CHAPTER – 1

INTRODUCTION

1.1 INTRODUCTION

It has been said that one of the main goals of mathematics is to isolate and study functions which are both interesting in their own rights and whose influence pervades large area of Mathematics and Physics. For Greeks such special functions were the circular functions. Since the time of Bernoulli, Euler, and Legendre, theta functions, classical orthogonal polynomials and other associated functions have been added to the list. These functions arise from the interplay of group theory and differential equations, and this may account for their ubiquity. There are in fact, many other functions which have been greately studied. Their origins are (appearing to be) mainly differential or other functional equations. Howevere the function which arose from the study of infinite (geometric) series is the Hypergeometric function ${}_2F_1(a,b;c;x)$ due to C. F. Gauss.

It is defined as

$$ {}_2F_1\left[\begin{matrix} a,b;x \\ c; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n, \quad |x|<1, \qquad \textbf{(1.1.1)} $$

where $(a)_n$ is Appell's symbol (or pochhamer symbol) defined by:

$$(a)_n = \begin{cases} a\,(a+1)\ldots.(a+n-1); & \text{if n is a positive integer} \\ 1; & \text{if } n = 0 \\ \Gamma(a+n)\,/\,\Gamma(a); & \text{for arbitrary non zero 'a' and n.} \end{cases} \qquad \textbf{(1.1.2)}$$

The series on the right hand side in (1.1.1) is called the Hypergeometric series or Gauss series.

The hypergeometric function ${}_2F_1 \equiv F$ is a solution of the second order linear differential equation:

$$x(1-x)\frac{d^2F}{dx^2} + [c-(a+b+1)x]\frac{dF}{dx} - abF = 0,$$

called the hypergeometric differential equation or Gauss' equation. Its equivalent form is:

$$[\,\theta\,(\theta+c-1) - x\,(\theta+a)\,(\theta+b)\,]\,F = 0, \qquad \theta = x\frac{d}{dx}. \qquad \textbf{(1.1.3)}$$

A generalized hypergeometric function has a series represenatation $\sum_{n=0}^{\infty} c_n$ with c_{n+1}/c_n a rational function of n. The ratio c_{n+1}/c_n can be factored and it is usually written as:

$$\frac{c_{n+1}}{c_n} = \frac{(a_1+n)\ldots\ldots.(a_p+n)\;x}{(b_1+n)\ldots\ldots.(b_q+n)\,(n+1)}. \qquad \textbf{(1.1.4)}$$

If $c_0 = 1$, then using the pochhamer symbol given in (1.1.2), the equation (1.1.4) can be solved for c_n as:

$$c_n = \frac{(a_1)_n\ldots\ldots(a_p)_n\;x^n}{(b_1)_n\ldots\ldots(b_q)_n\,n!},$$

and the series form becomes:

$$ {}_pF_q\left[\begin{matrix} a_1, & a_p, & x \\ b_1 \ldots\ldots b_q; & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \quad (a_p)_n \, x^n}{(b_1)_n \cdots (b_q)_n \, n!} . \qquad \textbf{(1.1.5)} $$

The infinite series above in (1.1.5) converges in one of the following situations.

(i) $|x| < \infty$, if $p \leq q$ (ii) $|x| < 1$ if p = q+1

(iii) $|x| = 1$ *if* $\text{Re}\,(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i) > 0$

When at least one of the numerator parameters is a negative integer then the above infinite series terminates and thus represents a polynomial.

The differential equation satisfied by w = ${}_pF_q[x]$ is:

$$ [\theta \prod_{j=1}^{q} (\theta + b_j - 1) - x \prod_{i=1}^{p} (\theta + a_i)]\, w = 0, \quad \theta = x\frac{d}{dx}, \qquad \textbf{(1.1.6)} $$

which is extension of equation (1.1.3).

The field of special functions is rich in polynomials. Amongst these polynomials, the Laguerre polynomial $L_n^{(\alpha)}(x)$, the Hermite polynomial $H_n(x)$, the Legendre polynomial $P_n(x)$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, the Gegenbauer polynomial $C_n^{\upsilon}(x)$ etc. and the family of Bessel functions are known as the fundamental functions of Mathematical physics; as they arise from particular phenomenon (Andrews [1], Lebedev [1], M.L. Boas [1], Simmons [1]). These polynomials were subsequently taken up for further study from the point of view of examining various properties (Olver [1], Rainville [1], Wang & Guo [1], Chihara [1]) such as generating function relations, Orthogonality, Rodrigue's formula, recurrance relations, zeros, inverse series relations (i.e. expansion of x^n in a series of the polynomial), various integral representations, summation formulas, differential equation etc.

