \right]_\lambda \psi(-k,n,q^\lambda)\, g(k), \\ g(n) = \sum_{k=0}^{n} (-1)^k q^{k\lambda(k-1)/2} \left[{n \atop k} \right]_\lambda (a_{k+1}+ q^{k\lambda}\, b_{k+1}) \cdot \dfrac{f(k)}{\psi\,(-n,k+1,q^\lambda\,)} \end{cases}$$

with $$\left[{n \atop k} \right]_\lambda = \frac{(1-q^{n\lambda})(1-q^{(n-1)\lambda})\ldots(1-q^{(n-k+1)\lambda})}{(1-q^{k\lambda})(1-q^{(k-1)\lambda})\ldots(1-q^{\lambda})}, \quad (\lambda\neq 0)$$

and

$$\psi(k,n,q) = \prod_{i=1}^{n} (a_i+q^{-k\lambda}b_i).$$

When $\lambda=-br-1$, $a_i=1$, and $b_i=-q^{p-ar+i-1}$, the products $\psi(-k,n,q^\lambda)$ and $\psi(-n,k+1,q^\lambda)$ get particularized respectively, to

$$\frac{[q^{p-ar-brk-k}]_\infty}{[q^{p-ar-brk+n-k}]_\infty}, \quad \text{and} \quad \frac{[q^{p-ar-brn-n}]_\infty}{[q^{p-ar-brn-n+k+1}]_\infty},$$

and consequently, the above pair (3.1.1) gets reduced to

$$(3.1.2)\quad \begin{cases} A_n = \sum_{k=0}^{n} (-1)^k q_1^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{-br-1} \dfrac{B_k}{[q^{p-ar-brk+n-k}]_\infty}, \\[2ex] B_n = \sum_{k=0}^{n} (-1)^k q_1^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{-br-1} (1-q^{p-ar-brk}) . \\ \qquad\qquad . [q^{p-ar-brn-n+k+1}]_\infty \; A_k , \end{cases}$$

where, $q_1 = q^{-br-1}$.

The inverse pair (3.1.2) provides a useful tool in obtaining a basic analogue of theorem-1 which is proved in section-2 of this chapter.

An investigation of this theorem for obtaining various particular cases, leads to certain seemingly new basic analogues of the polynomials $f_n^c(x,y,r,m)$, $P_n(m,x,y,p,C)$, $\Pi_{n,m}^{\nu}(x)$, $P_n(m,x)$ $p_n(x)$, $C_n^{\nu}(x)$, and the Legendre polynomial $P_n(x)$ along with their inverse series relations.

A complete list of the basic analogues of polynomials cited above, is given in the following table together with the notations used for them.

Table-13 : *Basic polynomials*

Ordinary polynomial (Name)	Basic analogue	Relations with their particular polynomials
$f_n^c(x,y,r,m)$ (of Singhal & S.Kumari)	$f_n^c(x,y,r,m\|q)$	———
$P_n(m,x,y,p,C)$ (Generalized Humbert)	$P_n(m,x,y,p,C\|q)$	$f_n^{-p}(x, \frac{1-q}{1-q^C}, y,1,m\|q)$
$\Pi_{n,m}^{\nu}(x)$ (Humbert)	$\Pi_{n,m}^{\nu}(x\|q)$	$f_n^{\nu}(x,1,1,m\|q), P_n(m,x,1,-\nu,1\|q)$
$P_n(m,x)$ (Kinney)	$P_n(m,x\|q)$	$f_n^{1/m}(x,1,1,m\|q)$, $\Pi_{n,m}^{1/m}(x\|q)$, $P_n(m,x,1,\frac{-1}{m},1\|q)$
$p_n(x)$ (Pincherle)	$p_{n,q}(x)$	$f_n^{1/2}(x,1,1,3\|q)$, $\Pi_{n,3}^{1/3}(x\|q)$, $P_n(3,x,1,\frac{-1}{2},1\|q)$
$C_n^{\nu}(x)$ (Gegenbauer)	$C_n^{\nu}(x\|q)$	$f_n^{\nu}(x,1,1,2\|q)$, $\Pi_{n,2}^{\nu}(x\|q)$, $P_n(2,x,1,-\nu,1\|q)$
$P_n(x)$ (Legendre)	$P_n(x\|q)$	$f_n^{1/2}(x,1,1,2\|q)$, $\Pi_{n,2}^{1/2}(x\|q)$, $P_n(2,x,1,\frac{-1}{2},1\|q)$, $P_n(2,x\|q)$

The explicit representations and the inverse series relation of these polynomials are given below.

$$
(3.1.3)\quad \begin{cases} f_n^c(x,y,r,m|q) = \sum_{k=0}^{[n/m]} \dfrac{[q^{-c-nr+mrk-k+1}]_\infty \, \alpha_{n-mk} \, y^k \, x^{n-mk}}{[q^{-c-nr+mrk+1}]_\infty \, (q^{mr-1};q^{mr-1})} \\[2ex] \alpha_n x^n = \sum_{k=0}^{[n/m]} (-y)^k \, q^{(mr-1)k(k-1)/2} \, \dfrac{(1-q^{-c-nr+mrk})}{(q^{mr-1};q^{mr-1})_k} \\[2ex] \qquad \cdot \dfrac{[q^{-c-nr+1}]_\infty}{[q^{-c-nr+k}]_\infty} \, f_{n-mk}^c(x,y,r,m|q); \end{cases}
$$

$$
(3.1.4)\quad \begin{cases} P_n(m,x,y,p,C|q) = \sum_{k=0}^{[n/m]} \dfrac{[q^{p-n+mk-k+1}]_\infty \, (1-q^C)^{p-n+mk-k} \, y^k}{[q^{p-n+mk+1}]_\infty \, (q^{m-1};q^{m-1})_k \, [q]_{n-mk}} \\[2ex] \qquad \cdot (1-q)^{-p+k} \, ((q^m-1)x)^{n-mk}, \\[2ex] \left(\dfrac{q^m-1}{1-q^C} \right)^n \dfrac{x^n}{[q]_n} = \sum_{k=0}^{[n/m]} (-y)^k \, q^{(m-1)(k-1)k/2} \, \dfrac{1-q^{p-n+mk}}{(q^{m-1};q^{m-1})_k} \\[2ex] \qquad \cdot \dfrac{[q^{p-n+1}]_\infty}{[q^{p-n+k}]_\infty} \left(\dfrac{1-q^C}{1-q} \right)^{-p-k} P_{n-mk}(m,x,y,p,C|q); \end{cases}
$$

$$
(3.1.5)\quad \begin{cases} \Pi_{n,m}^p(x|q) = \sum_{k=0}^{[n/m]} \dfrac{[q^{-\nu-n+mk-k+1}]_\infty \, (x(q^m-1)/(1-q))^{n-mk}}{[q^{-\nu-n+mk+1}]_\infty \, (q^{m-1};q^{m-1})_k \, [q]_{n-mk}}, \\[2ex] \left(\dfrac{q^m-1}{1-q} \right) \dfrac{x^n}{[q]_n} = \sum_{k=0}^{[n/m]} (-1)^k \, q^{(m-1)k(k-1)/2} \, \dfrac{(1-q^{-\nu-n+mk})}{[q^{\nu-n+k}]_\infty} \\[2ex] \qquad \cdot \dfrac{[q^{-\nu-n+1}]_\infty \, \Pi_{n-mk,m}^\nu(x|q)}{(q^{m-1};q^{m-1})_k}, \end{cases}
$$

