3

# CHAPTER - 3

# METHOD OF SOLUTION

## 3.1 INTRODUCTION :

Exact solution of the partial differential equations governing fluid flow and heat transfer in free convection from rectangular enclosures having either of the Dirichlet, Neumann or mixed boundary conditions, is not available, even after employing Boussinesq approximation and normalisation. This is because the equations are non-linear and, additionally, are coupled with each other and hence, are to be solved simultaneously. Thus, obtaining an approximate solution using appropriate numerical method, is the only analytical approach left to an investigator.

Before attempting a solution, the partial differential equations are categorised in order to select an appropriate method of solution. For this, a second order partial differential equation for steady, two-dimensional problems, is defined in general form, with $\psi$ as a dependent variable as :

$$A\frac{\partial^2\psi}{\partial x^2}+B\frac{\partial^2\psi}{\partial x\partial y}+C\frac{\partial^2\psi}{\partial y^2}+D\frac{\partial\psi}{\partial x}+E\frac{\partial\psi}{\partial y}+F\psi+G=0 \quad \ldots \quad (3.1)$$

In the above, if the coefficients A to G are functions of only x and y, then the equation is said to be linear while if they also depend additionally on $\psi$ or its derivatives, the equation is termed as non-linear. These coefficients play an important role in categorising the equation and hence in selecting the method of solution, as is shown by Crandall[77]. These coefficients are grouped into a parameter $\lambda$ defined as :

$$\lambda = B^2-4\cdot AC \quad \ldots \quad (3.2)$$

Now, according to whether $\lambda$ is negative, zero or positive, the partial differential equation (equation 3.1) is termed elliptic equation, parabolic equation or hyperbolic equation. While it may be conceived that the values of A, B and C and hence that of $\lambda$ may change from point to point, thus changing the type of equation and hence the method of solution, within the region of interest, it is improbable in practical applications. This observation helps us in selecting a unique method of solution covering all the points in the solution domain.

Elliptic equations normally occur in equilibrium problems where the boundary is closed and the boundary conditions are prescribed around the entire boundary. Such problems are called boundary value problems[78]. Close study of the governing equations in normalised form, obtained earlier (eqns. 2.5, 2.6, 2.7), reveals that they are elliptic equations and represent a boundary value problem, whose boundary conditions are prescribed all around the boundary (eqns.2.8,2.9,2.10,2.11).

The basic principle behind the numerical methods is discretisation, where continuous functions such as temperature, stream function etc. are represented approximately by their values at a pre-decided finite number of points (called nodes) within the solution domain. These values are obtained by solving the sets of simultaneous, algebraic equations, derived from the partial differential equations, appropriate to the problem. The accuracy of the solution increases with the number of points used, particularly if they are concentrated in regions where the functions vary most rapidly.

Of the various numerical methods available, to assemble the relevant sets of algebraic equations from the governing partial differential equations, finite difference methods and finite element methods are most popular. Avduyevský et al[67], while comparing the two methods, confirmed the superiority of finite element method. Fenner[79] in 1975, discussed finite difference method at length and listed its advantages and disadvantages. He also showed how finite element method differs from finite difference method. An up-to-date comparision of about nine contemporary finite difference and finite element methods was given by Shih and Chen[80]. An excellent treatment on both the methods for conduction heat transfer is given by Myers[81], who also, after giving thorough comparision with illustrations, confirms that finite element method is superior.

A somewhat recent treatise by Davies[82], gives a historical background of finite element method with number of bibliographical references. Finite element methods began to be used in structural engineering, as early as in 1941[83], while Turner et al[84] in 1956, refined it to its present form. Its application to non-linear problems was demonstrated by Turner et al[85] in 1960, while its use for solving three-dimensional structural problems required simple extensions and was first described by Argyris[86] in 1964.

The fact that the method is capable of dealing with a variety of problems is proved by many. Zienkiewicz and Cheung[87] applied it to Poisson equation in 1965, while in 1966, Wilson and Nickell[88] applied it to solve transient heat conduction prblems. Doctors[89] in 1970, used it for obtaining potential flow solution. The method has also been applied to viscous fluid flow problems[90], non-linear field problems in electromagnetic theory[91], bio-medical engineering[92] etc.