The flavour of ${}_pF_q[x]$ is that a number of orthogonal polynomials / functions (and non-orthogonal ones too!) which are hypergeometric in character, are contained in it.

The following is the list of the explicit forms of the aforementioned polynomials.

Laguerre Polynomial

$$L_n^{\alpha}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n;1+\alpha,x) = \sum_{k=0}^{n} \frac{(-n)_k (1+\alpha)_n}{(1+\alpha)_k\, n!\, k!} x^k . \tag{1.1.7}$$

Konhauser Polynomial

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(n+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)}, \tag{1.1.8}$$

where $\alpha > -1$, and k is positive integer.

Hermite Polynomial

$$H_n(x) = (2x)^n\, {}_2F_0\left(\frac{-n}{2}, \frac{-n+1}{2}; \_\_\_\_; -x^{-2}\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!\, (2x)^{n-2k}}{k!\,(n-2k)!}, \tag{1.1.9}$$

Jacobi Polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (1+\alpha+\beta+n)_k (1+\alpha)_n}{(1+\alpha)_k\, n!\, k!\, 2^k}(1-x)^k . \qquad \textbf{(1.1.10)}$$

Legendre Polynomials

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k (1-x)^k}{k!\,k!\;2^k} . \qquad \textbf{(1.1.11)}$$

Alternatively,

$$P_n(x) = \frac{(1/2)_n (2x)^n}{n!}\,{}_2F_1\left(\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}; \frac{1}{2}-n; -x^{-2}\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (1/2)_{n-k} (2x)^{n-2k}}{k!(n-2k)!} . \qquad \textbf{(1.1.12)}$$

Gegenbauer Polynomial

$$C_n^{\nu}(x) = \frac{(\nu)_n (2x)^n}{n!}\,{}_2F_1\left(\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}; 1-\nu-n; -x^{-2}\right)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (1/2)_{n-k} (2x)^{n-2k}}{k!(n-2k)!} . \qquad \textbf{(1.1.13)}$$

Hahn Polynomial (Andrews, G.E.[1], Gasper, G.[1], W.Hahn [1])

$$Q_n(x;\alpha,\beta,N) = {}_3F_2(-n, 1+\alpha+\beta+n, -x; 1+\alpha, -N; 1)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (1+\alpha+\beta+n)_k (-x)_k}{(1+\alpha)_k (-N)_k\, k!} \qquad (n = 0,1,2,\ldots,N) \qquad \textbf{(1.1.14)}$$

Racah Polynomial (Askey and Wilson [1])

$$R_n(x(x+\nu+\delta+1);\alpha,\beta,\nu,\delta) = {}_4F_3\left[\begin{matrix} -n, 1+\alpha+\beta+n, x+\nu+\delta+1, -x, \\ \\ 1+\alpha, \beta+\delta+1, \nu+1; \end{matrix} \; 1\right]$$

$$= \sum_{k=0}^{n} \frac{(-n)_k (1+\alpha+\beta+n)_k (x+\nu+\delta+1)_k (-x)_k}{(1+\alpha)_k (\beta+\delta+1)_k (\nu+1)_k \; k!}.$$

(1.1.15)

Wilson Polynomial (Askey and Wilson [1])

$$P_n(x^2) = (a+b)_n (a+c)_n (a+d)_n \, {}_4F_3\left[\begin{matrix} -n, a+b+c+d+n-1, a+ix, a-ix, \\ \\ a+b, a+c, a+d; \end{matrix} \; 1\right]$$

$$= (a+b)_n (a+c)_n (a+d)_n \sum_{k=0}^{n} \frac{(-n)_k (a+b+c+d+n-1)_k (a+ix)_k (a-ix)_k}{(a+b)_k (a+c)_k (a+d)_k}.$$

(1.1.16)

