CHAPTER - 6

BASIC TRANSFORMATION FORMULAS

6.1 INTRODUCTION

In this chapter, the q-Beta integral and q-Gamma integral due to W.Hahn[1, Eq.(3.12) and (3.16)] will be considered in which the integrand will be modified by introducing the q-polynomials $S_n(l,m,\alpha,\beta,x\,|\,q)$ and $M_n(s,A,\beta,x\,|\,q)$ in turn, and then these integrals will be simplified. The final expressions thus obtained, will be further exploited by means of the relation

$$e_q(x)E_q(x)=1,$$

where (Hahn [2, Eq. (6.1), p. 361])

$$e_q(x)={}_1\phi_0[0;-;q,x]=\sum_{n=0}^{\infty}\frac{x^n}{[q]_n}=\frac{1}{[x]_\infty} \tag{6.1.1}$$

and

$$E_q(x) = {}_1\phi_1[0,0,q,x]={}_0\phi_0[-;-;q,x] \tag{6.1.2}$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2}\frac{x^n}{[q]_n}=[x]_\infty .$$

This process finally leads us to the series transformation formulas. This will be taken up in the subsequent sections.

6.2 BASIC TRANSFORMATION FORMULA (I):

Consider the q-Gamma integral (W.Hahn[1])

$$\int_0^{\infty} t^{\nu-1}\, e_q(-t)\, d_q t = \frac{(1-q)\,[q]_\infty}{[q^{\nu}]_\infty} q^{-\nu(\nu-1)/2}. \tag{6.2.1}$$

Here introducing the basic Laguerre polynomial (D.S. Moak[1])

$$L_n^{(\nu)}(tq\,|\,q) = \frac{[\nu q]_n}{[q]_n} \sum_{k=0}^{n} q^{\nu k+nk+k(k+1)/2} \frac{[q^{-n}]_k\, t^k}{[\nu q]_k [q]_k} \tag{6.2.2}$$

in the integrand of this integral then we have the following simplification towards its evaluation. Writing p for q^{α} ($\alpha \neq 0$),

$$\int_0^{\infty} t^{\nu-1}\, e_p(-t)\, L_n^{(\mu)}(tp\,|\,p)\, d_p t$$

$$= \frac{(\mu p;p)_n}{(p;p)_n} \sum_{k=0}^{n} p^{\mu k+nk+k(k+1)/2} \frac{(p^{-n};p)_k}{(\mu p;p)_k\,(p;p)_k} \int_0^{\infty} t^{\nu+k-1}\, e_p(-t)\, d_p t$$

$$= \frac{(\mu p,p)_n}{(p;p)_n} \sum_{k=0}^{n} p^{\mu k+nk+k(k+1)/2} \frac{(p^{-n},p)_k\,(1-p)\,(p;p)_\infty\, p^{-(\nu+k)(\nu+k-1)/2}}{(\mu p;p)_k\,(p;p)_k\,(\nu p^k;p)_\infty}$$

$$= \frac{(\mu p;p)_n}{(p;p)_n} \sum_{k=0}^{n} p^{(\mu+n-\nu+1)k-\nu(\nu-1)/2} \frac{(1-p)\,(p;p)_\infty (p^{-n};p)_k}{(\mu p;p)_k\,(p;p)_k\,(\nu p^k;p)_\infty}$$

$$= \frac{(1-p)(\mu p;p)_n\,(p;p)_\infty}{(p;p)_n\,(\nu;p)_\infty} p^{-\nu(\nu-1)/2} \sum_{k=0}^{n} \frac{(p^{-n},p)_k\,(\nu,p)_k}{(\mu p;p)_k\,(p;p)_k} p^{(\mu+n-\nu+1)k}$$

$$= \frac{(1-p)\,(\mu p;p)_n (p;p)_\infty}{(p;p)_n\,(\nu;p)_\infty} p^{-\nu(\nu-1)/2}\; {}_2\phi_1\left[\begin{matrix} p^{-n}, \nu; p, p^{\mu+n-\nu+1} \\ \mu p; \end{matrix}\right]. \tag{6.2.3}$$