Use of finite element methods to solve complex fluid flow and convective heat transfer problems, is of recent origin. Abdel-Khalik et al[53] obtained a finite element solution of natural convection from a horizontal enclosure with compound parabolic side walls, in 1978. Marshall et al[93], concurrently employed penalty function finite element method for a square enclosure, while Taylor and Ijam[94] also obtained a numerical solution for enclosed cavities using finite element method. Strada and Heinrich[60] in 1982, used penalty function finte element method for high Ra convection in rectangular enclosures at all possible orientations for three different aspect ratios. Shih[95] in 1982, surveyed exhaustively, published literature on numerical methods in heat transfer from 1977 to 1981. Ozoe et al[96] used finite element method to obtain solution for horizontally confined infinite layer of fluid. Baliga et al[97] in 1983, used control volume finite element method to study fluid flow and heat transfer problems. Upson et al[98] employed a modified finite element method to study three-dimensional convection problems. Donea[99] studied convective transport problems using a Taylor-Galerkin method while Razzaque et al[100] in 1984, studied coupled radiative and conductive heat transfer in an enclosure using finite element method.

## 3.2 FINITE ELEMENT METHOD (FEM) :

In what follows, we shall discuss Rayleigh-Ritz variational formulation of the finite element method for solving two-dimensional steady conduction problems with internal heat generation (Poisson equation) and then extend it for solving two-dimensional, steady, Boussinesq approximated free convection problems.

3.2.1 **FEM applied to conduction problems :** Two-dimensional, steady conduction problems with internal generation are methematically expressed by Poisson equation e.g.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = Q(x, y) \qquad \ldots \qquad (3.3.)$$

Above equation can be normalised i.e. non-dimensionalised to obtain :

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} = Q(\xi, \eta) \quad \ldots \qquad (3.4)$$

In the above, $\xi$ and $\eta$ are dimensionless cartesian co-ordinates, whereas, F and Q are normalised temperature and source term respectively.

Above equation can be re-arranged to obtain,

$$\lambda = \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} - Q(\xi, \eta) = 0 \quad \ldots \qquad (3.5)$$

Now, it may be assumed that the function F to be determined, can be represented as a function of its values at a pre-decided number of nodal points, within the solution domain, including its boundary. Thus,

$$F = F\,(F_1,\ F_2\ \ldots,\ F_i, \ldots\ F_n) \quad \ldots \qquad (3.6)$$

Therefore,

$$\lambda \frac{\partial F}{\partial F_i} = \frac{\partial F}{\partial F_i} \cdot \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial F}{\partial F_i} \cdot \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial F_i} \cdot Q(\xi, \eta) = 0 \ \ldots \qquad (3.7)$$

Integrating above, over the region of interest,

$$\iint_A \left[ \frac{\partial F}{\partial F_i} \cdot \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial F}{\partial F_i} \cdot \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial F_i} \cdot Q(\xi, \eta) \right] dA = 0 \ \ldots \qquad (3.8)$$

Now,

$$\frac{\partial F}{\partial F_i}\cdot\frac{\partial^2 F}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \xi}\right) - \frac{1}{2}\cdot\frac{\partial}{\partial F_i}\left[\left(\frac{\partial F}{\partial \xi}\right)^2\right]$$

and

$$\frac{\partial F}{\partial F_i}\cdot\frac{\partial^2 F}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \eta}\right) - \frac{1}{2}\cdot\frac{\partial}{\partial F_i}\left[\left(\frac{\partial F}{\partial \eta}\right)^2\right]$$

Substitution of the above in eqn.(3.8) results in

$$\iint \frac{\partial}{\partial F_i}\left[\frac{1}{2}\left(\frac{\partial F}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial F}{\partial \eta}\right)^2 + F\cdot Q(\xi,\eta)\right] d\xi\, d\eta - I = 0 \qquad (3.9)$$

In the above,

$$I = \iint \left[\frac{\partial}{\partial \xi}\left(\frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \eta}\right)\right] d\xi\, d\eta \quad \ldots \qquad (3.10)$$

