CHAPTER IV

AVERAGE SAMPLE NUMBER FOR CURTAILED TWO CLASS ATTRIBUTES SAMPLING PLAN

4.1 In this chapter we discuss an important aspect of curtailed sampling plan, namely, the Average Sample Number (ASN). The expressions of the ASN for curtailed single and double two class attributes sampling plans are obtained under two probability laws, hypergeometric and binomial. The percent saving in inspection due to curtailment is explained by numerical examples.

4.2

4.2.1 Curtailment in the Inspection :

Girshick, Mosteller, and Savage [12] introduced the curtailed sampling plan while studying the problem of the unique unbiased estimation of the parameter p (fraction defective) for samples drawn from a binomial distribution. Two forms of curtailed single sampling plan can be distinguished. The sampling plan which considers curtailment in the inspection arising due to observing enough defectives to reject a lot is termed, a semi-curtailed sampling plan, following the terminology of the Statistical Research Group [44] . Similarly the sampling plan which considers curtailment in the inspection arising due to observing either enough defectives to reject a lot or enough nondefectives to accept a lot is called fully-curtailed sampling plan. If a double sampling plan is administered, one can have different variations in curtailment other than these forms. A list of possible variations in curtailment, when double sampling plan is executed, is given below :

(a) no curtailment in the inspection of the first sample and semi-curtailment in the inspection of the second sample,

(b) no curtailment in the inspection of the first sample and full-curtailment in the inspection of the second sample,

(c) semi-curtailment in the inspection of the first sample and semi-curtailment in the inspection of the second sample,

(d) semi-curtailment in the inspection of the first sample and full-curtailment in the inspection of the second sample,

(e) full-curtailment in the inspection of the first sample and full-curtailment in the inspection of the second sample.

Craig [8] has discussed (a) and (c). According to him the curtailment described in (a) is called partial truncation and the curtailment described in (c) is called complete truncation. In the following sections of this chapter we have considered only (c) and (e). If we follow the terminology of Statistical Research Group [44], only the variations (c) and (e) are meaningful and with reference to that terminology variation (c) is termed here as semi-curtailed double sampling plan and variation (e) is termed here as fully-curtailed double sampling plan.

4.2.2 Average Sample Number :

An important characteristic of curtailed sampling plan is a reduction in the ASN. This problem was dealt with, as early as 1948, in Chapter 17 of Sampling Inspection by the Statistical Research Group, Columbia University [44]. Later Burr (1957) [5], Patil (1953) [38], Phatak and Bhatt (1967) [40], and Craig (1968) [8], worked out the ASN for some curtailed sampling plans. For instance, Patil obtained the ASN of a curtailed single sampling plan in terms of the inverse binomial sampling plan. Recently Cohen [7], Guenther [16], and many others also have discussed the problem of the ASN in curtailed sampling plan. In the following sections of this chapter we have obtained the expressions of the ASN for fully-curtailed single and double sampling plans under two probability laws, hypergeometric probability law and binomial probability law. However, for comparative study of the percent saving in inspection we have also considered the expressions of the ASN under semi-curtailed sampling plan. The percent saving in inspection as one passes from an un-curtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan is illustrated with numerical examples in certain cases.

4.3 The ASN of the Curtailed Sampling Plan under the Hypergeometric Probability Law :

In the usual acceptance sampling plan by attributes the practice is to take units from a lot without replacement and to use the binomial probability law, under the assumption that the lot size is sufficiently large, to derive the various expressions such as the probability of acceptance (O.C. function), the ASN etc. However, when the lot size is not sufficiently large the use of the hypergeometric probability law is more realistic. Using the hypergeometric probability law, the expressions of the ASN for a fully-curtailed single two class attributes sampling plan and a semi-curtailed double two class attributes sampling plan are given by Guenther in two separate papers [14], [16]. In case of a double two class attributes sampling plan, Guenther [16] has considered a common rejection number for both the samples, i.e. $r_1=r_2=r$. In the following sections we have obtained the expressions of the ASN for a semi-curtailed single two class attributes sampling plan and a fully-curtailed double two class attributes sampling plan. In case of a fully-curtailed double sampling plan we have given the expressions of the ASN when (i) both samples have a common rejection number and (ii) they have different rejection numbers. We have also given the expression of the ASN for a semi-curtailed double sampling plan when the two samples have different rejection numbers. Thus the expressions obtained by us will fill in the gaps and our results along with Guenther's results will help us in obtaining the saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan. This also helps in finding the saving in inspection as one passes from a semi-curtailed sampling plan to a fully-curtailed sampling plan. Using these results the percent saving in inspection is illustrated with numerical examples.

4.3.1 Some Results of the Hypergeometric Distribution :

We state some results of the hypergeometric distribution and those of inverse hypergeometric distribution. Let a lot contains N units of which M units are defectives. Let n be the size of a random sample drawn from this lot without replacement. Then the probability of obtaining x defectives in the sample is given by

$$p(N,n,M,x) = \binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n}, \quad a \leq x \leq b \qquad \ldots(4.3.1.)$$

where $a = \max.\ [0, n-(N-M)]$ and $b = \min.\ [n,M]$.

Furthermore, it may be noted that

$$p\ (N,n,M,x) = p\ (N,M,n,x) \qquad \ldots(4.3.2)$$

$$P\ (N,n,M,r) = P\ (N,M,n,r) \qquad \ldots(4.3.3)$$

and $$P\ (N,n,M,r) = 1-P(N,n,N-M,n-r-1) \qquad \ldots(4.3.4)$$

where $P(N,n,M,r) = \sum_{x=0}^{r} p(N,n,M,x)$ represents the probability of obtaining at the most r defectives in the said sample. The expression (4.3.1) is known as the hyper-geometric probability function. The notations and the results given above can be found in Lieberman and Owen [30] .