(3.1.6)

$$P_n(m,x|q)=\sum_{k=0}^{[n/m]} \frac{[q^{-(1/m)-n+mk-k+1}]_\infty (x(q^m-1)/(1-q))^{n-mk}}{[q^{-(1/m)-n+mk+1}]_\infty \ (q^{m-1};q^{m-1})_k \ [q]_{n-mk}}$$

$$\left(\frac{q^m-1}{1-q}\right) \frac{x^n}{[q]_n} = \sum_{k=0}^{[n/m]} (-1)^k \ q^{(m-1)k(k-1)/2} \ \frac{(1-q^{-(1/m)-n+mk})}{[q^{-(1/m)-n+k}]_\infty} \cdot \frac{[q^{-(1/m)-n+1}]_\infty \ P_{n-mk}(m,x|q)}{(q^{m-1};q^{m-1})_k} \ ;$$

(3.1.7)

$$P_{n,q}(x)=\sum_{k=0}^{[n/3]} \frac{[q^{(1/2)-n+2k}]_\infty \ (q^2+q+1)^{n-3k} \ (-x)^{n-3k}}{[q^{1/2-n+3k}]_\infty \ (q^2;q^2)_k \ [q]_{n-3k}}$$

$$(q^2+q+1)^n(-x)^n = [q]_n \sum_{k=0}^{[n/3]} (-1)^k \ q^{k(k-1)} \ \frac{(1-q^{-(1/2)-n+3k})}{[q^{-(1/2)-n+k}]_\infty} \cdot \frac{[q^{(1/2)-n}]_\infty \ P_{n-3k,q}(x)}{(q^2;q^2)_k} \ ;$$

(3.1.8)

$$C_n^\nu(x|q) = \sum_{k=0}^{[n/2]} \frac{[q^{-\nu-n+k+1}]_\infty \ (1+q)^{n-2k} \ (-x)^{n-2k}}{[q^{-\nu-n+2k+1}]_\infty \ [q]_{n-2k} \ [q]_k} \ ,$$

$$\frac{(-x)^n}{[q]_n} = \sum_{k=0}^{[n/2]} (-1)^k \ q^{k(k-1)/2} \ \frac{(1-q^{-\nu-n+2k}) \ [q^{-\nu-n+1}]_\infty}{[q^{-\nu-n+k}]_\infty \ [q]_k \ (1+q)^n} \cdot C_{n-2k}^\nu(x|q) \ ;$$

$$
(3.1.9)\quad \begin{cases} P_n(x|q) = \sum_{k=0}^{[n/2]} \dfrac{[q^{(1/2)-n+k}]_\infty \, (1+q)^{n-2k}(-x)^{n-2k}}{[q^{(1/2)-n+2k}]_\infty \, [q]_{n-2k} \, [q]_k} , \\[2ex] \dfrac{(1+q)^n(-x)^n}{[q]_n} = \sum_{k=0}^{[n/2]} (-1)^k \, q^{k(k-1)/2} \, \dfrac{(1-q^{-(1/2)-n+2k})}{[q^{-(1/2)-n+k}]_\infty \, [q]_k} \\[2ex] \qquad \cdot \, [q^{(1/2)-n}]_\infty \, P_{n-2k}(x|q) \; ; \end{cases}
$$

(cf. (1.2.26) to (1.2.30)).

3.2 MAIN RESULT

The proposed basic analogue of theorem-1, in terms of the notations of section-1.4, may be stated in the form of

THEOREM-2. If $br \neq -1$, then

$$
(3.2.1)\quad F(a) = \sum_{k=0}^{M} y^k \frac{[q^{p-ar-brk-k+1}]_\infty}{(q^{-br-1};q^{-br-1})_k} \, G(a+bk)
$$

if, and only if

$$
(3.2.2)\quad G(a) = \sum_{k=0}^{M} (-y)^k q^{-(br+1)(k-1)k/2} \frac{(1-q^{p-ar-brk}) \, F(a+bk)}{[q^{p-ar+k}]_\infty (q^{-br-1};q^{-br-1})_k},
$$

where

$$
M = \begin{cases} [-a/b], & \text{if 'a' is non negative integer and 'b' is a negative integer} \\ \infty, & \text{if 'a' and 'b' are positive integers.} \end{cases}
$$

If $br = -1$, then the following relations hold true,

$$(3.2.3)\quad f(a) = \sum_{k=0}^{M} y^k \frac{[q^{p-ar+1}]_\infty}{(q;q)_k} g(a+bk)$$

if, and only if

$$(3.2.4)\quad g(a) = \sum_{k=0}^{M} (-y)^k q^{(k-1)k/2} \frac{f(a+bk)}{[q^{p-ar+k+1}]_\infty (q;q)_k}\ ,$$

wherein M is same as above.

The proofs of theorem-2 corresponding to the two separate cases : M = ∞ and M = a positive integer, are as given below.

When both a and b are positive intergers, i.e. when $M = \infty$, the proof of 'if' part uses the method due to Carlitz[3], wherein the following well known relation will be employed.

$$(3.2.5)\quad \sum_{k=0}^{n} q^{k(k-1)/2} \left[{n \atop k} \right] x^k = \prod_{k=1}^{n} (1+xq^{k-1}),\ n=1,2,3,\ldots\ .$$

Denoting the right hand side of (3.2.1) by F^*, and using (3.2.2) for G(a+bk), one gets

$$F^* = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k y^{k+j} q_1^{j(j-1)/2} \frac{[q^{p-ar-brk-k+1}]_\infty}{[q^{p-ar-brk+j}]_\infty}\ .$$

$$\cdot \frac{1-q^{p-ar-brk-brj}}{(q_1;q_1)_j (q_1;q_1)_k} F(a+bk+bj),$$

$$(q_1 = q^{-br-1}).$$

This may be put in the form :

$$(3.2.6)\quad F^* = F(a) + \sum_{j=1}^{\infty} (-y)^j q_1^{j(j-1)/2} \frac{1-q^{p-ar-brj}}{(q_1;q_1)_j} F(a+bj) \;.$$

$$\cdot \sum_{k=0}^{j} (-1)^k \left[\begin{matrix} j \\ k \end{matrix} \right]_{-br-1} q_1^{k(k-2j+1)/2} \frac{[q^{p-ar-brk-k+1}]_\infty}{[q^{p-ar-brk-k+j}]_\infty} \;.$$

Since,

$$\frac{[q^{p-ar-brk-k+1}]_\infty}{[q^{p-ar-brk-k+j}]_\infty} = \prod_{m=1}^{j-1} (1-q^{p-ar-brk-k+m})$$
$$= \sum_{i=0}^{j-1} A_i\, q_1^{ki}$$

is a polynomial in q_1^k of degree $(j-1)$, the inner series in (3.2.6) transforms to

$$\sum_{i=0}^{j-1} A_i \sum_{k=0}^{j} \left[\begin{matrix} j \\ k \end{matrix} \right]_{-br-1} q_1^{k(k-1)/2} (-q_1^{i-j+1})^k \;,$$

which by means of the result (3.2.5) simplifies to the form

$$\sum_{i=0}^{j-1} A_i \prod_{k=1}^{j} (1-q_1^{i-j+k})$$
$$= \sum_{i=0}^{j-1} A_i\, (1-q_1^{i-j+1})\,(1-q_1^{i-j+2}) \;\ldots\; (1-q_1^{i-1})(1-q_1^{i}) \;.$$

As this last expression vanishes for all $i=0,1,2,\ldots,j-1$, it follows from (3.2.6) that $F^* = F(a)$; which proves that (3.2.2) implies (3.2.1).

In establishing the 'only if' part, when $M = \infty$, the method employed here runs parallel to that of Gould [4].