Just as the Racah Polynomail $R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$ and the Wilson polynomial $P_n(x^2)$, given in (1.1.15) and (1.1.16), are extensions of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, another two worth mentioning extensions of the Jacobi polynomial due to H.M. Srivastava [5], which he referred to as "Extended Jacobi polynomials" are:

$$\mathcal{F}^{(c)}_{n,l,s}[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q : x] = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk} (c+n)_{lk} (\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_q)_k \, k!} x^k, \quad \textbf{(1.1.17)}$$

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[a_1,\ldots,a_p;b_1,\ldots,b_q : x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (a_1)_k \cdot (a_p)_k}{(1+\beta-n\alpha)_{lk} (b_1)_k \ldots (b_q)_k} \frac{x^k}{k!}. \quad \textbf{(1.1.18)}$$

These polynomials can also be written in an elegant form, using the hypergeometric function notation.

$$\mathcal{F}^{(c)}_{n,l,s}[\alpha_1,...,\alpha_p;\beta_1,..,\beta_q : x] = {}_{l+s+p}F_q\left[\begin{matrix}\Delta(s;-n),\Delta(l;c+n),\alpha_1,..,\alpha_p, \\ \beta_1,..,\beta_q;\end{matrix}\ x\right],$$

(1.1.19)

$$\mathcal{H}^{(\alpha,\beta)}_{n,l,m}[a_1,...,a_p;b_1,...,b_q : x] = {}_{m+p}F_{q+l}\left[\begin{matrix}\Delta(m;-n),\alpha_1,..,\alpha_p, \\ \Delta(l;1+\beta-n\alpha),\beta_1,...,\beta_q,\end{matrix}\ x\right],$$

(1.1.20)

where $\Delta(m,\lambda)$ denotes the sequence of m parameters $\frac{\lambda}{m},\frac{\lambda+1}{m},...,\frac{\lambda+m-1}{m}$.

1.2 INVERSE SERIES RELATIONS

Let $\{A_n\}$ and $\{B_n\}$ be two sequences such that

$$A_n = \sum_{k=0}^{N} a(n,k)B_k \quad \text{and} \quad B_n = \sum_{k=0}^{N} b(n,k)A_k, \qquad \textbf{(1.2.1)}$$

where N may be either finite of infinite.

The pair of series relations in (1.2.1) is known as a pair of inverse series relations, wherein each one of the series is called an inverse series (companion series) of the other. Inverse series relations are extensively used in the study of combinatorial identities (Riordan [2]). The use of inverse series relations can also be seen in Approximation theory, Distribution theory, Partition theory, Coding theory (Sloane [1]) and in Probability theory (Feller [1]). Some utilities of the inverse series relations are as below:

(1) Each pair implies a series orthogonal relation which itself may generate one or more identities.

(2) An identity having the form of one member of an inverse pair has a companion (which may be itself). If the identity is known and distinct from the companion, a new identity is found through the inverse series relation.

(3) If the identity is to be proved, the prover has a choice, that the proof of the inverse may be simpler, then inverse relations offer an alternative proof of a given identity.

(4) Particular choices of the variables (parameters) in a given inverse pair may serve to generate new identities, these choices may be suggested by one of the several methods of proving an inverse relation.

(5) The inverse relations may be transformed to the pairs giving the explicit representation of a polynomial and its inverse, expressing the expansion of x^n as a finite series of the corresponding polynomial.

(6) The expansion of x^n in the series of various orthogonal and non-orthogonal polynomials may be used to approximate various functions as a series of polynomials under consideration.

The well-known binomial expansion formulae

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad and \quad x^n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (x+1)^k$$

suggest a general inversion pair, which is very simple in the form:

$$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} v_k; \qquad v_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_k. \qquad \textbf{(1.2.2)}$$

With the above introduction of the inverse series relations, it is now straight forward to note that the defining relations of the polynomials in (1.1.7) to (1.1.16) are one of the series relations of the inverse pair of the type (1.2.1). Their corresponding inverse series relations have been obtained through various methods such as: generating function relation, summation formula, orthogonal property, difference and shift operators and recurrence relations (Rainville [1], Riordan [2]).

The pairs of inverse series relations of the polynomials defined by (1.1.7) to (1.1.16) are listed below.