If the units of a lot are selected and inspected in succession without replacement, then the probability that y inspections are required to obtain the kth defective is the product of the probability of having (k-1) defectives in (y-1) inspections and the probability of having a defective at the yth inspection. This probability denoted by $p^*(N,M,k,y)$ is given by

$$p^*(N,M,k,y) = \binom{y-1}{k-1}\binom{N-y}{M-k}\Big/\binom{N}{M} \qquad y=k,k+1,\ldots,N-M+k \qquad \ldots(4.3.5)$$

The above probability function is known as the inverse (or negative) hypergeometric probability function. The probability that r or less inspections are required to have the kth defective is

$$P^*(N,M,k,r) = \sum_{y=k}^{r} p^*(N,M,k,y) \qquad \ldots(4.3.6)$$

The probability that the kth defective is obtained at the kth inspection, the (k+1)th inspection,..., or rth inspection is equal to the probability that one obtains k,k+1,..., or r defectives in r inspections. Hence

$$P^*(N,M,k,r) = 1 - P(N,r,M,k-1) \qquad \ldots(4.3.7)$$

Furthermore, it is found that

$$\frac{k(N+1)}{(M+1)}\, p^*(N+1,M+1,k+1,y+1) = y\, p^*(N,M,k,y) \qquad \ldots(4.3.8)$$

The results (4.3.7) and (4.3.8) are due to Guenther [16]. However, the result (4.3.7) which is the relation between the hypergeometric distribution and the inverse hypergeometric distribution can be found established independently in Lieberman and Owen [30,pp.7] .

4.3.2 Curtailed Single Sampling Plan :

We recall here the statements of a semi-curtailed and a fully-curtailed forms of a single sampling plan from Phatak and Bhatt [40] for continuity and completeness. These forms are designated as Plan 2 and Plan 3 respectively in [40]. Plan 1 is the usual uncurtailed single sampling plan.

Semi-curtailed Sampling Plan : Inspect randomly selected units of a lot one at a time until either k defectives have been observed or until n units have been inspected. Reject the lot if k defectives are observed. Accept the lot if n units are inspected, provided that the number of defectives observed is less than k.

Fully-curtailed Sampling Plan : Inspect randomly selected units of a lot one at a time until either k defectives have been observed or g (n-k+1) nondefectives have been observed. Accept the lot if there are g nondefectives. Reject the lot if there are k defectives.

In these plans g, k (and hence n) are predetermined numbers. k is known as rejection number and it is related with the acceptance number, a, through the relationship $k = a + 1$.

4.3.3 Probability Functions :

Let the joint probability function associated with a semi-curtailed sampling plan be denoted by $h_2(x,t;N,M)$ and that of a fully-curtailed sampling plan be denoted by $h_3(x,t;N,M)$. These probability functions are given below :

4.3.3(i) Semi-Curtailed Sampling Plan :

$$p\{X=x,\ T=t\} = h_2(x,t;N,M)$$

$$= \begin{cases} p\,(N,n,M,x) & x = 0,1,\ldots,k-1;\ t=1, \\ p^*(N,M,k,x) & x=k,k+1,\ldots,n;\ t=2, \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(4.3.9)$$

4.3.3(ii) Fully-Curtailed Sampling Plan :

$$P\{X=x,\ T=t\} = h_3(x,t;N,M)$$

$$= \begin{cases} p^*(N,N-M,g,x) & x=g,g+1,\ldots,n;t=1, \\ p^*(N,M,k,x) & x=k,k+1,\ldots n;t=2 \\ 0 & \text{elsewhere.} \end{cases} \qquad \ldots(4.3.10)$$

where

(i) the part of the probability function associated with t=1 is implied by acceptance of a lot and the part of the probability function associated with t=2 is implied by the rejection of a lot,

(ii) in the case of a semi-curtailed sampling plan the physical meaning of the random variable X is the number of defectives observed when t=1 and the number of units inspected when t=2,

(iii) in the case of a fully-curtailed sampling plan the physical meaning of the random variables X is the number of units inspected for both the cases t=1 and t=2,

and (iv) $\binom{a}{b}$ is taken as zero whenever $a < b$.

4.3.4 Average Sample Number (ASN) :

We denote the ASN of a semi-curtailed sampling plan by ASN_2 and the ASN of a fully-curtailed sampling plan by ASN_3. Then the expressions for the ASN are given below :

4.3.4(i) Semi-curtailed Sampling Plan :

$$ASN_2 = n \sum_{x=o}^{k-1} h_2(x,1;N,M) + \sum_{x=k}^{n} x\, h_2(x,2;N,M)$$

$$= n \sum_{x=o}^{k-1} p\ (N,n,M,x) + \sum_{x=k}^{n} x\ p^*(N,M,k,x)$$

$$= nP\ (N,n,M,k-1) + \frac{k(N+1)}{M+1} \left[1-P(N+1,n+1,M+1,k)\right] \qquad \ldots(4.3.11)$$

4.3.4(ii) Fully-Curtailed Sampling Plan :

$$ASN_3 = \sum_{x=g}^{n} x\ h_3\ (x,1;N,M) + \sum_{x=k}^{n} x\ h_3(x,2;N,M)$$

$$= \sum_{x=g}^{n} x\ p^*(N,N-M,g,x) + \sum_{x=k}^{n} x\ p^*(N,M,k,x)$$

$$= \frac{g(N+1)}{(N-M+1)}\ P\ (N+1,n+1,M,n-g)$$

$$+ \frac{k(N+1)}{(M+1)} \left[1- P(N+1,n+1,M+1,k)\right] \qquad \ldots(4.3.12)$$

where it may be noted that (4.3.8) and (4.3.7) are used to obtain (4.3.11) whereas (4.3.8), (4.3.7), and (4.3.4) are used to obtain (4.3.12). The term of the form P(a,b,c,d) involved in (4.3.11) and (4.3.12) is readily available in the table given by Lieberman and Owen [30] .