Now, in view o the Vandermond's theorem

$$ {}_2\phi_1(a,q^{-n};c,q,cq^n/a)=\frac{(c/a,q)_n}{(c,q)_n} $$

the function ${}_2\phi_1$ in (6.2.3) gives $(\mu p/v;p)_n$ / $(\mu p;p)_n$, consequently (6.2.3) further simplifies to

$$ \int_0^\infty t^{v-1}\, e_q(-t)\, L_n^{(\mu)}(tp\,|\,p)\, d_p t=\frac{(1-p)(p;p)_\infty\,(\mu p/v;p)_n}{(p;p)_n\,(v;p)_\infty}\, p^{-v(v-1)/2} \qquad \textbf{(6.2.4)} $$

which we call as the q-Laguerre integral formula. In Chapter-3, there was defined a polynomial

$$ S_n(l,m,\alpha,\beta;x\,|\,p)=\sum_{k=0}^{[n/m]}(-1)^{mk}\, p^{mk(mk-2n+1)/2}\,\frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p,p)_{n-mk}}\,\sigma_k\, x^k . $$

Let $S_n^*(l,m,\alpha,\beta;x\,|\,p)=S_n(l,m,\alpha,\beta;x\,|\,p)(\beta q^{1-n\alpha};p)_\infty/(p;p)_n$

then

$$ S_n^*(l,m,\alpha,\beta,x\,|\,p)=\sum_{k=0}^{[n/m]}\frac{p^{mk}(p^{-n};p)_{mk}\,\sigma_k}{(\beta q^{1-n\alpha};p)_{lk}}\,x^k . \qquad \textbf{(6.2.5)} $$

When this polynomial is introduced into the integrand of (6.2.4), one gets

$$ \int_0^\infty t^{v-1}\; e_q(-t)\; L_n^{(\mu)}(tp\,|\,p)\; S_r^*(l,m,\alpha,\beta;xt\,|\,p)\; d_p t $$

$$ =\sum_{k=0}^{[r/m]} p^{mk}\,\frac{(p^{-r};p)_{mk}}{(\beta q^{1-r\alpha};p)_{lk}}\,\sigma_k\, x^k \int_0^\infty t^{v+k-1}\, e_q(-t)\; L_n^{(\mu)}(tp\,|\,p)\, d_p t . $$

In view of the formula (6.2.4), this assumes the form

$$\int_0^\infty t^{\nu-1}\, e_p(-t)\; L_n^{(\mu)}(tp\,|\,p)\;\; S_r^*(l,m,\alpha,\beta;xt\,|\,p)\; d_p t$$

$$=\sum_{k=0}^{[r/m]} p^{(-\nu(\nu-1)/2)-(k(k-1)/2)-\nu k+mk}\,\frac{(1-p)(p,p)_\infty (p^{-r},p)_{mk}(\nu,p)_k}{(\beta q^{1-r\alpha},p)_{lk}\,(p,p)_n(\nu;p)_\infty}(\mu p^{1-k}/\nu;p)_n\,\sigma_k\,x^k$$

Now by making use of the formulas (Slater [1, p.241])

$$(a;q)_{n-k}=\frac{(a;q)_n}{(q^{1-a-n};q)_k}(-1)^k\,q^{-ak+k(k+1)/2-nk}$$

and

$$(a;q)_{-k}=(-1)^k\,q^{-ak+k(k+1)/2}\,\frac{1}{(q^{1-a},q)_k}$$

with a is replaced by $\mu p/\nu\,(\equiv p^{\mu+1-\nu})$, one finds

$$(\mu p^{1-k}/\nu,p)_n=\frac{(\mu p/\nu;p)_{n-k}}{(\mu p/\nu;p)_{-k}}$$

$$=\frac{(\mu p/\nu;p)_n\,(\nu/\mu;p)_k}{(\nu p^{-n}/\mu;p)_k\,p^{nk}}.$$

Hence

$$\int_0^\infty t^{\nu-1}\, e_p(-t)\; L_n^{(\mu)}(tp\,|\,p)\;\; S_r^*(l,m,\alpha,\beta;xt\,|\,p)\; d_p t$$