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{(1+\alpha)_k (n-k)! k!}, \quad x^n = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n}{(n-k)!(1+\alpha)_k} L_k^{(\alpha)}(x) \qquad \textbf{(1.2.3)}$$

(inverse pair of Laguerre polynomial),

$$\left.\begin{aligned} Z_n^{\alpha}(x,k) &= \frac{\Gamma(n+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)} \\ x^{nk} &= \Gamma(kn+\alpha+1)\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{Z_j^{\alpha}(x,k)}{\Gamma(j+\alpha+1)} \end{aligned}\right\} \quad (1.2.4)$$

(inverse pair of Konhauser polynomial)

$$H_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!\,(2x)^{n-2k}}{k!\,(n-2k)!}; \qquad x^n=\sum_{k=0}^{[n/2]}\frac{n!\,H_{n-2k}(x)}{2^n k!(n-2k)!} \quad (1.2.5)$$

(inverse pair of Hermite polynomial)

$$\left.\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}\frac{(-n)_k(1+\alpha+\beta)_k(1+\alpha)_n}{(1+\alpha)_k 2^k k!\,n!}(1-x)^k, \\ \frac{(1-x)^n}{2^n(1+\alpha)_n} &= \sum_{k=0}^{n}\frac{(-n)_k(1+\alpha+\beta)_k(1+\alpha+\beta+2k)}{(1+\alpha+\beta)_{n+k+1}(1+\alpha)_k}P_k^{(\alpha,\beta)}(x) \end{aligned}\right\} \quad (1.2.6)$$

(inverse pair of Jacobi polynomial)

$$\left.\begin{aligned} P_n(x) &= \sum_{k=0}^{[n/2]}\frac{(-1)^k(1/2)_{n-k}(2x)^{n-2k}}{k!(n-2k)!}, \\ (2x)^n &= \sum_{k=0}^{[n/2]}\frac{(2n-4k+1)\,n!\,P_{n-2k}(x)}{(3/2)_{n-k}\,k!} \end{aligned}\right\} \quad (1.2.7)$$

(inverse pair of Legendre polynomial)

$$\left.\begin{aligned} C_n^{v}(x) &= \sum_{k=0}^{[n/2]}\frac{(-1)^k(v)_{n-k}(2x)^{n-2k}}{k!\,(n-2k)!}, \\ (2x)^n &= \sum_{k=0}^{[n/2]}\frac{(v+n-2k)\,n!}{(v)_{n-k+1}\,k!}C_{n-2k}^{v}(x) \end{aligned}\right\} \quad (1.2.8)$$

(inverse pair of Gegenbauer polynomial)

$$
\left.\begin{aligned}
Q_n(x,\alpha,\beta,N) &= \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+n)_k(-x)_k}{(1+\alpha)_k(-N)_k k!}, \\
\frac{(-x)_n}{(1+\alpha)_n(-N)_n} &= \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)}{(1+\alpha+\beta+2k)_{n+1} k!} Q_k(x,\alpha,\beta,N)
\end{aligned}\right\} \quad \textbf{(1.2.9)}
$$

(inverse pair of Hahn polynomial (Gasper [1])

$$
R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+n)_k(x+\gamma+\delta+1)_k(-x)_k}{(1+\alpha)_k(\beta+\delta+1)_k(\gamma+1)_k k!}
$$

(1.2.10)

$$
\frac{(-x)_n(x+\gamma+\delta+1)_n}{(1+\alpha)_n(1+\beta+\delta)_n(1+\gamma)_n} = \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)}{(\alpha+\beta+k+1)_{n+1} k!} R_k(x(x+\gamma+\delta+1),\alpha,\beta,\gamma,\delta)
$$

(inverse pair of Racah polynomial (Askey and Wilson [1])

$$
P_n(x^2) = (a+b)_n(a+c)_n(a+d)_n
$$

$$
= \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+n-1)_k(a+ix)_k(a-ix)_k}{(a+b)_k(a+c)_k(a+d)_k(a+b+c+d+k-1)_{n+1} k!},
$$

(1.2.11)

$$
\frac{(a+ix)_n(a-ix)_n}{(a+b)_n(a+c)_n(a+d)_n} = \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+2k-1)_k P_k(x^2)}{(a+b)_k(a+c)_k(a+d)_k(a+b+c+d+k-1)_{n+1} k!}
$$