4.3.5 Curtailed Double Sampling Plan :

A double sampling plan (DSP) and an usual double sampling plan (UDSP) are described in Section 2.3.1 of Chapter II. In the following sections we have considered the ASN of a semi-curtailed and a fully-curtailed DSP and a fully-curtailed UDSP under the hypergeometric probability law. The ASN of a semi-curtailed UDSP given by Guenther [16] is expressed here for continuity.

4.3.6 Probability Functions :

4.3.6(i) (i) Fully-Curtailed DSP :

Recall the definitions of the random variable Y and T, and sets A_i $(i = 1,2,3,4)$ of possible values attained by Y given in Section 2.4.1 of Chapter II. Then the joint probability function of the random variables Y and T under hypergeometric probability law can be expressed as

$$P\,(Y=y,\ T=i) = h\,(y,i;N,M) \qquad y \in A_i,\ i=1,2,3,4$$

$$= 0 \qquad \text{elsewhere} \qquad \ldots(4.3.13)$$

where $h\,(y,1;N,M) = p^*(N,M,r_1,y)$ ...(4.3.14)

$$h\,(y,2;N,M) = \sum_{j=1}^{b_1} p(N,n_1,M,n_1-g_1+j)$$

$$p^*(N-n_1,M-(n_1-g_1)-j,\ b_2+1-j,y-n_1) \qquad \ldots(4.3.15)$$

$$h(y,3;N,M) = p^*(N,N-M,g_1,y) \qquad \ldots(4.3.16)$$

$$h(y,4;N,M) = \sum_{j=1}^{b_1} p\,(N,n_1,M,n_1-g_1+j)$$

$$p^*(N-n_1,N-M-g_1+j,g_2-g_1+j,y-n_1) \qquad \ldots(4.3.17)$$

and $b_1 = g_1+r_1-n_1-1$, $b_2 = g_1+r_2-n_1-1$

4.3.6(ii) Fully-curtailed UDSP:

The probability function of a fully-curtailed UDSP can easily be obtained from the probability function of a fully-curtailed DSP given above by substituting $r_1=r_2=r$ and $b_1=b_2=b$. In this case the expression (4.3.16) remains same. Expressions (4.3.14), (4.3.15) and (4.3.17) are modified by taking $r_1=r$ and $b_1=b_2=b$.

4.3.6(iii) Semi-curtailed DSP :

The probability function of a semi-curtailed DSP can easily be obtained with an appropriate modification from the expressions of the probability of acceptance and the probability of rejection given by Guenther [16] .

4.3.7 The Expressions of ASN :

4.3.7(i) Fully-Curtailed DSP :

Use the symbol $ASN(D)_3$ to represent the ASN of a fully-curtailed DSP. Then the expression for the ASN under fully-curtailed DSP is

$$
\begin{aligned}
ASN(D)_3 &= E\,(Y) \\
&= \sum_y y \sum_{i=1}^{4} h(y,i;N,M) \\
&= \sum_{y\in A_1} y\, h(y,1;N,M) + \sum_{y\in A_2} y h(y,2;N,M) \\
&+ \sum_{y\in A_3} y h(y,3;N,M) + \sum_{y\in A_4} y h(y,4;N,M) \\
&= \frac{r_1(N+1)}{(M+1)} \left[1-P(N+1,n_1+1,M+1,r_1)\right] \\
&+ \frac{g_1(N+1)}{(N-M+1)} P(N+1,n_1+1,M,n_1-g_1) \\
&+ \sum_{j=1}^{b_1} p(N,n_1,M,n_1-g_1+j) \Big[\frac{(b_2+1-j)(N-n_1+1)}{(M+1-(n_1-g_1)-j)} \\
&\cdot\{1-P(N-n_1+1,n_2+1,M-(n_1-g_1)-j+1,b_2+1-j)\} + n_1 \\
&+ \frac{(g_2-g_1+j)(N-n_1+1)}{(N-M-g_1+j+1)} P\,(N-n_1+1,n_2+1,M-(n_1-g_1) \\
&-j,\ b_2-j)\Big] \qquad \ldots(4.3.18)
\end{aligned}
$$

4.3.7(ii) Fully-curtailed UDSP :

The expression of the ASN under fully-curtailed UDSP can easily be obtained from the expression (4.3.18) by substituting $r_1=r_2=r$ and hence $b_1=b_2=b$. Use the symbol $ASN\,(UD)_3$ for the ASN under fully-curtailed UDSP. Then the expression for the ASN is given below :

$$ASN(UD)_3 = \frac{r(N+1)}{(M+1)}[1-P(N+1,n_1+1,M+1,r)] + \frac{g_1(N+1)}{N-M+1} P(N+1,n_1+1,M,n_1-g_1)$$
$$+ \sum_{j=1}^{b} p(N,n_1,M,n_1-g_1+j)\ \Big[\frac{(b+1-j)(N-n_1+1)}{(M+1-(n_1-g_1)-j)}$$
$$\cdot\{1 - P(N-n_1+1,n_2+1,M-(n_1-g_1)-j+1,b+1-j)\} +n_1$$
$$+ \frac{(g_2-g_1+j)(N-n_1+1)}{(N-M-g_1+j+1)} P(N-n_1+1,n_2+1,M-(n_1-g_1)-j,$$
$$b-j)\Big] \qquad \ldots(4.3.19)$$

4.3.7 (iii) <u>Semi-curtailed DSP</u> :