$$=\frac{(1-p)(p;p)_\infty(\mu p/\nu;p)_n}{(p;p)_n(\nu;p)_\infty\,p^{\nu(\nu-1)/2}}\sum_{k=0}^{[r/m]} p^{-\nu k-nk-k(k-1)/2}\,\frac{(p^{-r};p)_{mk}\,(\nu;p)_k\,(\nu/\mu;p)_k}{(\beta q^{1-r\alpha};p)_{lk}\,(\nu p^{-n}/\mu;p)_k}\sigma_k\,x^k.$$

(6.2.6)

Here, if the series on the right hand side is denoted by $T_{r,n}^m(l,m,\nu,\beta;x\,|\,p)$

then one finds

$$\int_0^\infty t^{\nu-1} e_p(-t)\, L_n^{(\mu)}(tp\,|\,p)\, S_r^*(l,m,\alpha,\beta,xt\,|\,p)\, d_p t$$

$$= \frac{(1-p)(p;p)_\infty (\mu p/\nu;p)_n}{(p;p)_n (\nu;p)_\infty\, p^{\nu(\nu-1)/2}}\, T_{r,n}^m(\mu,\nu;x\,|\,p),$$

which is the desired result.

In case when μ and ν are same and equal to b say then the q-Laguerre integral formula (6.2.4) would reduce to

$$\int_0^\infty t^{b-1} e_p\, L_n^{(b)}(tp\,|\,p)\, d_p t = \frac{(1-p)(p;p)_\infty}{(\nu,p)_\infty}\, p^{-b(b-1)/2}$$

and therefore

$$\int_0^\infty t^{b-1} e_p(-t)\, L_n^{(b)}(tp\,|\,p)\, S_r^*(l,m,\alpha,\beta;xt\,|\,p)\, d_p t$$

$$= \sum_{k=0}^{[r/m]} p^{-b(b-1)/2-k(k-1)/2-bk+mk} \frac{(1-p)(p;p)_\infty (p^{-r};p)_{mk} (b,p)_k}{(\beta q^{1-r\alpha};p)_{lk} (b;p)_\infty} \sigma_k\, x^k$$

$$= \frac{(1-p)(p;p)_\infty}{(b;p)_\infty\, p^{b(b-1)/2}}\, T_r^m(l,b,\beta,x\,|\,p),$$

where

$$T_r^m(b;x\,|\,p) = \sum_{k=0}^{[r/m]} p^{-k(k-1)/2-bk+mk} \frac{(p^{-r};p)_{mk} (b;p)_k}{(\beta q^{1-r\alpha};p)_{lk}} \sigma_k\, x^k .$$

The q-Beta integral is given by (W. Hahn[1])

$$\int_0^1 t^{\lambda-1} E_q(tq)\, d(q,t) = (1-q)\frac{[q]_\infty}{[q^\lambda]_\infty}, \quad |q|<1, \quad \mathrm{Re}(\lambda)>0. \qquad \textbf{(6.2.8)}$$

Consider now

$$\int_0^1 t^{\lambda-1} E_p(tq)\ S_n(l,m,\alpha,\beta,xt\,|\,p)\ d(p,t)$$

$$=\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\frac{(\beta q^{\ell+k-n\alpha+1};p)_\infty}{(p;p)_{n-mk}}.$$

$$.\int_0^1 t^{\lambda+k-1}\,E_p(tq)\,d(p,t)$$

$$=\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\frac{(\beta q^{lk-n\alpha+1};p)_\infty\,(1-p)(p;p)_\infty}{(p;p)_{n-mk}\,(p^{\lambda+k};p)_\infty}\sigma_k\,x^k$$

$$=\frac{(1-p)(p,p)_\infty}{(p^{\lambda};p)_\infty}\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\frac{(\beta q^{lk-n\alpha+1};p)_\infty(p^{\lambda},p)_k\,\sigma_k\,x^k}{(p;p)_{n-mk}}.$$