(Inverse pair of Wilson polynomial (Askey and Wilson [1])

$$
\mathcal{F}^{(c)}_{n,l,s}[\alpha_1,\ldots\alpha_p;\beta_1,\ldots\beta_q : x] = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(c+n)_{lk}(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k k!} x^k
$$

(1.2.12)

$$
\frac{(sn)!(\alpha_1)_n\ldots(\alpha_p)_n}{n!(\beta_1)_n\ldots(\beta_q)_n} x^n = \sum_{k=0}^{sn} \frac{(-sn)_k(c+k+(lk/s))}{(c+k)_{lk+1} k!} \mathcal{F}^{(c)}_{k,l,s}[\alpha_1,\ldots\alpha_p;\beta_1,\ldots\beta_q : x]
$$

(Inverse pair of Extended Jacobi polynomial)

A systematic study of the inverse series relations was taken up for the first time in the middle of the 20th century. Actually, it appears from the works of Gould ([1] to [6]) that initially such relations were merely an out come of a study of binomial series transformations; but later on, an independent development took place, and as a result of that a number of inverse pairs were discovered and also studied at length by Gould ([1] to [6]), Gould and Hsu [1], Carlitz ([1], [2]), Riordan [2], and others. The main aim however, of their study were to obtain combinatorial identities and/or to obtain inverse series relations of particular polynomials. A brief account of this development of the subject is given below.

In 1956, in an attempt to generalize the Vandermonde's convolution identity

$$\sum_{j=0}^{n}\binom{r}{j}\binom{m}{n-j}=\binom{r+m}{n}$$

Gould [1,Eq. (7), p. 85] proved that

$$\sum_{k=0}^{n} A_k(a,b)z^k = x^a\ , \qquad \textbf{(1.2.13)}$$

where

$$A_k(a,b)=\frac{a}{a+bk}\binom{a+bk}{k} \text{ and } z=(x-1)x^{-b}. \qquad \textbf{(1.2.14)}$$

By making use of the result (1.2.13), he obtained a binomial series transformation as well as its inverse transformation ([3, theorems 1 and 2]), where he deduced that,

$$\left.\begin{array}{l}\phi(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{n}\psi(k)\\ \text{if and only if}\\ \psi(n)=\sum_{k=0}^{n}(-1)^k\binom{a+bn}{k}^{-1}A_{n-k}(a+bk-k,b)\,\phi(k)\end{array}\right\}\tag{1.2.15}$$

wherein $A_k(a,b)$ is same as defined in (1.2.14).

The orthogonal series relation viz.

$$\sum_{k=0}^{n}(-1)^k A_{n-k}(a+bk-k,b)\binom{a}{k}=\binom{0}{n}\tag{1.2.16}$$

supplied by the pair (1.2.15) was further used by Gould who, in 1962, proved a more general pair of inverse relation ([4,p.394]) which is given below,

$$\left.\begin{array}{l}F(a)=\sum_{k=0}^{M}(-1)^k A_k(a,b)f(a+bk-k)\\ \text{if and only if}\\ f(a)=\sum_{k=0}^{M}\binom{a}{k}F(a+bk-k)\end{array}\right\}\tag{1.2.17}$$

where M=[a/(1-b)] is finite if 'a' is positive and 'b' is zero or a negative integer, otherwise $M=\infty$.

This general pair possesses a number of particular cases, for example when b = 2, one finds

$$\left.\begin{array}{l}F(a)=\sum_{k=0}^{M}(-1)^k A_k(a,2)\,f(a+k)\\ \Leftrightarrow\\ f(a)=\sum_{k=0}^{M}\binom{a}{k}F(a+k).\end{array}\right\}\tag{1.2.18}$$

Gould has discussed other special cases in [4, p.395].