Use the symbol ASN $(D)_2$ to denote the ASN for a semi--curtailed DSP. Then the expression is

$$ASN(D)_2 = \frac{r_1(N+1)}{(M+1)}[1-P(N+1,n_1+1,M+1,r_1)] +n_1P(N,n_1,M,r_1-1)$$
$$+ \sum_{j=1}^{b_1} p(N,n_1,M,n_1-g_1+j)\Big[\frac{(b_2+1-j)(N-n_1+1)}{(M+1-(n_1-g_1)-j)}$$
$$\cdot\{1-P(N-n_1+1,n_2+1,M-(n_1-g_1)-j+1,b_2+1-j)\}$$
$$+n_2\ P(N-n_1,n_2,M-(n_1-g_1)-j,b_2-j)\Big] \qquad \ldots(4.3.20)$$

4.3.7 (iv) <u>Semi-curtailed UDSP</u> :

Let ASN $(UD)_2$ be the symbol for the expression of the ASN under semi-curtailed UDSP. Then the expression is as given below :

$$ASN(UD)_2 = \frac{r(N+1)}{(M+1)} [1-P(N+1, n_1+1, M+1, r)]$$

$$+ n_1 P(N, n_1, M, r-1)$$

$$+ \sum_{j=1}^{b} p(N, n_1, M, n_1-g_1+j) \left[\frac{(b+1-j)(N-n_1+1)}{(M+1-(n_1-g_1)-j)}\right.$$

$$\cdot \{1-P(N-n_1+1, n_2+1, M-(n_1-g_1)-j+1, b+1-j)\}$$

$$\left. + n_2 P(N-n_1, n_2, M-(n_1-g_1)-j, b-j)\right] \quad ...(4.3.21)$$

This expression is same as the expression (2.1) of [16].

4.3.8 Numerical Examples :

Guenther [16] states that the termination of the inspection on finding enough nondefectives complicates the formulas (of the ASN) with no appreciable reduction in the ASN. Suprisingly he makes this statement without calculation or derivation of the ASN for a fully-curtailed sampling plan. Firstly we observe that in case of single sampling plan as such there is no complication in using (4.3.12) giving ASN_3 instead of using (4.3.11) giving ASN_2. As such, the second term of (4.3.11) and that of (4.3.12) are exactly identical and the first term of (4.3.12) and that of (4.3.11), though differ from each other, have identical nature so far the process of referring to the hypergeometric tables given by Lieberman and Owen [30] is concerned. In case of the UDSP, the expression (4.3.19) giving $ASN(UD)_3$ and (4.3.21) giving $ASN(UD)_2$ are to be compared from the complication point of view. First term and first part of the third term of (4.3.19) are exactly identical with the first term and first part of the third term of (4.3.21). Second term of (4.3.19) and that of (4.3.21) have identical nature so far the process of referring to the hypergeometric Tables [30] is concerned. Remaining terms of (4.3.19) and (4.3.21) appear complicated but evaluation go quickly with Lieberman and Owen Tables [30].

Regarding the question of reduction in the inspection it may be stated that the reduction depends upon the constants of the sampling plan and the number of defectives in the lot. The percent saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan may be defined in the case of single sampling plan as

$$S_{12} = 100\ (n-ASN_2)/n \qquad \ldots(4.3.22)$$

$$S_{13} = 100\ (n-ASN_3)/n \qquad \ldots(4.3.23)$$

where n stands for the ASN of the usual uncurtailed single sampling plan. The percent saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan may be defined in the case of an UDSP as

$$S_{12.d} = 100(ASN(UD)_1 - ASN(UD)_2)/ASN(UD)_1 \qquad ...(4.3.24)$$

$$S_{13.d} = 100\ (ASN(UD)_1 - ASN(UD)_3)/ASN(UD)_1 \qquad ...(4.3.25)$$

where $ASN(UD)_1 = n_1 + n_2 \quad P(N,n_1,M,r-1) - P(N,n_1,M,n_1-g_1)$

$$...(4.3.26)$$

The differences between (4.3.22) and (4.3.23) and (4.3.24) and (4.3.25) reflect the advantage in going from a semi--curtailed sampling plan to a fully-curtailed sampling plan.

For the following usual single sampling plan

$N = 25$, $n=10$, $a=k-1 = 2$

ASN_2, ASN_3 respectively given by (4.3.11) and (4.3.12) and the percent saving in inspection given by (4.3.22) and (4.3.23) are calculated and are presented in Table 4.1.

Similarly for the following UDSP

$N = 25$, $n_1 = 5$, $n_2 = 10$, $a_1 = 1$, $a_2 = 3$.

$ASN(UD)_1$, $ASN(UD)_2$, $ASN(UD)_3$ respectively given by (4.3.26), (4.3.21), and (4.3.19) and the percent saving in inspection given by (4.3.24) and (4.3.25) are calculated and are presented in Table 4.2. It is revealed from Table 4.1 and Table 4.2 that there is appreciable saving in inspection in going from uncurtailed to a semi-curtailed or to a fully-curtailed sampling plan. The mentionable saving in inspection exists in going from a semi-curtailed sampling plan to a fully--curtailed sampling plan for certain values of M. We have made use of Lieberman and Owen Tables [30] for the calculations presented in Tables 4.1 and 4.2.

Table 4.1

N = 25, n=10, a = 2 (k = 3)

M	P_a	ASN_2	ASN_3	S_{12}	S_{13}
4	0.841107	9.69	8.96	3.1	10.4
5	0.698814	9.36	8.86	6.4	11.4
6	0.544664	8.93	8.60	10.7	14.0
7	0.398627	8.44	8.23	15.6	17.7
8	0.273684	7.92	7.79	20.8	22.1
9	0.175690	7.39	7.32	26.1	26.8
10	0.104819	6.88	6.84	31.2	31.6

Table 4.2

$N=25,\ n_1=5,\ n_2=10,\ a_1=1,\ a_2=3$

M	P_a	$ASN(UD)_1$	$ASN(UD)_2$	$ASN(UD)_3$	$S_{12.d}$	$S_{13.d}$
4	0.956127	6.66	6.53	6.01	1.95	9.76
5	0.863524	7.50	7.04	6.62	6.13	11.73
6	0.742970	8.38	7.38	7.05	11.93	15.87
7	0.617724	9.23	7.55	7.30	18.20	20.91
8	0.502406	10.02	7.59	7.40	24.25	26.15
9	0.402408	10.69	7.53	7.39	29.56	30.87
10	0.317615	11.22	7.40	7.29	34.05	35.02