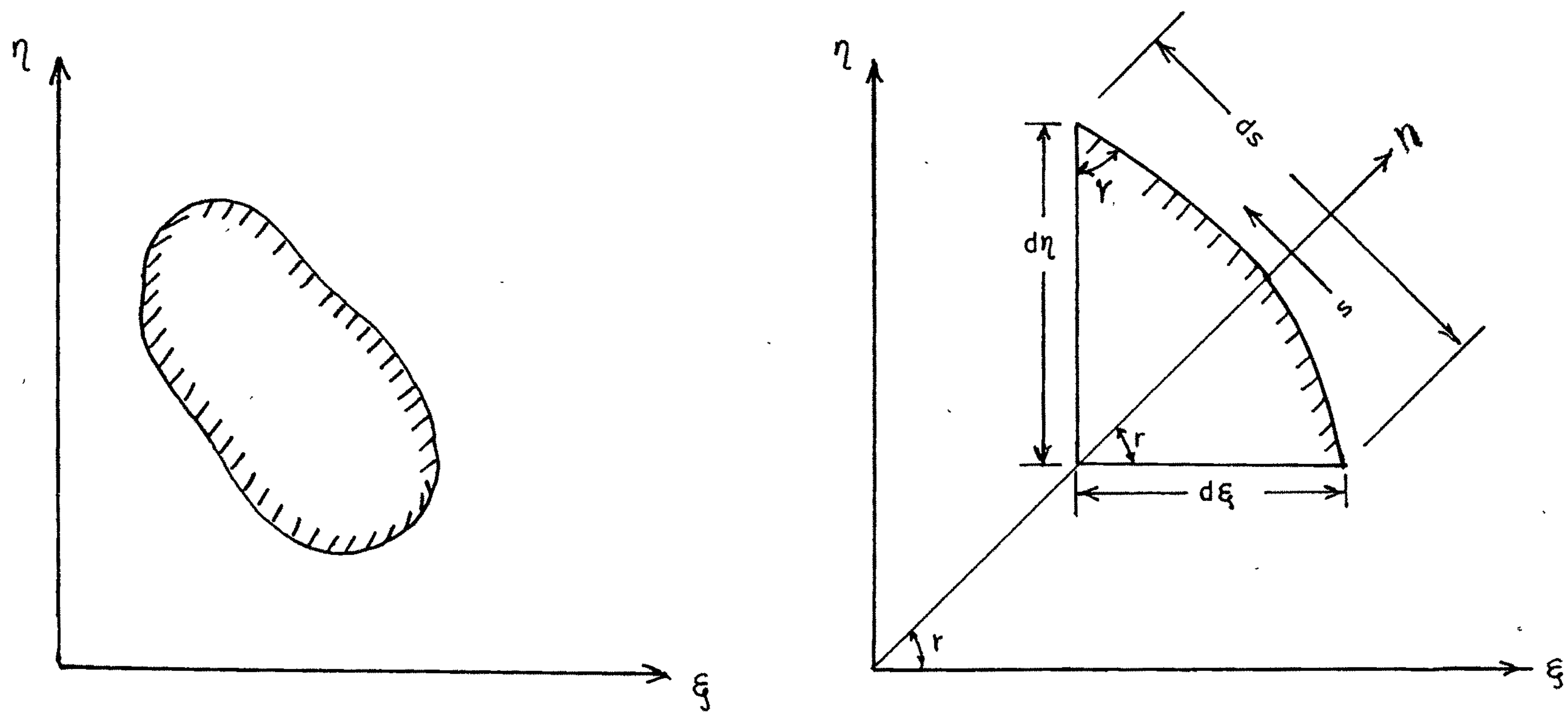
Applying Green's theorem, above equation reduces to :

$$I = \oint \left(\frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \xi}\cdot d\eta - \frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial \eta}\cdot d\xi\right) \quad \ldots \qquad (3.11)$$

In the above, I is the line integral over the boundary of the region in anticlockwise direction.