By doing a slight modification in (1.2.18) Gould introduced in 1964 yet another inversion pair [5,p.326]:

$$\left.\begin{aligned} &G(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+n+bk}{n}f(k) \\ &\text{if and only if} \\ &f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bn+k}{k}^{-1}\frac{a+bk+k+1}{a+bn+k+1}G(k) \end{aligned}\right\} \qquad \textbf{(1.2.19)}$$

and thereby showed that the Bessel polynomial

$$\binom{a+n}{n}Y_n^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{a+n+k}{k}\binom{a+k}{k}k!(-x/2)^k\ , \qquad \textbf{(1.2.20)}$$

the Legendry polynomial

$$P_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\left(\frac{1-x}{2}\right)^k \qquad \textbf{(1.2.21)}$$

and the Chebyshev polynomial $U_n(x) = \sin(n+1)\theta/\sin\theta$, where $x=\cos\theta$, possess the inverse series relations given by:

$$\binom{a+2n}{n}\binom{a+n}{n}n!\left(\frac{x}{2}\right)^n=\sum_{k=0}^{n}(-1)^k\frac{a+2k+1}{a+2n+1}\binom{a+2n+1}{n-k}\binom{a+k}{k}Y_k^{(a)}(x), \qquad \textbf{(1.2.22)}$$

$$\binom{2n}{n}\left(\frac{1-x}{2}\right)^n=\sum_{k=0}^{n}\frac{2k+1}{2n+1}\binom{2n+1}{n-k}P_k(x), \qquad \textbf{(1.2.23)}$$

$$2^n(1-x)^n=\sum_{k=0}^{n}(-1)^k\frac{k+1}{n+1}\binom{2n+2}{n-k}U_k(x) \qquad \textbf{(1.2.24)}$$

respectively.

In 1965, he introduced a generalized Humbert polynomial

$$P_n(m,x,y,p,C)=\sum_{k=0}^{[n/m]}\binom{p-n+mk}{k}\binom{p}{n-mk}C^{p-n-k+mk}y^k(-mx)^{n-mk} \quad \textbf{(1.2.25)}$$

and obtained its inverse series in the form (Gould [6]):

$$\binom{p}{n}(-mx)^n=\sum_{k=0}^{[n/m]}(-1)^k\binom{p-n+k}{k}\frac{p-n+mk}{p-n+k}C^{n-k-p}y^kP_{n-mk}(m,x,y,p,C) \quad \textbf{(1.2.26)}$$

by establishing a novel type of inversion pair:

$$\left.\begin{aligned} &F(n)=\sum_{k=0}^{[n/m]}A_k(p-n,m)f(n-mk)\\ &\text{if and only if}\\ &f(n)=\sum_{k=0}^{[n/m]}(-1)^kA_k(p-n,1)F(n-mk). \end{aligned}\right\} \quad \textbf{(1.2.27)}$$

The polynomials of Humbert: $\Pi_{n,m}^{\upsilon}(x)=P_n(m,x,1,\upsilon,1)$,

Kinney: $P_n^{\upsilon}(m,x)=P_n(m,x,-1/m,1)$, Pincherle: $P_n(x)=P_n(3,x,1,1/2,1)$,

Gegenbauer: $C_n^{\upsilon}(x)=P_n(2,x,1,-\upsilon,1)$, Legendre: $P_n(x)=P_n(2,x,1,-1/2,1)$ etc.

are specializations of the generalized Humbert polynomial (1.2.25), hence their inverse series relations follow immediately from the inverse series (1.2.26).

In 1973, the inversion pairs (1.2.15) and (1.2.19) were further extended in an elegant form by Gould and Hsu [1] who proved that if $\{a_i\}$ and $\{b_i\}$ be two sequences of numbers such that $\prod_{i=1}^{n}(a_i+xb_i)\equiv\psi(x,n)\neq 0$ for all non negatives x and n and $\psi(x,0)=1$ then

$$\left.\begin{array}{l} F(n)=\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k}\psi(k,n)\,G(k) \\ \text{if and only if} \\ G(n)=\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k}(a_{k+1}+kb_{k+1})\,\psi(n,k+1)^{-1}F(k). \end{array}\right\} \qquad \textbf{(1.2.28)}$$

In their work, Gould and Hsu [1] however do not discuss the reducibilities of (1.2.28) to the inverse series relations of various particular polynomials, although it can be shown that the inverse relations of the polynomials quoted in (1.1.7) to (1.1.16) are obtainable from this general pair.