4.4 The ASN of the Curtailed Sampling Plan under Binomial Probability Law :

In this section we consider the ASN of the curtailed sampling plan under binomial probability law. It is assumed that the process average proportion of defectives be p and for sufficiently large lots it can be considered as the probability of selecting a defective in a single trial. Furthermore, it (p) remains constant from trial to trial and the trials are stochastically independent. This applies to the Type B situation of Dodge and Romig [10]. Under these assumptions, in the following sections, we have considered the problem of the ASN under curtailed single and double sampling plans. The expressions of the ASN under curtailed single sampling plan are given by various authors [44], [5] [38] ,[40], [8]. etc. We have given, following the Craig's procedure, a simplified form of the ASN under fully-curtailed single sampling plan. Expressions of the ASN under semi--curtailed and fully-curtailed double sampling plans are also derived. The percent saving in inspection under curtailed DSP is illustrated by numerical example.

4.4.1 A Simplied form of the ASN under Curtailed Single Sampling Plan :

4.4.2 Remarks on Craig's and Cohen's Results :

Craig [8] gives a simplified form of the ASN under semi-curtailed single sampling plan. He remarks in his paper [8] that neither Sampling Inspection by Statistical Research Group [44] nor Burr [5] has given any numerical example to illustrate how far there is a reduction in the ASN if one administers a curtailed sampling plan instead of a complete sampling plan. As stated earlier, Craig [8] considers only one type of curtailment and ignores the other type of curtailment, i.e. full-curtailment, merely stating that the effect on the ASN due to full-curtailment is small. But, surprisingly, he does not confirm this fact numerically. Numerical examples illustrating the reduction in the ASN were, however, given earlier by Phatak and Bhatt [40] . In their paper [40] uncurtailed, semi-curtailed, and fully-curtailed single sampling plans are termed as Plan-1, Plan-2, and Plan-3 respectively. They considered both, Plan-2 and Plan-3. The last two columns of Table 1 of the paper [40] give indirectly the percent saving in inspection as one passes from Plan-1 to Plan-2 and Plan-1 to Plan-3. The difference between these two columns gives saving in inspection as one passes from Plan-2 to Plan-3. There is no appreciable saving in inspection if one administers Plan-3 instead of Plan-2, particularly for large values of the fraction defective, p, of a lot. There is a substantial saving in inspection for small values of p since a small value of p means a greater probability of acceptance and thereby curtailment in inspection in a greater number of cases under Plan-3.

Secondly, we would like to point out that we can calculate the ASN of Plan-2 and Plan-3 for all the typical examples worked out by Craig[8] using expression (22) and (23) of [40] and the cumulative binomial probability tables [43]. At this stage the recurrence relation

$$B(k;n+1,p) = p\,B(k-1;n,p) + qB(k;n,p) \qquad \ldots(4.4.1)$$

where $B(k;n,p) = \sum_{x=0}^{k} \binom{n}{x} p^x q^{n-x}$ may be found useful. It is very likely that, for large values of n the binomial tables may not give the cumulative probability at unit interval for n. In that case one may have to use the above recurrence relation.

Lastly Cohen [7] has given a different presentation of the probability function associated with Plan-3 and continued to find the ASN for this sampling plan. He, furthermore, remarks that the ASN derived by him {(18) of [7]} and that derived by Phatak and Bhatt {(23) of [40]} are the same. This is quite an evident. But he further remarks that the simplified form of these expressions is given by Craig [8]. This statement is misleading. In fact Craig [8] did not consider Plan-3 at all. In this section, following the Craig's procedure, we have given a simplified form of the ASN under fully-curtailed single sampling plan.

4.4.3 Probability of Acceptance :

Consider a single attributes sampling plan in which individual units randomly selected from a lot are inspected in a sequence until either

(a) an accumulated total of k defectives is found, in which case the lot is rejected or

(b) an accumulated total of g nondefectives is found, in which case the lot is accepted.

The number of units inspected to reach a decision with respect to acceptance or rejection of a given lot is thereby a discrete random variable which assumes the values

$$k,\ k+1, \ldots, n$$

where $$n = k+g-1$$

This is the description of a fully-curtailed sampling plan given by Cohen [7] and resembles the statement of Plan-3 given by Phatak and Bhatt with n-k+1 replaced by g.

The probability of acceptance, p_a, for a semi-curtailed or a fully-curtailed, sampling plan is the same as that for the usual uncurtailed sampling plan [44],[40]. Therefore, P_a can be expressed in a number of ways such as

$$\sum_{z=g}^{n} \binom{z-1}{g-1} q^{g} p^{z-g} \qquad \ldots(4.4.2)$$

$$1 - \sum_{y=k}^{n} \binom{y-1}{k-1} p^{k} q^{y-k} \qquad \ldots(4.4.3)$$

$$q^{g} \sum_{r=o}^{k-1} \binom{n-r-1}{k-1-r} p^{k-1-r} \qquad \ldots(4.4.4)$$

$$1 - p^{n-g+1} \sum_{r=o}^{g-1} \binom{n-r-1}{g-1-r} q^{g-1-r} \qquad \ldots(4.4.5)$$

$$\sum_{x=o}^{k-1} \binom{n}{x} p^{x} q^{n-x} \qquad \ldots(4.4.6)$$

where p = probability of selecting a defective in a single trial,

$q = 1-p$,

y = Number of items inspected when the kth defective is found,

and z = Number of items inspected when the gth nondefective is found.