Now, along the boundary, as is evident from Fig. 3.1,

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial \xi}\cdot\cos r + \frac{\partial F}{\partial \eta}\cdot\sin r$$




(a) REGION OF INTEREST (b) PART OF THE REGION

SOLUTION DOMAIN

FIG: 3·1

But, $\cos\gamma = d\eta/ds$ and $\sin\gamma = -d\xi/ds$ where s is the distance along the boundary measured in anti-clock wise direction. Thus,

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial \xi}\cdot\frac{d\eta}{ds} - \frac{\partial F}{\partial \eta}\cdot\frac{d\xi}{ds}$$

$$\therefore \frac{\partial F}{\partial \xi}\cdot d\eta - \frac{\partial F}{\partial \eta}\cdot d\xi = \frac{\partial F}{\partial n}\cdot ds$$

In light of above, eqn. (3.11) simplifies to :

$$I = \oint \frac{\partial F}{\partial F_i}\cdot\frac{\partial F}{\partial n}\cdot ds \qquad \ldots \qquad (3.12)$$

In the above, I = 0 when F is prescribed on the boundary i.e. F = constant with respect to $F_i$ (Dirichlet condition) or when its first derivative normal to the boundary vanishes i.e. $\partial F/\partial n = 0$ (Neumann condition).

Above important observation regarding the boundary integral I, reduces eqn.(3.9), for Dirichlet or Neumann boundary value problems, into :

$$\frac{\partial}{\partial F_i}\iint\left[\frac{1}{2}\left(\frac{\partial F}{\partial \xi}\right)^2+\frac{1}{2}\left(\frac{\partial F}{\partial \eta}\right)^2+F\cdot Q(\xi,\eta)\right]d\xi d\eta = 0$$

That is,

$$\frac{\partial \chi}{\partial F_i} = 0 \qquad \ldots \qquad (3.13)$$

where,

$$\chi = \iint \left[ \frac{1}{2} \left( \frac{\partial F}{\partial \xi} \right)^2 + \frac{1}{2} \left( \frac{\partial F}{\partial \eta} \right)^2 + F \cdot Q(\xi, \eta) \right] d\xi \, d\eta \quad \ldots \qquad (3.14)$$

It may be noted that $\chi$ is the functional to be obtained by comparing the above relation with eqn.(3.4), stated earlier. THis functional is to be extremised using eqn. (3.13) for each subregion or the element of the solution domain.

Extremisation of the above functional for all the elements e.g. eqn.(3.13) when solved for i = 1,2,... m where m are the number of elements considered in the solution domain, leads to :

$$[K] \cdot [F] = [Q] \quad \ldots \qquad (3.15)$$

Where,

$$[K] = \sum_{e=1}^{E} [D^e] \cdot [K^e] \cdot [D^e]^T \ldots \qquad (3.16)$$

In the above,

$[K]$ is the m x m global conduction matrix,
$[D^e]$ is the m x 3 element displacement matrix,
$[K^e]$ is the 3 x 3 element conduction matrix
$[Q]$ is the m x 1 generation vector and
$[F]$ is the m x 1 function vector, to be determined.

Myers[81] in his treatise, explained at length, the extremisation procedure leading to eqn. (3.15) and eqn.(3.16) and defined various matrices as under :