(6.2.9)

Now using the relation $e_q(x)\ E_q(x)=1$ in this formula, one can obtain a new result. In deed, applying the integral on the left hand side in (6.2.9) to this relation, one gets,

$$\int_0^1 t^{\lambda-1}\ E_p(tq)\ e_p(xtp)\ E_p(xtp)\ S_n(l,m,\alpha,\beta;xt\,|\,p)\ d(p,t)$$

$$=\int_0^1 t^{\lambda-1}\ E_p(tq)S_n(l,m,\alpha,\beta;xt\,|\,p)\ d(p,t). \qquad \textbf{(6.2.10)}$$

If R denotes the right hand side of (6.2.10) then in the light of (6.2.9),

$$R=\frac{(1-p)(p;p)_\infty}{(p^{\lambda};p)_\infty}\sum_{k=0}^{[n/m]}(-1)^{mk}\,p^{mk(mk+1)/2-mnk}\frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p;p)_{n-mk}}(p^{\lambda};p)_k\,\sigma_k\,x^k.$$

Similarly, if L stands for the left hand member of (6.2.10), then in view of the definitions (6.1.1) and (6.1.2),

$$L=\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j p^{v+j+j(j-1)/2} x^{v+j}}{(p;p)_v(p;p)_j}\int_0^1 t^{\lambda+v+j-1} E_p(tq) S_n(l,m,\alpha,\beta,xt\,|\,p)\, d(p,t)$$

which in view of (6.2.9) assumes the form

$$L=\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j p^{v+j+j(j-1)/2} x^{v+j}(1-p)(p,p)_\infty}{(p;p)_v(p,p)_j(p^{\lambda+v+j};p)_\infty}\cdot$$

$$\cdot\sum_{k=0}^{[n/m]}(-1)^{mk} p^{mk(mk+1)/2-mnk}\frac{(\beta q^{lk-n\alpha+1};p)_\infty(p^{\lambda+v+j};p)_k\,\sigma_k\, x^k}{(p;p)_{n-mk}}.$$

Using $(p^\lambda;p)_\infty = (p^\lambda;p)_{v+j}\,(p^{\lambda+v+j};p)_\infty$, and replacing the series forms in L & R above in (6.2.10), one obtains the transformation formula

$$\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j p^{v+j+j(j-1)/2} x^{v+j}(p^\lambda;p)_{v+j}}{(p;p)_v(p;p)_j(p^\lambda;p)_\infty}\sum_{k=0}^{[n/m]}(-1)^{mk} p^{mk(mk+1)/2-mnk}$$

$$\cdot\frac{(\beta q^{lk-n\alpha+1};p)_\infty\,(p^{\lambda+v+j};p)_k\,\sigma_k\, x^k}{(p;p)_{n-mk}}$$

$$=\sum_{k=0}^{[n/m]}(-1)^{mk} p^{mk(mk+1)/2-mnk}\frac{(\beta q^{lk-n\alpha+1};p)_\infty\,(p^\lambda;p)_k\,\sigma_k\, x^k}{(p;p)_{n-mk}}.$$

(6.2.11)

It is interesting to note that specializing the sequence σ_k suitably, one can express the extended Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b):x\,|\,p]$ in a series of itself.

Therefore selecting

$$\sigma_k = \frac{(a_2,p)_k \cdots (a_r)_k}{(b_1,p)_k \cdots (b_s;p)_k (p;p)_k}$$

in (6.2.11), one arrives at

$$\sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j p^{v+j+j(j-1)/2} x^{v+j} (p^{\lambda};p)_{v+j}}{(p,p)_v (p;p)_j (p^{\lambda};p)_\infty} \mathcal{H}_{n,l,m}^{(\alpha,\beta)}[a_1 \ldots, a_r, b_1 \ldots, b_s : x \mid p]$$

$$= \mathcal{H}_{n,l,m}^{(\alpha,\beta)}[p_1, a_2 \ldots, a_r; b_1 \ldots, b_s \cdot x \mid p],$$

where $a_1 = p^{\lambda+v+j}$ and $p_1 = p^{\lambda}$, are the first numerator parameters in the extended Jacobi polynomial on the left hand and right hand sides respectively.