Expressions (4.4.2) and (4.4.3) follow from the probability function of the number of items inspected in reaching the decision. Expression (4.4.6) results from the usual uncurtailed single sampling plan. Expression (4.4.4) is given by Craig [8] in relation to a semi-curtailed sampling plan. However, even for a fully-curtailed sampling the use of (4.4.4) is recommended for the calculation of P_a if the binomial tables do not give a direct entry for (4.4.6) or its complements and if k is small. Expression (4.4.5) may be derived from (4.4.2) using the following identity.

$$\sum_{r=n+1}^{\infty} \binom{r-1}{k-1} p^k q^{r-k} = q^g \sum_{r=0}^{k-1} \binom{n-r-1}{k-1-r} p^{k-1-r} \quad \ldots(4.4.7)$$

established in [8]. Expression (4.4.5) is used later in this paper in deriving the simplified form of the ASN for a fully-curtailed sampling plan.

4.4.4 The ASN :

As usual, denote the ASN of a semi-curtailed sampling plan by ASN_2 and the ASN of a fully-curtailed sampling plan by ASN_3. Craig's [8] simplified form of ASN_2 is

$$ASN_2 = (n-k/p)\, P_a + (k/p) \left\{1 - q \binom{n}{k} p^k q^{n-k}\right\} \quad \ldots(4.4.8)$$

We now proceed to derive a simplified form of ASN_3. Let y and z be defined as above. Noting that neither y nor z would exceed n, the ASN for a fully-curtailed sampling plan is

$$ASN_3 = E(y/y \le n) + E(z/z \le n) \quad \ldots(4.4.9)$$

From the expression (10)of Craig [8] it follows that

$$E(y/y \le n) = k/p - k\binom{n}{k} p^{k-1} q^g - (k/p)P_a \quad \ldots(4.4.10)$$

Furthermore,

$$E(z|z \leq n) = E(z) - E(z|z \geq n+1)$$

$$= \sum_{z=g}^{\infty} z\binom{z-1}{g-1} q^g p^{z-g} - \sum_{z=n+1}^{\infty} z\binom{z-1}{g-1} q^g p^{z-g}$$

$$= g/q - (g/q)p^{n-g+1} \sum_{r=o}^{g} \binom{n-r}{g-r} q^{g-r}$$

where we use (4.4.7) on the last step. Comparing the series in the second term of the above expression with (4.4.5) we find that the above expression can be written as

$$g/q-(g/q)\ p^{n-g+1} \binom{n}{g} q^g - (g/q)\ (1-P_a)$$

$$= (g/q)\ P_a-(g/q)\binom{n}{g} q^g\ p^{n-g+1} \qquad \ldots(4.4.11)$$

Adding (4.4.10) and (4.4.11) we have

$$ASN_3 = \left[\frac{p(n+1)-k}{pq}\right] P_a + \left(\frac{k}{p}\right)\left[1-\binom{n}{k}p^k\ q^{n-k}\right] \ldots(4.4.12)$$

This is the simplified form for the ASN of a fully-curtailed sampling plan. It appears that the principle in expressing the ASN in the above forms, namely (4.4.8) and (4.4.12) lies in expressing the ASN in terms of a simple multiple of P_a and an individual term of a binomial distribution.

We note that ASN_3 given by (4.4.12) can be shown to be equivalent to the corresponding expression given by the

Statistical Research Group [44] (c.f. equation (43) on page 214). Obviously this ought to be the case.

4.4.5 Saving in Inspection :

It follows from (4.4.8) that ASN_3 can be expressed as

$$ASN_3 = (ASN_2 + P_a - k)/q \qquad ...(4.4.13)$$

The above expression easily yields the following instructive form

$$ASN_2 - ASN_3 = k - P_a - p(ASN_3) \qquad ...(4.4.14)$$

which leads to an obvious conclusion that the saving in going from a semi-curtailed sampling plan to a fully-curtailed sampling plan can never exceed k and should be small, even for small p.

Secondly, the percent saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan may be defined respectively as

$$S_{12} = 100(n - ASN_2)/n \qquad ...(4.4.15)$$

and

$$S_{13} = 100(n - ASN_3)/n \qquad ...(4.4.16)$$

where n stands for the ASN of the usual uncurtailed sampling plan. The expressions (4.4.15) and (4.4.16) represent the respective loss in efficiency in estimation of p as one passes from the uncurtailed sampling plan to the respective curtailed sampling plans [40]. Thus saving in inspection is counterbalanced by the loss in efficiency in estimation.

4.4.6 Numerical Example :

Consider the plan with n=80, k=5 and g=76. This example was considered by Phatak and Bhatt [40]. It may be noted that Table 1 of [40]. was prepared with the help of the Tables of the Cumulative Binomial Probability Distribution [43] where formula 17(b) of [43] or (16) of[40] was found extremely useful. In Table 4.3, the ASN for a semi-curtailed and the ASN for a fully-curtailed single sampling plan, using (4.4.8) and (4.4.12) respectively, are given. Last two columns of the table give percent saving in inspection as one passes from the usual uncurtailed sampling plan to a semi-curtailed sampling plan and a fully-curtailed sampling plan respectively. The difference between these columns give percent saving in inspection as one passes from a semi--curtailed to a fully-curtailed sampling plan.

Table 4.3

$n=80$, $k=5$, and $g=76$

p	p_a	ASN_2	ASN_3	S_{12}	S_{13}
0.03	0.90721	78.43	76.63	1.96	4.21
0.04	0.78358	75.91	74.68	5.11	6.65
0.05	0.62888	72.19	71.39	9.76	10.76
0.06	0.47174	67.63	67.13	15.46	16.09
0.07	0.33333	62.67	62.37	21.66	22.04
0.08	0.22350	57.70	57.52	27.88	28.10
0.09	0.14311	52.97	52.87	33.79	33.91
0.10	0.08797	48.64	48.58	39.20	39.28

4.4.7 The ASN under Curtailed Double Sampling Plan :

In this section we consider the ASN of a semi-curtailed and a fully-curtailed DSP under the assumptions described in Section 4.4.