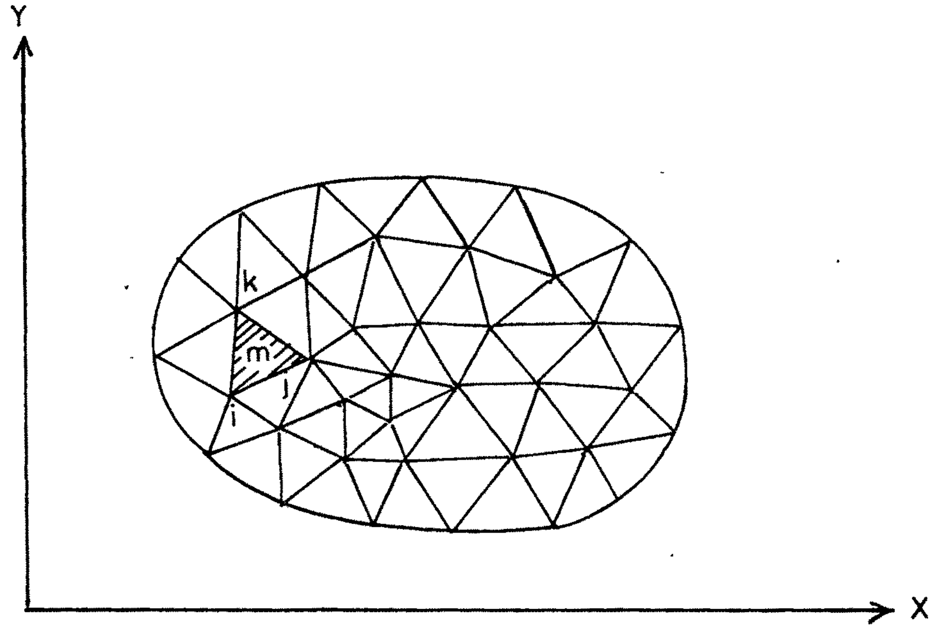
With reference to Fig.3.2, which shows a triangular finite element for two-dimensional problems,

$[D^e]$ is the m x 3 element displacement matrix and is given by :

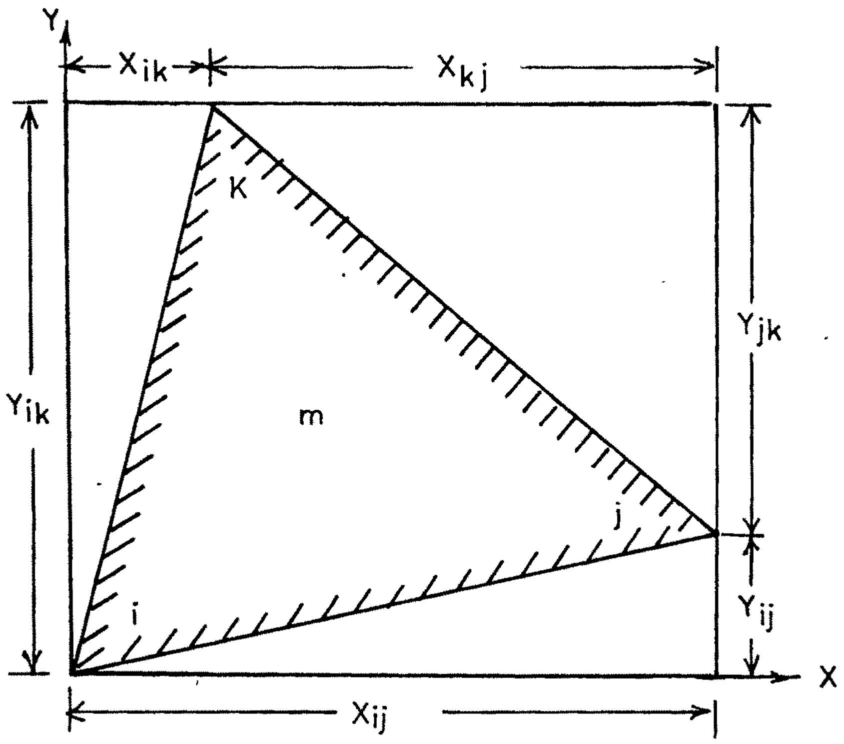
$$[D^e] = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \leftarrow \text{ith row} \\ \leftarrow \text{jth row} \\ \leftarrow \text{kth row} \\ \\ \end{array}$$

$[D^e]^T$ is the transpose of the element displacement matrix $[D^e]$ and is given by :

$$[D^e]^T = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & \uparrow & \uparrow & \uparrow & \\ & \text{ith column} & \text{jth column} & \text{kth column} & \end{array} \end{array}$$