In fact, writing (3.2.1) and (3.2.2) in the forms

$$F(a) = \sum_{k=0}^{\infty} C_{a,k}\, G(a+bk) \;;\; G(a) = \sum_{k=0}^{\infty} D_{a,k}\, F(a+bk) ,$$

it is easy to observe that the validity of the 'only if' part is established if the following orthogonal relation holds true :

$$(3.2.7)\quad \delta_{jo} = \sum_{k=0}^{j} D_{a,k}\; C_{a+bk,j-k} = \begin{cases} 1, & j=0 \\ 0, & j\neq 0 \end{cases} .$$

Here, the expression corresponding to the series in (3.2.7) is equal to

$$\sum_{k=0}^{j} (-1)^k q_1^{k(k-1)/2} \left[{j \atop k} \right]_{-br-1} (1-q^{p-ar-brk}) \frac{[q^{p-ar-brj-j+k+1}]_{\infty}}{[q^{p-ar+k}]_{\infty}}$$

wherein replacing $1/[q^{p-ar+k}]_{\infty}$ by B_k and then denoting the series by A_j, one gets

$$(3.2.8)\quad A_j = \sum_{k=0}^{j} (-1)^k q_1^{k(k-1)/2} \left[{j \atop k} \right]_{-br-1} (1-q^{p-ar-brk}) \,.$$

$$. [q^{p-ar-brj-j+k+1}]_{\infty} B_k .$$

The inverse series of (3.2.8) is easily obtainable from the pair (3.1.2) in the form :

$$(3.2.9)\quad B_j = \sum_{k=0}^{j} (-1)^k q_1^{k(k-2j+1)/2} \left[{j \atop k} \right]_{-br-1} \frac{A_k}{[q^{p-ar-brk-k+j}]_{\infty}} .$$

On making substitution $A_N = \left[{0 \atop N} \right]$ in (3.2.9), one gets

$$B_N = 1/[q^{p-ar+N}]_{\infty} .$$

while (3.2.8), with these A_N and B_N leads to the above mentioned orthogonality relation (3.2.7), which completes the proof of the 'only if' part and hence, the proof of the theorem for $M = \infty$.

If 'a' is a non-negative integer: n, and 'b' is a negative integer : $-m$ ($m=1,2,3 \ldots$), in which case $M = [n/m]$, the proof of the theorem, which runs on the same lines as the proof for the case $M=\infty$, is summarized as below.

In order to prove the 'if' part, put

$$f^{*} = \sum_{k=0}^{[n/m]} y^{k} \frac{[q^{p-nr+mrk-k+1}]_{\infty}}{(q^{mr-1};\, q^{mr-1})_{k}} G(n-mk).$$

Then in view of (3.2.2), this becomes

$$f^{*} = \sum_{k=0}^{[n/m]} \sum_{j=0}^{[n/m]-k} (-1)^{j} q^{(mr-1)(j-1)j/2} \frac{[q^{p-nr+mrk-k+1}]_{\infty}}{[q^{p-nr+mrk+j}]_{\infty}}$$

$$\cdot \frac{1-q^{p-nr+mrj+mrk}}{(q^{mr-1};q^{mr-1})_{j}\,(q^{mr-1};q^{mr-1})_{k}} F(n-mk-mj) ,$$

wherein using the known relation (see Gould [6]):

$$(3.2.10) \qquad \sum_{k=0}^{[n/m]} \sum_{j=0}^{[n/m]-k} A(k,j) = \sum_{j=0}^{[n/m]} \sum_{k=0}^{j} A(k,j-k)$$

one obtains

$$(3.2.11) \quad f^{*} = F(n) + \sum_{j=1}^{[n/m]} (-y)^{j} \frac{1-q^{p-nr+mrj}}{(q_2;q_2)_{j}} q_2^{j(j-1)/2} F(n-mj) \; .$$

$$\cdot \sum_{k=0}^{j} (-1)^{k} q_2^{k(k-2j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_{mr-1} \frac{[q^{p-nr+mrk-k+1}]_{\infty}}{[q^{p-nr+mrk+j-k}]_{\infty}} ,$$

where $q_2 = q^{mr-1}$.

Evidently,

$$\frac{[q^{p-nr+mrk-k+1}]_\infty}{[q^{p-nr+mrk-k+j}]_\infty} = \sum_{i=0}^{j-1} B_i\, q_2^{ki} .$$

with which, (3.2.11) gets transformed to

$$f^* = F(n) + \sum_{j=1}^{[n/m]} (-y)^j \frac{1-q^{p-nr+mrj}}{(q_2;q_2)_j} q_2^{j(j-1)/2} .$$

$$. \sum_{i=0}^{j-1} B_i \sum_{k=0}^{j} q_2^{k(k-2j+1)/2} \left[{j \atop k} \right]_{mr-1} (-q_2^{i-j})^k .$$

This, with an apeal to the formula (3.2.5), assumes the form :

$$f^* = F(n) + \sum_{j=1}^{[n/m]} (-y)^j \frac{1-q^{p-nr+mrj}}{(q_2;q_2)_j} q_2^{j(j-1)/2} F(n-mj) .$$

$$. \sum_{i=0}^{j-1} B_i \prod_{k=1}^{j} (1-q_2^{i-j+k}) .$$

It is obvious that the inner series in this last experesion vanishes for all i=0,1,2,...,j-1, and therefore, one finally gets $f^* = F(n)$; thus, (3.2.2) implies (3.2.1).

Conversely, in view of (3.2.1), the right hand member of (3.2.2) denoted for brevity by g^*, can be expressed as

$$g^* = \sum_{k=0}^{[n/m]} \sum_{j=0}^{[n/m]-k} (-1)^k y^{k+j} q_2^{(k-1)k/2} \frac{[q^{p-nr+mrk+mrj}]_\infty}{[q^{p-nr+k}]_\infty} .$$

$$. \frac{1-q^{p-ar+brk}}{(q_2;q_2)_k (q_2;q_2)_j} G(n-mk-mj)$$

$$= \sum_{j=0}^{[n/m]} \frac{G(n-mj)}{(q_2;q_2)_j} \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \left[{j \atop k} \right]_{mr-1} (1-q^{p-nr+mrk}) \cdot \frac{[q^{p-nr+mrj-j+k+1}]_\infty}{[q^{p-nr+k}]_\infty} .$$

If V_j denotes the inner series in the last expression given above, then it can be easily seen that, with $1/[q^{p-nr+k}]_\infty$ replaced by W_k, it reads as

$$(3.2.12)\quad V_j = \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \left[{j \atop k} \right]_{mr-1} (1-q^{p-nr+mrk}) \cdot [q^{p-nr+mrj-j+k+1}]_\infty \, W_k .$$

The inverse series of this follows readily from (3.1.2) in the form :

$$W_j = \sum_{k=0}^{j} (-1)^k q_2^{k(k-2j+1)/2} \left[{j \atop k} \right]_{mr-1} \frac{V_k}{[q^{p-nr+mrk-k+j}]_\infty} ,$$

which with

$$V_k = \left[{0 \atop k} \right]$$

gives

$$W_j = 1/[q^{p-nr+j}]_\infty ,$$

and censequently the series in (3.2.12) leads to the orthogonality relation :

$$(3.2.13)\quad \left[{0 \atop j} \right] = \delta_{j0} = \sum_{k=0}^{j} (-1)^k q_2^{k(k-1)/2} \left[{j \atop k} \right]_{mr-1} \frac{(1-q^{p-nr+mrk})}{[q^{p-nr+k}]_\infty} \cdot [q^{p-nr+mrj-j+k+1}]_\infty .$$

thus,

$$g^{*} = \sum_{j=0}^{[n/m]} \frac{G(n-mj)}{(q_2;q_2)_j} \delta_{j0}$$

$$= \begin{cases} G(n), & \text{if } j=0 \\ 0, & \text{if } j\neq 0 \end{cases}$$

which completes the proof of the 'only if' part, and the proof of the theorem for M = [n/m].