Another transformation formula is derived for the polynomial $S_n(l,m,\alpha,\beta;x \mid p)$ using q-Gamma integral of W. Hahn sated in (6.2.1):

$$\int_0^\infty t^{v-1} e_q(-t)\, d_q t = (1-q) \frac{[q]_\infty}{[q^v]_\infty} q^{-v(v-1)/2}.$$

For that consider,

$$\int_0^\infty t^{h-1} e_p(-t)\; S_n(l,m,\alpha,\beta;xt \mid p)\; d_p t$$

$$= \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-mnk} \frac{(\beta q^{lk-n\alpha+1};p)_\infty \sigma_k x^k}{(p;p)_{n-mk}} \int_0^\infty t^{h+k-1} e_p(-t) d_p t$$

$$= \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-mnk} \frac{(\beta q^{lk-n\alpha+1};p)_\infty (1-p)(p;p)_\infty \sigma_k x^k}{(p;p)_{n-mk} (p^{h+k};p)_\infty}.$$

$$\cdot p^{-(h+k-1)(h+k)/2}.$$

$$\int_0^\infty t^{h-1}\, e_p(-t)\; S_n(l,m,\alpha,\beta;xt\,|\,p)\; d_p t$$

$$= \frac{p^{-h(h-1)/2}(1-p)(p,p)_\infty}{(p^h;p)_\infty} \sum_{k=0}^{[n/m]} (-1)^{mk}\, p^{mk(mk+1)/2-k(k-1)/2-hk-mnk} .$$

$$\frac{(\beta q^{lk-n\alpha+1};p)_\infty\,(p^h,p)_k\,\sigma_k\,x^k}{(p;p)_{n-mk}}. \qquad \textbf{(6.2.12)}$$

The integral on the left hand side above when applied to the relation $e_q(x)\;\; E_q(x)=1$, one gets

$$\int_0^\infty t^{h-1}\;\; e_p(-t)e_p(xtp)E_p(xtp)\; S_n(l,m,\alpha,\beta;xt\,|\,p)d_p t$$

$$= \int_0^\infty t^{h-1} e_p(-t) S_n(l,m,\alpha,\beta,xt\,|\,p) d_p t \qquad \textbf{(6.2.13)}$$

= I, say.

From (6.2.12) the right hand member I of (6.2.13) simplifies to

$$\mathrm{I} = \frac{p^{-h(h-1)/2}(1-p)(p;p)_\infty}{(p^h;p)_\infty} \sum_{k=0}^{[n/m]} \frac{(-1)^{mk}\, p^{mk(mk+1)/2-k(k-1)/2-hk-mnk}}{(p,p)_{n-mk}}.$$

$$(\beta q^{lk-n\alpha+1};p)_\infty (p^h;p)_k\, \sigma_k\, x^k.$$

If J denotes the integral on the left hand side in (6.2.13), then with the aid of the definitions (6.1.1) and (6.1.2) of q-exponential functions, one gets

$$J = \sum_{v=0}^\infty \sum_{j=0}^\infty \frac{(-1)^j\, p^{v+j+j(j-1)/2}\; x^{v+j}}{(p;p)_v\,(p,p)_j} \int_0^\infty t^{h+v+j-1}\; e_p(-t)\, S_n(l,m,\alpha,\beta;xt\,|\,p)\, d_p t$$

$$= \sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j p^{v+j+j(j-1)/2} x^{v+j} p^{-(h+v+j-1)(h+v+j)/2} (1-p)(p;p)_\infty}{(p;p)_v (p,p)_j (p^{h+j+v};p)_\infty} \cdot$$

$$\cdot \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-k(k-1)/2-mnk-(h+v+j)k} (p^{h+v+j},p)_k \cdot$$

$$\cdot \frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p,p)_{n-mk}} \sigma_k x^k$$