(a) TRIANGULAR ELEMENTS IN THE DOMAIN




(b) A TYPICAL TRIANGULAR ELEMENT

FIG: 3·2

$[K^e]$ is the 3 x 3 element conduction matrix and is given by :

$$[K^e] = \frac{K^e A^e}{(x_{ij} y_{jk} - x_{jk} y_{ij})^2} \cdot \begin{bmatrix} x_{jk}^2 + y_{jk}^2 & -(x_{ik}x_{jk} + y_{ik}y_{jk}) & x_{ij}x_{jk} + y_{ij}y_{jk} \\ & x_{ik}^2 + y_{ik}^2 & -(x_{ij}x_{ik} + y_{ij}y_{ik}) \\ \text{Symmetric} & & x_{ij}^2 + y_{ij}^2 \end{bmatrix}$$

where,

$$A^e = \frac{1}{2} / x_{ij} y_{jk} - x_{jk} y_{ij} /$$

$[K]$ is the m x m global conduction matrix and is given by :

$$[K] = \sum_{e=1}^{E} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^e \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix}^e \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^e$$

It is apparent from the above that global conduction matrix is assembled element wise till all the elements are considered for their matrix operation $[D^e] \cdot [K^e] \cdot [D^e]^T$. The assembled matrix $[K]$ for any element e is given by :

$$[K] = \begin{array}{c} \\ \left[\begin{array}{ccc} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{Ki} & K_{Kj} & K_{kK} \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{ith column} & \text{jth column} & \text{Kth column} \end{array} \end{array} \begin{array}{l} \\ \leftarrow \text{ ith row} \\ \leftarrow \text{ jth row} \\ \leftarrow \text{ Kth row} \\ \\ \end{array}$$

As the operation is continued from e=1 to e=E, the above matrix gets gradually filled up at respective locations and eventually gets completely assembled.

$[Q]$ in eqn. (3.15) is the m x 1 generation vector and is given by :

$$[Q] = \sum_{e=1}^{E} [D^e] \cdot [Q^e] = \sum_{e=1}^{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^e \begin{bmatrix} Q_i \\ Q_j \\ Q_K \end{bmatrix}^e$$

It is clear from the above, that vector $[Q]$ is gradually filled up in much the same way as matrix $[K]$, that is, the contributions from each element is computed and added to the sum of the contributions from the previous elements.

Lastly $[F]$ is the $m \times 1$, function vector to be determined by solving eqn. (3.15) subject to appropriate boundary conditions, using a suitable numerical method. This is given by :

$$[F] = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_i \\ \vdots \\ F_m \end{bmatrix}$$

Before attempting to solve the system of simultaneous, linear, algebraic equations to be obtained from eqn. (3.15), the relevant boundary conditions are to be embedded in them without loosing their symmetry.

If Dirichlet condition is to be applied where the function F is prescribed on the boundary e.g. if $F_p = \alpha =$ constant at a nodal point P on the boundary, we can modify the global conduction matrix $[K]$ and generation vector $[Q]$ in eqn. (3.15) such that,

$$K^*_{pp} = M K_{pp} \quad \text{and} \quad Q^*_p = \alpha K^*_{pp} \quad \ldots \qquad (3.17)$$

In the above, asterisked coefficients are modified coefficients replacing the older ones and M is a suitably large number, say of the order of $10^{10}$. The effect of these modifications is to make the non-diagonal coefficients in the pth row of the conduction matrix negligible compared with the diagonal one, reducing the pth equation of matrix eqn.(3.15) into:

$$K^{*}_{pp} \cdot F_p = \propto K^{*}_{pp} \qquad \text{i.e.} \quad F_p = \propto \qquad \ldots \qquad (3.18)$$

On the other hand, if Neumann condition is to be applied where the first derivative of the function vanishes at the boundary nodes e.g. if $\partial F/\partial \eta = 0$ at a nodal point p on the boundary, we can simply treat this boundary node as an internal node, while applying a numerical method. This will result in automatic satisfaction of this type of boundary condition as was observed by Fenner[101].

After incorporating relevant boundary conditions in the system of simultaneous equations represented by eqn. (3.15), they are solved by a suitable numerical method. Two methods are most popular. One is the Gaussian ellimination method, described fully by Fenner[102] with a computer subroutine while the other is the Gauss-Seidal Method of matrix iterations which is more popular with computers because of its limited memory storage requirements and it is this method which is used in the present investigation. The method is described in detail in Appendix: A-4.