The proofs of the relations (3.2.3) and (3.2.4) corresponding to the cases M = ∞ and M = [n/m] as outlined below, make use of the formula (3.2.5) :

$$\sum_{k=o}^{n} q^{k(k-1)/2} \left[{n \atop k} \right] x^{k} = \prod_{k=0}^{n} (1+x\, q^{k-1}).$$

It may be noted here that if, in (3.2.3) and (3.2.4), f(a) is replaced by $[q^{p-ar+1}]_{\infty}$ f(a), then one obtains an elegant (and convenient) form :

$$(3.2.14)\quad f(a) = \sum_{k=0}^{M} y^{k} \frac{g(a+bk)}{[q]_k}, \quad g(a) = \sum_{k=0}^{M} (-y)^{k} q^{k(k-1)/2} \frac{f(a+bk)}{[q]_k}.$$

In order to prove the inverse series relations when M = ∞, take

$$\sum_{k=0}^{\infty} y^{k} \frac{g(a+bk)}{[q]_k} = \rho \quad .$$

Then with the aid of the second relation in (3.2.14) one gets,

$$\rho = f(a) + \sum_{j=1}^{\infty} y^{j} q^{j(j-1)/2} \frac{f(a+bj)}{[q]_j} \sum_{k=0}^{j} q^{k(k-1)/2} \left[{j \atop k} \right] (-q^{1-j})^{k},$$

this in view of the formula (3.2.5) further simplifies to

$$\rho = f(a) + \sum_{j=1}^{\infty} y^j q^{j(j-1)/2} \frac{f(a+bj)}{[q]_j} \prod_{k=1}^{j} (1-q^{k-j}),$$

which readily gives $\rho = f(a)$ and thus the second relation in (3.2.14) implies the first.

Likewise, with

$$\sum_{k=0}^{\infty} (-y)^k q^{k(k-1)/2} \frac{f(a+bk)}{[q]_k} = \sigma,$$

and making an appeal to the first relation in (3.2.14) one arrives at

$$\sigma = g(a) + \sum_{j=1}^{\infty} y^j \frac{g(a+bj)}{[q]_j} \sum_{k=0}^{j} (-1)^k q^{k(k-1)/2} \left[\begin{matrix} j \\ k \end{matrix} \right].$$

Here, the formula (3.2.5) gives

$$\sigma = g(a) + \sum_{j=1}^{\infty} y^j \frac{g(a+bj)}{[q]_j} \prod_{k=1}^{j} (1-q^{k-1}),$$

which ultimately reduces to $\sigma = g(a)$ as the product term in the last expression vanishes for $j \geq 1$. Hence, the second part and, the proof for $M = \infty$.

For $M = [n/m]$, the inverse relations may be justified with the help of the formulas (3.2.10) and (3.2.5) .

In fact, if ω denotes the right hand side of (3.2.3), then making an appeal to (3.2.4) and the relation in (3.2.10), one finds

$$\omega = \sum_{j=0}^{[n/m]} (-y)^j \, q^{j(j-1)/2} \, \frac{f(n-mj)}{[q]_j} \sum_{k=0}^{j} q^{k(k-1)/2} \left[{j \atop k} \right] (-q^{1-j})^k ,$$

which may be written as

$$\omega = f(n) + \sum_{j=1}^{[n/m]} (-y)^j \, q^{j(j-1)/2} \, \frac{f(n-mj)}{[q]_j} \prod_{k=1}^{j} (1-q^{k-j})$$

$$= f(n) ,$$

since the product term vanishes for $j \geq 1$. Thus, (3.2.4) implies (3.2.3).

Similarly, denoting by ψ the right hand side of (3.2.4) and making use of (3.2.3) and (3.2.10), one arrives at

$$\psi = g(n) + \sum_{j=1}^{[n/m]} \frac{g(n-mj)}{[q]_j} \sum_{k=0}^{j} q^{k(k-1)/2} \left[{j \atop k} \right] (-1)^k$$

$$= g(n) + \sum_{j=1}^{[n/m]} \frac{g(n-mj)}{[q]_j} \prod_{k=0}^{j} (1-q^{k-1}) ,$$

$$= g(n),$$

which completes the proof of the inverse relations when $M = [n/m]$.

3.3 PARTICULAR CASES : POLYNOMINALS

In this section, theorem-2 is particularized first, so as to yield a basic analogue of the class of polynomials $\{f_n^c(x,y,r,m)\}$ together with its inverse relations. The fact that $f_n^c(x,y,r,m)$ includes a large number of polynomials as discussed in section-3.1 leads to the basic analogues of all those polynomials and their corresponding inverse relations which are also discussed in this section. It also includes a basic analogue of the Gould-Hopper polynominal.

In order to obtain a basic analogue of the polynomial $f_n^c(x,y,r,m)$, it may be seen that when $a = n$ ($n = 0,1,2,\ldots$) and $b = -m$ ($m = 1,2,3\ldots$) then, theorem-2 assumes the form

$$(3.3.1)\quad \begin{cases} F(n) = \sum_{k=0}^{[n/m]} y^k \dfrac{[q^{p-nr+mrk-k+1}]_\infty}{(q^{mr-1};\, q^{mr-1})_k} \quad G(n-mk) \\ \text{if, and only if} \\ G(n) = \sum_{k=0}^{[n/m]} (-y)^k \; q^{(mr-1)(k-1)k/2} \dfrac{(1-q^{p-nr+mrk})\; F(n-mk)}{[q^{p-nr+k}]_\infty (q^{mr-1};q^{mr-1})_k} \end{cases}$$

This inverse pair, with the aid of the substitutions $p = -c$, and $G_n = \alpha_n x^n / [\, q^{-c-nr+1}\,]_\infty$, defines a (seemingly new) basic analogue of the polynomial $f_n^c(x,y,r,m)$ which is denoted by $f_n^c(x,y,r,m|q)$, and mentioned in (3.1.3) along with its inverse series relation. This basic pair (i.e.(3.1.3)) with an appeal to the formula :

$$\frac{(1-q^\alpha)(1-q^{\alpha-1})\ \ldots\ (1-q^{\alpha-k+1})}{(1-q^k)(1-q^{k-1})\ \ldots\ (1-q)} = \left[\begin{matrix} \alpha \\ k \end{matrix} \right] .$$

admits an alternative form:

$$(3.3.2)\quad \begin{cases} f_n^c(x,y,r,m|q) = \sum_{k=0}^{[n/m]} y^k \left[\begin{matrix} -c-nr+mrk \\ k \end{matrix} \right] \dfrac{[q]_k\ \alpha_{n-mk}\ x^{n-mk}}{(q^{mr-1};\, q^{mr-1})_k} , \\ \\ \alpha_n x^n = \sum_{k=0}^{[n/m]} (-y)^k \dfrac{q^{(mr-1)(k-1)k/2}\ (1-q^{-c-nr+mrk})}{(1-q^{-c-nr+k})\ (q^{mr-1};q^{mr-1})_k} \\ \qquad \cdot \left[\begin{matrix} -c-nr+k \\ k \end{matrix} \right] f_{n-mk}^c(x,y,r,m|q), \end{cases}$$

which is more convenient in examining its limiting case $q \to 1$.