$$= \frac{p^{-h(h-1)/2}(1-p)(p;p)_\infty}{(p^h;p)_\infty} \sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j p^{(1-h)(v+j)-v(v-1)/2-vj}}{(p;p)_v (p;p)_j (p^h;p)_\infty} \cdot$$

$$. x^{v+j} (p^h;p)_{v+j} \sum_{k=0}^{[n/m]} \frac{(-1)^{mk} p^{mk(mk+1)/2-k(k-1)/2-(h+v+j)k-mnk}}{(p,p)_{n-mk}} \cdot$$

$$. (\beta q^{lk-n\alpha+1};p)_\infty (p^{h+v+j};p)_k \sigma_k x^k .$$

Since I=J, one gets the transformation formula given by:

$$\sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j p^{(1-h)(v+j)-v(v-1)/2-vj} x^{v+j} (p^h;p)_{v+j}}{(p;p)_v (p;p)_j (p^h;p)_\infty} \cdot$$

$$\cdot \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-k(k-1)/2-(h+v+j)k-mnk} .$$

$$\cdot \frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p;p)_{n-mk}} (p^{h+v+j};p)_k \sigma_k x^k .$$

$$= \sum_{k=0}^{[n/m]} (-1)^{mk} p^{mk(mk+1)/2-k(k-1)/2-hk-mnk} \frac{(\beta q^{lk-n\alpha+1};p)_\infty}{(p;p)_{n-mk}} (p^h;p)_k \sigma_k x^k .$$

(6.2.14)

As it is mentioned in Chapter–3 that, the extended Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b):x|q]$ is a special case of $S_n(l,m,\alpha,\beta,x|p)$, it can be seen that from (6.2.14) a relation can be obtained wherein $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b):x|q]$ is expressed in a series of itself. For that take

$$\sigma_k = \frac{p^{k(k-1)/2}(a_2,p)_k \ldots (a_r;p)_k}{(p;p)_k(b_1;p)_k \ldots (b_s;p)_k}$$

then from (6.2.14) we have,

$$\sum_{v=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j p^{(1-h)(v+j)-v(v-1)/2-vj} x^{v+j} (p^h;p)_{v+j}}{(p;p)_v (p;p)_j (p^h;p)_\infty} .$$

$$.\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[a_1,a_2,\ldots,a_r;b_1,\ldots,b_s:xp^{-(h+v+j)}\,|\,p]$$

$$=\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[c,a_2,\ldots,a_r,b_1,\ldots,b_s:xp^{-h}\,|\,p],$$

where $a_1 = p^{h+v+j}$ and $c = p^h$.

6.3 BASIC TRANSFORMATION FORMULAS: (II)

In this section, the q-Beta and q-Gamma integrals due to W.Hahn which are already considered in the above section ((6.2.8) and (6.2.1) respectively), will be used once again as a tool, with reference to the polynomial $M_n(s,A,\beta;x|q)$ introduced in Chapter–5. The technique adopted in deducing the formulas is same as that of the section-6.2.

The polynomial $M_n(s,A,\beta;x|q)$ defined by

$$M_n(s,A,\beta;x\,|\,q)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta},q^{\beta})_{n-sk}}{[q]_{n-sk}}\xi_k\,x^k \qquad \textbf{(6.3.1)}$$

will be first used in the q-Beta integral (6.2.8).

$$\int_0^1 t^{\lambda-1}E_q(tq)\,d(q,t)=(1-q)\frac{[q]_\infty}{[q^{\lambda}]_\infty}$$

with $q^{\beta}=q_2$,

$$\int_0^1 t^{\lambda-1}E_{q_2}(tq)M_n(s,A,\beta;xt\,|\,q)\,d(q_2,t)$$

$$=\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}\frac{(Aq^{sk+sk\beta},q_2)_{n-sk}}{[q]_{n-sk}}\xi_k\,x^k\int_0^1 t^{\lambda+k-1}E_{q_2}(tq)\,d(q_2,t)$$

$$=\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}\frac{(Aq^{sk+sk\beta};q_2)_{n-sk}}{[q]_{n-sk}}\xi_k\,x^k\frac{(1-q_2)(q_2;q_2)_\infty}{(q_2^{\lambda+k};q_2)_\infty}$$

$$=\frac{(1-q_2)(q_2,q_2)_\infty}{(q_2^{\lambda};q_2)_\infty}\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}\frac{(Aq^{sk+sk\beta};q_2)_{n-sk}(q_2^{\lambda};q_2)_k}{[q]_{n-sk}}\xi_k\,x^k \qquad \textbf{(6.3.2)}$$