4.4.8 The ASN under a Fully-Curtailed DSP :

We have given the statement of a fully-curtailed DSP in the Section 2.3.2 of Chapter II. Recall the definitions of the random variables Y and T, sets A_i (i=1,2,3,4) of possible values attained by Y, and the joint probability function of the random variables Y and T given in the Section 2.4.1 of Chapter II. Then the average sample number is merely the average number of units inspected thus

$$ASN = E(Y)$$

It follows from (2.5.6) of Chapter II that

$$E(Y) = \sum_{y} y \sum_{i=1}^{4} f_i(y;p)$$

$$= \sum_{y \in A_1} y\, f_1(y;p) + \sum_{y \in A_2} y\, f_2(y;p)$$

$$+ \sum_{y \in A_3} y\, f_3(y;p) + \sum_{y \in A_4} y\, f_4(y;p) \qquad \ldots(4.4.17)$$

Each of the four summation terms of the right hand side of the above expression can be expressed as given below :

$$\sum_{y \in A_1} y\, f_1(y;p) = \frac{r_1}{p} \{1-B(r_1;n_1+1,p)\} \qquad \ldots(4.4.18)$$

$$\sum_{y \in A_2} y\, f_2(y;p) = \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} p^{n_1-g_1+t} q^{g_1-t}$$

$$\cdot\Big[\frac{b_2+1-t}{p} \{1-B(b_2+1-t;\, n_2+1,p)\}$$

$$+ n_1 \{1-B(b_2-t;\, n_2,p)\}\Big] \qquad \ldots(4.4.19)$$

$$\sum_{y \in A_3} y\, f_3(y;p) = \frac{g_1}{q} B(n_1-g_1;\, n_1+1,p) \qquad \ldots(4.4.20)$$

$$\sum_{y \in A_4} y\, f_4(y;p) = \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} p^{n_1-g_1+t} q^{g_1-t}$$

$$\cdot\Big[\frac{g_2-g_1+t}{q} B(b_2-t;\, n_2+1,p)$$

$$+ n_1\, B(b_2-t;\, n_2,p)\Big] \qquad \ldots(4.4.21)$$

where $b_1 = g_1+r_1-n_1-1$, $b_2=g_1+r_2-n_1-1$ and

$$B(k;n,p) = \sum_{x=o}^{k} \binom{n}{x} p^x q^{n-x} .$$

These expressions have been obtained using the identity

$$\sum_{x=o}^{r} \binom{k+x-1}{x} p^k q^x = 1-B(k-1;\ r+k,p)$$

whose proof can be had in [32],[37] .

The ASN which is the sum of the above four terms given by (4.4.18) through (4.4.21) can be expressed as

$$ASN(D)_3 = \frac{r_1}{p} \{1-B(r_1;\ n_1+1,p)\} + \frac{g_1}{q} B(n_1-g_1;n_1+1,p)$$

$$+ \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} p^{n_1-g_1+t} q^{g_1-t} \Big[\frac{b_2+1-t}{p} \{1-B(b_2+1-t;n_2+1,p)\}$$

$$+ n_1 + \frac{g_2-g_1+t}{q} B(b_2-t;\ n_2+1,p)\Big] \qquad \ldots(4.4.22)$$

where $ASN(D)_3$ denotes the ASN of a fully-curtailed DSP.

4.4.9 The ASN under Semi-Curtailed DSP :

The statement of a semi-curtailed DSP is given below :

Inspect randomly selected units of a lot one at a time till one of the following four mutually exclusive events $B_i(i=1,2,3,4)$. occurs :

(B_1) r_1 defectives are observed and the number of units inspected is less than or equal to n_1,

(B_2) r_2 defectives are observed and the number of units inspected is greater than n_1 but less than or equal to n_1+n_2,

(B_3) $0 \leq D_1 \leq n_1-g_1$ defectives are observed and the number of units inspected is equal to n_1,

(B_4) $n_1-g_1+1 \leq D_1+D_2 \leq n_1+n_2-g_2$ defectiges are observed and the number of units inspected is equal to n_1+n_2,

where D_1 is the number of defectives observed during the inspection of first sample and D_2 is the number of defectives observed during the inspection of the second sample.

The decision rule is then to reject the lot if either (B_1) or (B_2) occurs and to accept the lot if either (B_3) or (B_4) occurs.

4.4.10 The Probability Function :

The probability function associated with the random phenomenon prevailing in the above curtailed sampling plan is as given below :

$$P(Y=y,\ D_1=d_1,\ D_2=d_2,\ T=i) =$$

$$\begin{cases} \binom{y-1}{r_1-1} p^{r_1} q^{y-r_1} & y=r_1,\ r_1+1,\dots,n_1;\\ & d_1=r_1\ ;\\ & i=1.\\ \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} \binom{y-n_1-1}{b_2-t} p^{r_2} q^{y-r_2} & \\ & y=r_2-r_1+n_1+1,\dots,n_1+n_2;\\ & d_1=n_1-g_1+t;\\ & d_2=r_2-d_1\ ;\\ & i=2.\\ \binom{n_1}{d_1} p^{d_1} q^{n_1-d_1} & y=n_1\ ;\\ & d_1=0,1,\dots,\ n_1-g_1;\\ & i=3.\\ \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} \binom{n_2}{d_2} p^{n_1-g_1+t+d_2}\ q^{n_2+g_1+t-d_2} & \\ & y=n_1+n_2\\ & d_1=n_1-g_1+t\\ & d_2=0,1,\dots,\ r_2-1-d_1;\\ & i=4 \end{cases}$$

...(4.4.23)

where Y and T are the random variables defined as in fully curtailed DSP, $b_1=g_1+r_1-n_1-1$ and $b_2=g_1+r_2-n_1-1$.