In fact, if y is replaced by $(mr-1)y$ in (3.3.2) and then limit $q \to 1$ is taken, it would lead to the corresponding ordinary form :

$$(3.3.3)\quad \begin{cases} f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} \gamma_{n-mk}\, x^{n-mk}, \\ \gamma_n x^n = \sum_{k=0}^{[n/m]} (-y)^k \frac{-c-nr+mrk}{-c-nr+k} \binom{-c-nr+k}{k} f_{n-mk}^c(x,y,r,m). \end{cases}$$

In an analogous manner, it can be shown that under the process of replacing $G(a)$ by $(1-q)^{-p+ar}G(a)$, y by $(q-1)(br+1)y$, $[q^{p-ar-brk-k+1}]_\infty$ by

$$\frac{(1-q)^{-p+ar+brk+k}[q]_\infty}{\Gamma_q(p-ar-brk-k+1)},$$

and, $[q^{p-ar+k+1}]_\infty$ by

$$\frac{(1-q)^{-p+ar-k}\,[q]_\infty}{\Gamma_q(p-ar+1)},$$

and then on letting $q \to 1$, theorem-2 gets transformed to theorem-1 (Chapter-2).

In view of the fact that the class $\{f_n^c(x,y,r,m)\}$ of polynomials defined explicitly by (3.3.3) above admits a large number of polynomials, it would be interesting to take note of the corresponding particular cases obtainable from the set of polymomials $\{f_n^c(x,y,r,m|q)\}$.

When $r=1$, $c=-p$, $\alpha_n = ((q^m-1)/(1-q^c))^n/[q]_n$, and y is replaced by $\{(1-q)/(1-q^C)\}y$, then the pair in (3.1.3) defines a basic analogue of the class of polymomials $\{p_n(m,x,y,p,C)\}$ together with its inverse series relations as given in (3.1.4). Some further reducibilities of this basic pair, i.e. (3.1.4) give rise to the basic analogues of several other polynomials as discussed below.

The special case $p = -\nu$, $C = 1$, and $y = 1$, of the polynomial $P_n(m,x,y,p,C|q)$ defines a basic analogue of the Humbert polynomial $\Pi^{\nu}_{n,m}(x|q)$ whose explicit form and inverse relation are stated in (3.1.5). A basic analogue of the kinney polynomial (1.2.27) is obtainable from the particular case $P_n(m,x,1,-1/m,1|q)$, an inverse series of which follows readily from (3.1.4) (see (3.1.6)). On the other hand, if $m = 3$, $p = -1/2$ and $y = C = 1$, then the pair (3.1.4) yields a basic Pincherle polynomials (denoted by $P_{n,q}(x)$), along with its inverse series relation which are stated in (3.1.7). Yet another specialization viz. $P_n(2,x,1,-\nu,1|q)$ of $P_n(m,x,y,p,C|q)$ defines a basic analogue of the Gegenbauer polynomial which is mentioned together with its inverse series, in (3.1.8).

Lastly, the substitutions $y = 1$, $C = 1$, $m=2$, and $p=-1/2$ in (3.1.4) yields the pair of a basic Legendre polynomial and its inverse series relation (see (3.1.9)).

It is interesting to point out that the pair of inverse relation of basic Kinney polynomial $P_n(m,x|q)$, when written alternatively as

$$
(3.3.4)\quad
\begin{cases}
P_n(m,x|q) = \displaystyle\sum_{k=0}^{[n/m]} \begin{bmatrix} -1/m \\ k \end{bmatrix} \begin{bmatrix} -k-(1/m) \\ n-mk \end{bmatrix} \frac{[q]_k}{(q^{m-1};q^{m-1})_k} \cdot \left(\frac{q^m-1}{1-q}\, x \right)^{n-mk}, \\[2ex]
\begin{bmatrix} -1/m \\ n \end{bmatrix} \left(\dfrac{q^m-1}{1-q}\, x \right)^n = \displaystyle\sum_{k=0}^{[n/m]} (-1)^k q^{(m-1)(k-1)k/2} \frac{1-q^{mk-n-(1/m)}}{1-q^{k-n-(1/m)}} \cdot \begin{bmatrix} -n+k-(1/m) \\ k \end{bmatrix} P_{n-mk}(m,x|q)
\end{cases}
$$

transforms to the ordinary form (Gould [6,p. 707]) :

$$
(3.3.5)\quad
\begin{cases}
P_n(m,x) = \displaystyle\sum_{k=0}^{[n/m]} \binom{-1/m}{k} \binom{-k-(1/m)}{n-mk} (-mx)^{n-mk}, \\[2ex]
\dbinom{-1/m}{n} (-mx)^n = \displaystyle\sum_{k=0}^{[n/m]} (-1)^k \frac{-(1/m)-n+mk}{-(1/m)-n+k} \binom{-n+k-(1/m)}{k} \cdot P_{n-mk}(m,x),
\end{cases}
$$

as $q \to 1$.

Coming to the particular cases of (3.2.3) and (3.2.4) or equivalently those of (3.2.14), it is to be noted that when $a=n (n=0,1,2,\ldots)$, $b=-1$, and $y=1$, so that $M=n$, one gets the pair

$$
f(n) = \sum_{k=0}^{n} \frac{g(n-k)}{[q]_k} \; ; \qquad g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \frac{f(n-k)}{[q]_k},
$$

which, by reversing the series and writing $g^*(n)$ for $(-1)^n q^{n(n-1)/2} g(n)$, takes the form :

$$
f(n) = \sum_{k=0}^{n} (-1)^{n-k} q^{k(k-1)/2} \frac{g^*(k)}{[q]_{n-k}} \; ; g^*(n) = \sum_{k=0}^{n} q^{k(k-2n+1)/2} \frac{f(k)}{[q]_{n-k}}.
$$

This inversion pair is essentially the same pair as given in (1.5.12), whose particular cases are given by (1.5.13), (1.5.14) and (1.5.15). However, it is to be mentioned here that the pair (3.2.14) is capable of yielding a basic analogue of the Gould-Hopper polynomial (Gould and Hopper [1]) :

$$g_n^m(x,\lambda) = \sum_{k=0}^{[n/m]} \frac{n!\ \lambda^k\ x^{n-mk}}{k!\ (n-mk)!}$$

together with its inverse series relation.

In fact, setting a=n (n=0,1,2,...) as before, b=-m (m=1,2,3,...), $y = (1-q)^{1-m}\lambda$ and $g(n) = x^n/[q]_n$ in (3.2.14), and denoting by $g_n^m(x,\lambda|q)$ the polynomial thus obtained, one gets the following basic Gould-Hopper polynomial :

$$(3.3.6)\quad g_n^m(x,\lambda|q) = \sum_{k=0}^{[n/m]} \frac{[q]_n\ (1-q)^{k-mk}}{[q]_k\ [q]_{n-mk}}\ \lambda^k\ x^{n-mk}\ ,$$

along with its inverse relation :

$$(3.3.7)\quad x^n = [q]_n \sum_{k=0}^{[n/m]} (-1)^k\ q^{k(k-1)/2}\ \frac{\lambda^k\ (1-q)^{k-mk}}{[q]_k\ [q]_{n-mk}}\ g_{n-mk}^m(x,\lambda|q).$$

3.4 PARTICULAR CASES : RIORDAN'S INVERSE RELATIONS

Besides giving rise to the various basic polynomials, the inverse series relations given in (3.2.1) and (3.2.2); as well as those given by (3.2.3) and (3.2.4), also furnish the basic analogues of the inversion pairs belonging to the Riordan's classification (see tables 1 to 6).

With a view to obtain the basic analogues of the simplest inverse relations which are listed in Table-1, the following transformed version of the pair of relations (3.2.3) and (3.2.4) will be used here with a=n (a non-negative integer) and y=1 (see (3.2.14)).

$$(3.4.1)\quad f(n) = \sum_{k=0}^{M} \frac{g(n+bk)}{[q]_k} \;;\; g(n) = \sum_{k=0}^{M} (-1)^k\, q^{k(k-1)/2}\, \frac{f(n+bk)}{[q]_k}\,.$$

In this, putting b=-1 and then reversing the series, one finds

$$(3.4.2)\quad \begin{cases} f(n) = \displaystyle\sum_{k=0}^{n} q^{k(k-1)/2}\, \frac{g_k}{[q]_{n-k}} \;; \\ g_n = \displaystyle\sum_{k=0}^{n} (-1)^{n+k}\, q^{k(k-2n+1)/2}\, \frac{f(k)}{[q]_{n-k}}\,, \end{cases}$$

where $g_n = q^{n(n-1)/2}\, g(n)$.