A new basic series transformation formula can now be obtained by using the indentity $e_q(x)\,E_q(x)=1$ in (6.3.2) above. In fact, applying the integral on the left hand side in (6.3.2) to the relation $e_q(x)\,E_q(x)=1$, it gives

$$\int_0^1 t^{\lambda-1}E_{q_2}(tq)e_{q_2}(xtq_2)E_{q_2}(xtq_2)M_n(s,A,\beta;xt\,|\,q)\,d(q_2,t)$$

$$=\int_0^1 t^{\lambda-1}E_{q_2}(tq)M_n(s,A,\beta;xt\,|\,q)\,d(q_2,t). \qquad \textbf{(6.3.3)}$$

Let us put

$$L=\int_0^1 t^{\lambda-1} E_{q_2}(tq)e_{q_2}(xtq_2)E_{q_2}(xtq_2)M_n(s,A,\beta;xt|q)d(q_2,t)$$

and

$$R=\int_0^1 t^{\lambda-1} E_{q_2}(tq)M_n(s,A,\beta;xt|q)d(q_2,t).$$

Then using (6.3.2)

$$R=\frac{(1-q_2)(q_2,q_2)_\infty}{(q_2^\lambda;q_2)_\infty}\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta},q_2)_{n-sk}(q_2^\lambda;q_2)_k\,\xi_k\,x^k}{[q]_{n-sk}}$$

whereas the definitions (6.1.1) and (6.1.2) at once put the left hand side of (6.3.3) in the form:

$$L=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}x^{i+j}}{(q_2;q_2)_i(q_2;q_2)_j}.$$

$$\cdot\int_0^1 t^{(\lambda+i+j)-1}E_{q_2}(tq)M_n(s,A,\beta;xt|q)d(q_2,t).$$

Now applying the integral formula (6.3.2) here, one gets

$$L=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}x^{i+j}}{(q_2;q_2)_i(q_2;q_2)_j}\frac{(1-q_2)(q_2,q_2)_\infty}{(q_2^{\lambda+i+j};q_2)_\infty}\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}$$

$$\cdot\frac{(Aq^{sk+sk\beta};q_2)_{n-sk}(q_2^{\lambda+i+j};q_2)_k\,\xi_k\,x^k}{[q]_{n-sk}}$$

But since

$$(q_2^{\lambda v+i+j},q_2)_\infty=\frac{(q_2^{\lambda},q_2)_\infty}{(q_2^{\lambda},q_2)_{i+j}},$$

$$L=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}x^{i+j}(1-q_2)(q_2;q_2)_\infty(q_2^{\lambda};q_2)_{i+j}}{(q_2;q_2)_i(q_2;q_2)_j(q_2^{\lambda};q_2)_\infty}.$$

$$\cdot\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}\frac{(Aq^{sk+sk\beta};q_2)_{n-sk}(q_2^{\lambda+i+j};q_2)_k\xi_k x^k}{[q]_{n-sk}}.$$

Thus L=R leads us to the desired result:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}x^{i+j}(q_2^{\lambda},q_2)_{i+j}}{(q_2;q_2)_i(q_2;q_2)_j}\sum_{k=0}^{[n/s]}(-1)^{sk}q^{-snk}$$

$$\frac{(Aq^{sk+sk\beta};q_2)_{n-sk}(q_2^{\lambda+i+j};q_2)_k}{[q]_{n-sk}}\xi_k x^k$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}(q_2^{\lambda};q_2)_k\xi_k x^k}{[q]_{n-sk}}.$$