4.4.11 The ASN :

The ASN is defined as

$$ASN = E(Y).$$

It follows from (4.4.23) that

$$E(Y) = \sum_y y \sum_{d_1} \sum_{d_2} \sum_i P(Y=y, D_1=d_1, D_2=d_2, T=i)$$

$$= \sum_{y=r_1}^{n_1} y \binom{y-1}{r_1-1} p^{r_1} q^{y-r_1}$$

$$+ \sum_{y=r_2-r_1+n_1+1}^{n_1+n_2} y \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} \binom{y-n_1-1}{b_2-t} p^{r_2} q^{y-r_2}$$

$$+ \sum_{y=n_1} y \sum_{d_1=o}^{n_1-g_1} \binom{n_1}{d_1} p^{d_1} q^{n_1-d_1}$$

$$+ \sum_{y=n_1+n_2} y \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} p^{n_1-g_1+t} q^{g_1-t}$$

$$\cdot \sum_{d_2=o}^{r_2-1-d_1} \binom{n_2}{d_2} p^{d_2} q^{n_2-d_2}$$

$$\therefore \quad ASN(D)_2 = \frac{r_1}{p}\left\{1-B(r_1;n_1+1,p)\right\} + n_1\ B(n_1-g_1;n_1,p)$$

$$+ \sum_{t=1}^{b_1} \binom{n_1}{g_1-t} p^{n_1-g_1+t}\ q^{g_1-t}$$

$$\cdot\left[\frac{b_2+1-t}{p}\ \left\{1-B(b_2+1-t;\ n_2+1,p)\right\} + n_1\right.$$

$$\left. + n_2\ B(b_2-t;\ n_2,p)\ \right] \qquad \ldots(4.4.24)$$

where $ASN(D)_2$ denotes the ASN of a semi-curtailed DSP. Craig [8] has given the ASN of a semi-curtailed UDSP. His expression is given below :

$$ASN(UD)_2 = \frac{r}{p}\ \left\{1-B(r;n_1+1,p)\right\} + n_1 B(n_1-g_1;n_1,p)$$

$$+ \sum_{t=1}^{b} \binom{n_1}{g_1-t}\ p^{n_1-g_1+t}\ q^{g_1-t}$$

$$\cdot\left[\frac{b+1-t}{p}\ \left\{1-B(b+1-t;\ n_2+1,p)\right\}\right.$$

$$\left. + n_1 + n_2\ B(b-t;\ n_2,p)\right] \qquad \ldots(4.4.25)$$

It may be noted that one can calculate the ASN using (4.4.22), (4.4.24), and (4.4.25) with the help of the usual binomial tables such as the tables of the cumulative Binomial Probability Distribution [43] . However, for large values of n, the binomial tables including [43] do not give the cumulative probability at an unit interval for n. In that case the following recurrence relation

$$B(k;n+r,p)=\sum_{x=0}^{r}\binom{r}{x}p^{r-x}q^{x}B(k-r+x;n,p) \qquad \ldots(4.4.26)$$

may be found useful.

4.4.12 Numerical Example :

To illustrate the percent saving in inspection under curtailed DSP, we consider the following DSP :

$n_1=50$, $n_2=100$, $r_1=3$, $r_2=4$, $g_1=49$, and $g_2=147$.

The percent saving in inspection as one passes from an uncurtailed sampling plan to a semi-curtailed sampling plan or to a fully-curtailed sampling plan may be defined in the case of DSP as

$$S_{12.d}=100\,[ASN(D)_1 - ASN(D)_2\,]/ASN(D)_1 \qquad \ldots(4.4.27)$$

$$S_{13.d}=100\,[ASN(D)_1 - ASN(D)_3]\,/ASN(D)_1 \qquad \ldots(4.4.28)$$

where

$$ASN(D)_1 = n_1+n_2\,[B(r_1-1;n_1,p)-B(n_1-g_1;n_1,p)]\ldots(4.4.29)$$

The expressions of $ASN(D)_2$ and $ASN(D)_3$ are given by (4.4.24) and (4.4.22) respectively. $ASN(D)_1$, $ASN(D)_2$, $ASN(D)_3$ and the percent saving in inspection given by (4.4.27) and (4.4.28) are calculated and are presented in Table 4.4.

It is revealed from the Table 4.4 that there is appreciable saving in the inspection in going from an uncurtailed sampling plan to a semi-curtailed or to a fully-curtailed sampling plan. The difference between column (6) and column (7) reflects the advantage in going from a semi-curtailed sampling plan to a fully-curtailed sampling plan. The mentionable saving in inspection exists in going from a semi-curtailed DSP to a fully-curtailed DSP for smaller values of p.

Table 4.4

n_1=50, n_2=100, r_1=3, r_2=4, g_1=49, and g_2 = 147

p	P_a	$ASN(D)_1$	$ASN(D)_2$	$ASN(D)_3$	$S_{12.d}$	$S_{13.d}$
1	2	3	4	5	6	7
0.01	0.9662016	57.56180	56.61671[illegible]	55.97763	1.65	2.75
0.02	0.8106996	68.58009	57.03777	56.64104	16.83	17.41
0.03	0.6050094	75.55182	62.16715	61.92980	17.72	18.03
0.04	0.4245586	77.62328	57.91618	57.77604	25.39	25.57
0.05	0.2891136	76.11013	52.28129	52.19867	31.31	31.42
0.06	0.1934357	72.62442	46.61682	46.56809	35.81	35.88
0.07	0.1276016	68.42951	41.47878	41.45009	39.38	39.43
0.08	0.0830443	64.32623	37.02507	37.00822	42.44	42.47
0.09	0.0533322	60.73020	33.24186	33.23201	45.26	45.28
0.10	0.0338110	57.79429	30.05191	30.04618	48.00	48.01