Now, if f(n) is replaced by $f(n)/[q]_n$ and, g_n by $g_n\ /[q]_n$, then (3.4.2) results in the pair

$$(3.4.3)\quad \begin{cases} f(n) = \displaystyle\sum_{k=0}^{n} q^{k(k-1)/2} \left[{n \atop k} \right] g_k \;; \\ g_n = \displaystyle\sum_{k=0}^{n} (-1)^{n+k}\, q^{k(k-2n+1)/2} \left[{n \atop k} \right] f(k), \end{cases}$$

which provides a basic analogue of the pair (Table-1, No.(1)):

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} g_k \;\; ; \;\; g_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(k).$$

In a similar manner, the basic analogues of the other simplest type pairs can also be deduced. The following table embodies the basic analogues of those pairs which appear in Table-1.

Table-14 : *Basic simplest inverse relations*

$$f(n) = \sum_{k=0}^{n} q^{k(k-1)/2} C_{n,k}\, g_k;\quad g_n = \sum_{k=0}^{n} (-1)^{k+n} q^{k(k-2n+1)/2} D_{n,k} f(k)$$

	b	$C_{n,k}$	$D_{n,k}$	Basic analogue of class (No.) in Table-1
Inverse series relations given in (3.4.1)	-1	$\left[{n \atop k}\right]$	$\left[{n \atop k}\right]$	(1)
	1	$q^k \left[{k \atop n}\right]$	$q^k \left[{k \atop n}\right]$	(2)
	-1	$\left[{p-k \atop p-n}\right]$	$\left[{p-k \atop p-n}\right]$	(3)
	-1	$\left[{p+n \atop p+k}\right]$	$\left[{p+n \atop p+k}\right]$	(4)
	1	$q^k \left[{p+k \atop p+n}\right]$	$q^k \left[{p+k \atop p+n}\right]$	(5)
	-1	$\frac{[q]_n}{[q]_k} \left[{n-1 \atop k-1}\right]$	$\frac{[q]_n}{[q]_k} \left[{n-1 \atop k-1}\right]$	(6)

In order to obtain the basic analogues of the other classes namely, the Gould classes, simpler Chebyshev classes, Chebyshev classes, simpler Legendre classes, and the Legendre-Chebyshev classes, the following inverse pairs which are deduced from theorem-2, will be used.

First note that theorem-2 when r is replaced by $-r$, 'a' is replaced by a non-negative integer:n, and G(n) by $[q]_{p+rn}G_n/[q]_\infty$, assumes the form :

$$(3.4.4)\quad \begin{cases} F(n) = \sum_{k=0}^{M} y^k \dfrac{[q]_{p+rn+rbk}\ G(n+bk)}{[q]_{p+rn+rbk-k}\ (q_1;q_1)_k} , \\[2ex] G(n) = \sum_{k=0}^{M} (-y)^k\ q_1^{k(k-1)/2}\ \dfrac{1-q^{p+rn+rbk}}{1-q^{p+rn+k}} \\ \qquad \cdot \dfrac{[q]_{p+rn+k}\ F(n+bk)}{[q]_{p+rn}\ (q_1;q_1)_k} , \end{cases}$$

wherein $q_1=q^{br-1}$, and M is finite or infinite according as b is a negative or a positive integer.

Further, if F(n) and G(n) are replaced by $F(n)/1-q^{p+rn}$, and $G(n)/1-q^{p+rn}$ respectively then (3.4.4) gets transformed to the pair

$$(3.4.5)\quad \begin{cases} F(n) = \sum_{k=0}^{M} y^k \dfrac{1-q^{p+rn+1}}{1-q^{p+rn+rbk-k+1}}\ \dfrac{[q]_{p+rn+rbk}\ G(n+bk)}{[q]_{p+rn+rbk-k}\ (q_1;q_1)_k} , \\[2ex] G(n) = \sum_{k=0}^{M} (-y)^k\ q_1^{k(k-1)/2}\ \dfrac{[q]_{p+rn+k}\ F(n+bk)}{[q]_{p+rn}\ (q_1;q_1)_k} . \end{cases}$$

Next, if the base q is inverted, then theorem-2 with a=n as before, results in the form as given below.

$$(3.4.6)\quad \begin{cases} F(n) = \sum_{k=0}^{M} y^k \dfrac{(q^{-p+rn+rbk+k-1};q^{-1})_\infty}{(q_2;q_2)_k}\ G(n+bk) , \\[2ex] G(n) = \sum_{k=0}^{M} (-y)^k\ q_2^{k(k-1)/2}\ \dfrac{1-q^{-p+rn+rbk}}{1-q^{-p+rn-k}}\ \dfrac{F(n+bk)}{(q_2;q_2)_k} , \end{cases}$$

in which $q_2 = q^{br+1}$.

A little simplification in (3.4.6) leads to the pair

$$(3.4.7)\quad \begin{cases} F(n) = \sum_{k=0}^{M} y^k \dfrac{1-q^{-p+rn}}{1-q^{-p+rn+rbk+k}} \dfrac{[q]_{-p+rn+rbk+k}\, G(n+bk)}{[q]_{-p+rn+rbk}\, (q_2;q_2)_k} \\[2ex] G(n) = \sum_{k=0}^{M} (-y)^k q_2^{k(k-1)/2} \dfrac{[q]_{-p+rn}\, F(n+bk)}{[q]_{-p+rn-k}\, (q_2;q_2)_k}, \end{cases}$$

which on replacing $F(n)$ by $(1-q^{-p+rn})\,F(n)$ and, $G(n)$ by $(1-q^{-p+rn})\,G(n)$, gives

$$(3.4.8)\quad \begin{cases} F(n) = \sum_{k=0}^{M} y^k \dfrac{[q]_{-p+rn+rbk+k-1}}{[q]_{-p+rn+rbk-1}} \dfrac{G(n+bk)}{(q_2;q_2)_k}, \\[2ex] G(n) = \sum_{k=0}^{M} (-y)^k q_2^{k(k-1)/2} \dfrac{1-q^{-p+rn+rbk}}{1-q^{-p+rn-k}} \cdot \dfrac{[q]_{-p+rn-1}\, F(n+bk)}{[q]_{-p+rn-k-1}(q_2;q_2)_k}, \end{cases}$$

By appropriately specializing the parameters involved in these inverse series relations, one gets the seemingly new basic analogues of the aforementioned classes. For example, to get a basic analogue of the pair(2) of Table-2, put $b=-1$, and $y=1$ in (3.4.5) so that $M=n$. In this case, by reversing the series, one finds

$$(3.4.9)\quad \begin{cases} F(n) = \sum_{k=0}^{n} \dfrac{1-q^{p+rn+1}}{1-q^{p+rk-n+k+1}} \dfrac{[q]_{p+rk}\ G(k)}{[q]_{p+rk+k-n}\ (q^{-r-1};q^{-r-1})_{n-k}} \\[2ex] G(n) = q^{-(r+1)(n-1)n/2} \sum_{k=0}^{M} (-1)^{n+k}\ q^{-(r+1)(k-2n+1)/2} \ . \\[1ex] \qquad \cdot \dfrac{[q]_{p+rn+n-k}\ F(k)}{[q]_{p+rn}\ (q^{-r-1};q^{-r-1})_{n-k}} \end{cases}$$

Here, the substitution $G(n) = q^{-(r+1)(n-1)n/2}\ g(n)$ transforms the above pair in the form :

$$(3.4.10)\quad \begin{cases} F(n) = \sum_{k=0}^{n} q^{-(r+1)(k-1)k/2} \dfrac{1-q^{p+rn+1}}{1-q^{p+rk+k-n+1}} \dfrac{[q]_{p+rk}}{[q]_{p+rk+k-n}} \\[1ex] \qquad \cdot \dfrac{g(k)}{(q^{-r-1};q^{-r-1})_{n-k}} , \\[2ex] g(n) = \sum_{k=0}^{n} (-1)^{k+n}\ q^{-(r+1)(k-2n+1)/2} \ . \\[1ex] \qquad \cdot \dfrac{[q]_{p+rn+n-k}\ F(k)}{[q]_{p+rn}\ (q^{-r-1};q^{-r-1})_{n-k}} , \end{cases}$$

which with $r=m-1$, provides a basic analogue of Gould class(2) in Table-2.