Another basic transformation is obtained using

$$\int_0^\infty t^{\mu-1}e_q(-t)d_qt=\frac{(1-q)[q]_\infty}{[q^\mu]_\infty}q^{-\mu(\mu-1)/2},$$

the q-Gamma integral of W. Hahn[1]. For that consider

$$\int_0^\infty t^{p-1}e_{q_2}(-t)M_n(s,A,\beta;xt\,|\,q_2)\,d_{q_2}t$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}\xi_k x^k}{[q]_{n-sk}}\int_0^\infty t^{p+k-1}e_{q_2}(-t)d_{q_2}t$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\,x^k}{[q]_{n-sk}}$$

$$\frac{(1-q_2)(q_2;q_2)_\infty\,q_2^{-(p+k-1)(p+k)/2}}{(q_2^{p+k};q_2)_\infty}.$$

$$\int_0^\infty t^{\lambda-1}e_{q_2}(-t)M_n(s,A,\beta;xt\,|\,q_2)d_{q_2}t=\frac{(1-q_2)(q_2;q_2)_\infty\,q_2^{-p(p-1)/2}}{[q_2^p,q_2]_\infty}.$$

$$\cdot\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}q_2^{-k(k+1)/2-pk}(q_2^p;q_2)_k(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\,x^k}{[q]_{n-sk}}. \quad \textbf{(6.3.4)}$$

Here also, following the same method of using the identity $e_q(x)E_q(x)=1$, one finds at once

$$\int_0^\infty t^{p-1}e_{q_2}(-t)e_{q_2}(xtq_2)E_{q_2}(xtq_2)M_n(s,A,\beta,xt\,|\,q_2)d_{q_2}t$$

$$=\int_0^\infty t^{p-1}e_{q_2}(-t)M_n(s,A,\beta;xt\,|\,q_2)d_{q_2}t. \quad \textbf{(6.3.5)}$$

The integral on the right hand side is equal to

$$\frac{(1-q_2)(q_2;q_2)_\infty\,q_2^{-p(p-1)/2}}{(q_2^p;q_2)_\infty}\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}q^{-snk}q_2^{k(k+1)/2+pk}(q_2^p;q_2)_k}{[q]_{n-sk}}.$$

$$.(Aq^{sk+sk\beta},q_2)_{n-sk}\,\xi_k\,x^k$$

whereas the integral on the left hand side becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}\, x^{i+j}}{(q_2;q_2)_i (q_2;q_2)_j}\int_0^{\infty} t^{p+i+j-1}\, e_{q_2}(-t)\, M_n(s,A,\beta;xt\,|\,q_2)\, d_{q_2}t$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i+j+j(j-1)/2}\, x^{i+j}}{(q_2;q_2)_i (q_2;q_2)_j}\frac{(1-q_2)(q_2;q_2)_\infty}{(q_2^{p+i+j};q_2)_\infty} q_2^{-(p+i+j)(p+i+j-1)/2}.$$

$$\sum_{k=0}^{[n/s]}\frac{(-1)^{sk} q^{-snk} q_2^{k(k+1)/2+(p+i+j)k} (q_2^{p+i+j};q_2)_k}{[q]_{n-sk}}(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\, x^k$$

Hence from (6.3.5), one finally obtains

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q_2^{i(i+1)/2+j^2+p(i+j)+ij}(q_2^p;q_2)_{i+j}}{(q_2;q_2)_i\,(q_2;q_2)_j}.$$

$$\cdot\sum_{k=0}^{[n/s]}\frac{q^{-snk} q_2^{k(k+1)/2+k(p+i+j)} (q_2^{p+i+j};q_2)_k\,(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\, x^k}{[q]_{n-sk}}$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk} q^{-snk} q_2^{k(k+1)/2+pk}\,(q_2^p;q_2)_k\,(Aq^{sk+sk\beta};q_2)_{n-sk}\,\xi_k\, x^k}{[q]_{n-sk}}.$$