**3.2.2. Extension of FEM to Convection Problems :** Basic partial differential equations governing the conservation of mass, momentum and energy of fluid in free or forced convection can be converted into the form of eqn. (3.4), after appropriately normalising them. For example, in the present investigation, eqns. (2.1, 2.2, 2.3 and 2.4) represent the conservation equations for two-dimensional steady free convection problems, which, after normalisation, are reduced into eqns. (2.5, 2.6 and 2.7). These equations are of the form of eqn. (3.4), whose left hand sides represent Laplacian operations on the dimensionless functions like $\psi$, $\theta$ and $\omega$ , where central difference scheme may be used and whose righthand sides express generation terms which are non-linear functions of $\psi$, ~~and $\psi$~~ $\theta$ and $\omega$ , in addition to parameters like Ra and Pr. As these non-linear generation terms are of convective nature, forward difference scheme should be used here for stable solutions.

Just as eqn.(3.4) was converted into eqn.(3.15) employing Rayleigh -Ritz variational formulation of the finite element method, eqns. (2.5, 2.6 and 2.7) can be reduced into three sets of simultaneous, algebraic equations, respectively as :

$$[K] \cdot [\psi] = [Q_\psi] \quad \ldots \qquad (3.19)$$

$$[K] \cdot [\theta] = [Q_\theta] \quad \ldots \qquad (3.20)$$

$$[K] \cdot [\omega] = [Q_\omega] \quad \ldots \qquad (3.21)$$

In the above,

$[K]$ is the m x m global conduction matrix,

$[\psi]$ is the m x 1 stream function vector to be determined.

$[\theta]$ is the m x 1 temperature function vector to be determined,

$[\omega]$ is the m x 1 vorticity function vector to be determined,

$[Q_\psi]$ is the m x 1 stream function generation vector,

$[Q_\theta]$ is the m x 1 temperature function generation vector,

$[Q_\omega]$ is the m x 1 vorticity function generation vector.

It may be noted that the global conduction matrix $[K]$ is common for all the equations and depends only upon the geometry of the element and its relative location in the region of interest, while the generation vectors are different for different equations and are to be obtained by comparing eqns. (3.19, 3.20 and 3.21), with eqns. (2.5, 2.6 and 2.7) respectively. A close look at the generation vectors, so obtained, will reveal that the eqns. (3.19, 3.20 and 3.21) are inter-connected and hence, are required to be solved simultaneously, using some form of a computational algorithm.

## 3.3. COMPUTATIONAL ALGORITHM :

Following computational procedure has been used in the present investigation, to obtain a solution of the three sets of simultaneous, algebraic eqns. (3.19, 3.20 and 3.21) :

1. Generate and modify finite element mesh co-ordinates.

2 Obtain overall or global conduction matrix $[K]$.

3 Initalise values of $\theta$ at all the nodes as zero, for conduction.

4 Calculate values of $\theta$ at all the nodes for conduction using $[K][\theta] = 0$.

5 Initialise values of $\theta$ with conduction temperatures while values of $\Psi$ and $\omega$ as zero at all the nodes, for convection.

6 Evaluate generation vector $[Q_\Psi]$ using current values of $\omega$, modify $[K][\Psi] = [Q_\Psi]$ for the boundary condition for $\Psi$ and solve it for $\Psi$.

7 Evaluate generation vector $[Q_\theta]$ using current values of $\Psi$ and $\theta$, modify $[K][\theta] = [Q_\theta]$ for the boundary condition for $\theta$ and solve it for $\theta$.

8 Evaluate values of $\omega$ at the boundary nodes, evaluate generation vector $[Q_\omega]$ using current values of $\Psi$, $\theta$ and $\omega$, modify $[K][\omega] = [Q_\omega]$ for the boundary condition for $\omega$ and solve it for $\omega$.

9 Repeat steps **6** through **8** until all the nodal values of $\Psi$, $\theta$ and $\omega$ converge to within predecided tolerance levels.

A flow chart following the above algorithm is presented in Appendix : A-5, based on which a computer programme in **FORTRAN** was developed, for the present investigation.