Analogously, taking $b=1$, $y=1$ and, replacing p by $-p$ in (3.4.7), one readily gets the pair

$$(3.4.11)\quad \begin{cases} F(n) = \sum_{k=0}^{\infty} \dfrac{1-q^{p+rn}}{1-q^{p+rn+rk+k}} \dfrac{[q]_{p+rn+rk+k}\ G(n+k)}{[q]_{p+rn+rk}\ (q^{r+1};q^{r+1})_{k}} , \\[2ex] G(n) = \sum_{k=0}^{\infty} (-1)^{k}\ q^{(r+1)k(k-1)/2} \dfrac{[q]_{p+rn}\ F(n+k)}{[q]_{p+rn-k}\ (q^{r+1};q^{r+1})_{k}} \end{cases}$$

in which on replacing k by k-n, one gets

$$
(3.4.12)\begin{cases}
F(n) = \sum\limits_{k=n} \dfrac{1-q^{p+rn}}{1-q^{p+rk+k-n}} \quad \dfrac{[q]_{p+rk+k-n}}{[q]_{p+rk}} \quad \dfrac{G(k)}{(q^{r+1};q^{r+1})_{k-n}} \\[2ex]
G(n) = \sum\limits_{k=n} (-1)^{k+n} \quad q^{(r+1)(n(n+1)+k(k-2n-1))/2} \\
\qquad\qquad \cdot \dfrac{[q]_{p+rn} \; F(k)}{[q]_{p+rn+n-k} \; (q^{r+1};q^{r+1})_{k-n}}
\end{cases}
$$

In this pair, on putting $G(n) = q^{(r+1)n(n+1)/2}\, g(n)$, one arrives at the inverse relations

$$
(3.4.13)\begin{cases}
F(n) = \sum\limits_{k=n} \dfrac{(1-q^{p+rn}) \; q^{(r+1)(k+1)k/2} \; [q]_{p+rk+k-n} \; g(k)}{(1-q^{p+rk+k-n}) \quad [q]_{p+r} \quad (q^{r+1};q^{r+1})_{k-n}} \\[2ex]
g(n) = \sum\limits_{k=n} (-1)^{k+n} \quad q^{(r+1)k(k-2n-1)/2} \\
\qquad\qquad \cdot \dfrac{[q]_{p+rn} \qquad F(k)}{[q]_{p+rn+n-k} \; (q^{r+1};q^{r+1})_{k-n}}
\end{cases}
$$

which on replacing r by m-1, serves as a basic analogue of the Gould class (3) of Table-2.

Likewise, the other basic pairs may be obtained from (3.4.4) to (3.4.8) with the aid of the substitutions as indicated in the following tables.

Table-15 : *Basic analogues of Gould Classes*

$$F(n) = \sum q^{\alpha k(k-1)/2} C_{n,k}\, g(k) \;;\; g(n) = \sum (-1)^{n+k} q^{\alpha k(k-2n-1)/2} D_{n,k}\, F(k)$$

Inversion pair (citation) with y=1, r=m-1	b	p	α	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-2
(3.4.4)	-1	p	-m	$\dfrac{[q]_{p+mk-k}}{[q]_{p+mk-n}\,(q^{-m};q^{-m})_{n-k}}$	$\dfrac{q^{-mk}\,(1-q^{p+mk-k})\,[q]_{p+mn-k-1}}{[q]_{p+mn-n}\,(q^{-m};q^{-m})_{n-k}}$	(1)
(3.4.5)	-1	p	-m	$\dfrac{[q]_{p+mk-k}\,(1-q^{p+mn-n+1})}{[q]_{p+mk-n+1}\,(q^{-m};q^{-m})_{n-k}}$	$\dfrac{q^{-mk}\,[q]_{p+mn-k}}{[q]_{p+mn-n}\,(q^{-m};q^{-m})_{n-k}}$	(2)
(3.4.7)	1	-p	m	$\dfrac{q^{mk}\,[q]_{p+mk-n-1}\,(1-q^{p+mn-n})}{[q]_{p+mk-k}\,(q^{m};q^{m})_{k-n}}$	$\dfrac{[q]_{p+mn-n}}{[q]_{p+mn-k}\,(q^{m};q^{m})_{k-n}}$	(3)
(3.4.8)	1	-p-1	m	$\dfrac{q^{mk}\,[q]_{p+mk-n}}{[q]_{p+mk-k}\,(q^{m};q^{m})_{k-n}}$	$\dfrac{(1-q^{p+mk-k+1})\,[q]_{p+mn-n}}{[q]_{p+mn-k+1}\,(q^{m};q^{m})_{k-n}}$	(4)

Table-16 : *Basic analogues of simpler Chebyshev classes*

$$F(n) = \sum y^k C_{n,k}\, g(n+bk) \; ; \; G(n) = \sum (-y)^k q^{\alpha k(k-1)/2} D_{n,k} F(n+bk)$$

Inversion pair (citation) with r=1	b	p	α	y	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-3
(3.4.7)	-2	0	-1	-1	$\frac{[q]_{n-k-1}\,(1-q^n)}{[q]_{n-2k}\,(q^{-1};q^{-1})_k}$	$\frac{[q]_n}{[q]_{n-k}\,(q^{-1};q^{-1})_k}$	(1)
(3.4.8)	-2	-1	-1	-1	$\frac{[q]_{n-k}}{[q]_{n-2k}\,(q^{-1};q^{-1})_k}$	$\frac{[q]_n\,(1-q^{n-2k+1})}{[q]_{n-k+1}\,(q^{-1};q^{-1})_k}$	(2)
(3.4.4)	2	0	1	1	$\begin{bmatrix} n+2k \\ k \end{bmatrix}$	$\frac{1-q^{n+2k}}{1-q^{n+k}} \begin{bmatrix} n+k \\ k \end{bmatrix}$	(3)
(3.4.5)	2	0	1	1	$\frac{1-q^{n+1}}{1-q^{n+k+1}} \begin{bmatrix} n+2k \\ k \end{bmatrix}$	$\begin{bmatrix} n+k \\ k \end{bmatrix}$	(4)
(3.4.4)	-1	0	-2	1	$\frac{[q]_{n-k}}{[q]_{n-2k}\,(q^{-2};q^{-2})_k}$	$\frac{[q]_{n+k-1}\,(1-q^{n-k})}{[q]_n\,(q^{-2};q^{-2})_k}$	(5)
(3.4.5)	-1	0	-2	1	$\frac{[q]_{n-k}\,(1-q^{n+1})}{[q]_{n-2k+1}\,(q^{-2};q^{-2})_k}$	$\frac{[q]_{n+k}}{[q]_n\,(q^{-2};q^{-2})_k}$	(6)

Table-17 : Basic analogues of Chebyshev classes

$$F(n) = \sum C_{n,k} \frac{[q]_k}{(q^{\alpha};q^{\alpha})_k} G(n+bk); \quad G(n) = \sum (-1)^k q^{\alpha k(k-1)/2} D_{n,k} \frac{[q]_k}{(q^{\alpha};q^{\alpha})_k} F(n+bk)$$

Inversion pair (citation) with r=1	p	α	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-4
(3.4.7)	0	b+1	$\frac{1-q^n}{1-q^{n+bk+k}} \begin{bmatrix} n+bk+k \\ k \end{bmatrix}$	$\begin{bmatrix} n \\ k \end{bmatrix}$	(1)
(3.4.8)	-1	b+1	$\begin{bmatrix} n+bk+k \\ k \end{bmatrix}$	$\frac{1-q^{n+bk+1}}{1-q^{n-k+1}} \begin{bmatrix} n \\ k \end{bmatrix}$	(2)
(3.4.4)	0	b-1	$\begin{bmatrix} n+bk \\ k \end{bmatrix}$	$\frac{1-q^{n+bk}}{1-q^{n+k}} \begin{bmatrix} n+k \\ k \end{bmatrix}$	(3)
(3.4.5)	0	b-1	$\frac{1-q^{n+1}}{1-q^{n+bk-k+1}} \begin{bmatrix} n+bk \\ k \end{bmatrix}$	$\begin{bmatrix} n+k \\ k \end{bmatrix}$	(4)

Table-18 : *Basic analogues of simpler Legendre classes*

$$F(n) = \sum q^{\alpha(k^2+k)/2} C_{n,k}\, G(k) \; ; \; G(n) = \sum (-1)^{k+n} q^{\alpha k(k-2n+1)/2}\, D_{n,k}\, F(k)$$

Inversion pair (citation) with y=1,r=2	b	p	α	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-5
(3.4.8)	-1	-p-1	-1	$\frac{[q]_{p+n+k}}{[q]_{p+2k}\,(q^{-1};q^{-1})_{n-k}}$	$\frac{(1-q^{p+2k+1})\,[q]_{p+2n}}{[q]_{p+n+k+1}\,(q^{-1};q^{-1})_{n-k}}$	(1)
(3.4.7)	-1	-p	-1	$\frac{[q]_{p+n+k-1}\,(1-q^{p+2n})}{[q]_{p+2k}\,(q^{-1};q^{-1})_{n-k}}$	$\frac{[q]_{p+2n}}{[q]_{p+n+k}\,(q^{-1};q^{-1})_{n-k}}$	(2)
(3.4.5)	1	p	1	$\frac{1-q^{p+2n+1}}{1-q^{p+n+k+1}}\begin{bmatrix}p+2k\\k-n\end{bmatrix}$	$\begin{bmatrix}p+n+k\\k-n\end{bmatrix}$	(3)
(3.4.4)	1	p	1	$\begin{bmatrix}p+2k\\k-n\end{bmatrix}$	$\frac{1-q^{p+2k}}{1-q^{p+n+k}}\begin{bmatrix}p+n+k\\k-n\end{bmatrix}$	(4)

Table-18(a) : *Basic analogues of simpler Legendre classes*

$$F(n) = \sum_{k=0}^{[n/2]} (-1)^k C_{n,k} \frac{[q]_k\; G(n-2k)}{(q^{-3};q^{-3})_k} \; ; \; G(n) = \sum_{k=0}^{[n/2]} q^{-3k(k-1)/2}\, D_{n,k} \frac{[q]_k\; F(n-2k)}{(q^{-3};q^{-3})_k}$$

Inversion pair (citation) with y=-1,r=2	b	p	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-5
(3.4.7)	-2	-p	$\frac{1-q^{p+2n}}{1-q^{p+2n-3k}}\begin{bmatrix}p+2n-3k\\k\end{bmatrix}$	$\begin{bmatrix}p+2n\\k\end{bmatrix}$	(5)
(3.4.8)	-2	-p-1	$\begin{bmatrix}p+2n-3k\\k\end{bmatrix}$	$\frac{1-q^{p+2n-4k+1}}{1-q^{p+2n-k+1}}\begin{bmatrix}p+2n\\k\end{bmatrix}$	(6)

Table-19 . *Basic analogues of Legendre-Chebyshev Classes*

$$F(n) = \sum y^{n+k} q^{k(\alpha k+\beta)/2} C_{n,k} G(k) \; ; \; G(n) = \sum (-y)^{n+k} q^{k(\alpha k-2(n-\beta)/2} D_{n,k} F(k)$$

Inversion pair (citation) with r=c	b	p	α	β	y	$C_{n,k}$	$D_{n,k}$	Basic analogue of Class (No.) in Table-6
(3.4.7)	-1	-p	-c+1	c-1	-1	$\frac{[q]_{p+ck-k+n-1}(1-q^{p+cn})}{[q]_{p+ck}(q^{-c+1};q^{-c+1})_{n-k}}$	$\frac{[q]_{p+cn}}{[q]_{p+cn-n+k}(q^{-c+1};q^{-c+1})_{n-k}}$	(1)
(3.4.7)	1	-p	c+1	c+1	-1	$\frac{[q]_{p+ck+k-n-1}(1-q^{p+cn})}{[q]_{p+ck}(q^{c+1};q^{c+1})_{k-n}}$	$\frac{[q]_{p+cn}}{[q]_{p+cn+n-k}(q^{c+1};q^{c+1})_{k-n}}$	(2)
(3.4.4)	-1	p	-c-1	c+1	1	$\frac{[q]_{p+ck}}{[q]_{p+ck+k-n}(q^{-c-1};q^{-c-1})_{n-k}}$	$\frac{(1-q^{p+ck})[q]_{p+cn+n-k-1}}{[q]_{p+cn}(q^{-c-1};q^{-c-1})_{n-k}}$	(3)
(3.4.4)	1	p	c-1	c-1	1	$\frac{[q]_{p+ck}}{[q]_{p+ck-k+n}(q^{c-1};q^{c-1})_{k-n}}$	$\frac{(1-q^{p+ck})[q]_{p+cn-n+k-1}}{[q]_{p+cn}(q^{c-1};q^{c-1})_{k-n}}$	(4)
(3.4.8)	-1	-p-1	-c+1	c-1	-1	$\frac{[q]_{p+ck-k+n}}{[q]_{p+ck}(q^{-c+1};q^{-c+1})_{n-k}}$	$\frac{(1-q^{p+ck+1})[q]_{p+cn}}{[q]_{p+cn-n+k+1}(q^{-c+1};q^{-c+1})_{n-k}}$	(5)
(3.4.8)	1	-p-1	c+1	c+1	-1	$\frac{[q]_{p+ck+k-n}}{[q]_{p+ck}(q^{c+1};q^{c+1})_{k-n}}$	$\frac{(1-q^{p+ck+1})[q]_{p+cn}}{[q]_{p+cn+n-k+1}(q^{c+1};q^{c+1})_{k-n}}$	(6)
(3.4.5)	-1	p	-c-1	c+1	1	$\frac{[q]_{p+ck}(1-q^{p+cn+1})}{[q]_{p+ck+k-n+1}(q^{-c-1};q^{-c-1})_{n-k}}$	$\frac{[q]_{p+cn+n-k}}{[q]_{p+cn}(q^{-c-1};q^{-c-1})_{n-k}}$	(7)
(3.4.5)	1	p	c-1	c-1	1	$\frac{[q]_{p+ck}(1-q^{p+cn+1})}{[q]_{p+ck-k+n+1}(q^{c-1};q^{c-1})_{k-n}}$	$\frac{[q]_{p+cn-n+k}}{[q]_{p+cn}(q^{c-1};q^{c-1})_{k-n}}$	(8)