For instance the inverse relations involving the Laguerre polynomial $L_n^{(\alpha)}(x)$ (1.1.7) may be obtained from (1.2.28) by setting $a_i=1$, $b_i=0$ for all i. Similarly the Jacobi polynomial and its inverse series relation (1.2.6) follow from (1.2.28) when $a_i=\alpha+\beta+i$ and $b_i=1$ for all i.

1.3 BASIC HYPERGEOMETRIC SERIES AND ASSOCIATED POLYNOMIALS

In 1812, Gauss introduced the hypergeometric series

$$1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{c(c+1)1\cdot 2}x^2+\ldots\ldots \qquad (c\neq 0,-1,-2,\ldots\ldots\ldots\ldots,|x|<1). \qquad \textbf{(1.3.1)}$$

Almost thirty years after this, E.Heine ([1],[2]) introduced an interesting extension of this series in the form:

$$1+\frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)}x+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)}x^2+\ \ldots$$

$$(c\neq 0,-1,-2,\ldots\ \ldots,|x|<1). \qquad \textbf{(1.3.2)}$$

Before defining this series, he defined a 'basic analogue' of a number 'a' in the form $[a,q]=\frac{1-q^a}{1-q}$ where the arbitrary number q$(q\neq 1)$ is called the **base.**

From this definition it is clear that as q→1, [a;q]→a, and that the Heine's series (1.3.2) approaches the Gauss' hypergeometric series (1.3.1).

Thus, Heine's series defines 'a q – analogue' or 'a basic analogue' of the Gauss series and therefore the Heine's series is called a basic hypergeometric series (BHS) or a q – hypergeometric series.

Just as it happened for the Gauss series that it was known in other particular forms before its introduction, the q – series (1.3.2) was also known in special forms prior to its introduction.

For example, the identity

$$1+\sum_{n=1}^{\infty}(-1)^n\left\{q^{n(3n-1)/2}+q^{n(3n+1)/2}\right\}=\prod_{n=1}^{\infty}(1-q^n)$$

was given by Leonhard Euler in 1748 A.D; and also the triple product identity

$$\prod_{n=0}^{\infty}\left\{(1-xq^n)(1-q^{n+1}x^{-1})(1-q^{n+1})\right\}=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}x^n$$

and the Theta Functions : $\theta_i(z,q)$, i =1, 2, 3, 4, were given by Carl Gustav Jacob Jacobi in 1829 A.D. But it was not until about sixteen years later that the field of BHS acquired an independent status when Heine ([1], [2], [3]) introduced the q - series (1.3.2) and carried out a systematic study of it. During his study, a basic analogue of binomial theorem, basic transformation formulas, basic analogue of the Gauss' summation formula, and also the q-contiguous functions relations (Gasper and Rahman [1]) were given. A q-Gamma function defined by him in the form

$$\Gamma_q(x) = \prod_{n=1}^{\infty} \frac{1-q^x}{1-q^{x+n-1}}$$

differs slightly from Thomae's definition [1]:

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{x+n-1}}.$$

Since then remarkable development have taken place in the field of BHS by many eminent researchers, among whom the names of F. H. Jackson, W. N. Bailey, D.B.Sears, L. J. Rogers, W. Hahn, L. J. Slater, L. Carlitz, H. Exton, R. P. Agarwal, G. E. Andrews, R. Askey, W.A. Al-Salam, H.M. Srivastava, A. Verma, M. E. H. Ismail, T. H. Koornwinder, J. A. Wilson, G. Gasper, S. C. Milne, M. Rahman, V. K. Jain are worth mentioning. This list, although not exhaustive would seem to be incomplete without taking note of S. Ramanujan, as quite a good number of formulae given by him may be viewed as the special cases of the results involving BHS.

Heine used the notation ϕ(a,b,c,q,x) to denote series in (1.3.2). However, the other notations:

$$_2\phi_1(a,b,c;q,x), \qquad \textit{and} \qquad {}_2\phi_1\begin{bmatrix} a,b;q,x \\ c; \end{bmatrix}$$

are frequently used, in terms of which the series (1.3.2) can be represented in the form:

$$_2\phi_1(a,b,c,q,x) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b,q)_n}{(c;q)_n\,(q,q)_n} x^n , \qquad \textbf{(1.3.3)}$$

where $(a;q)_n \equiv [a]_n$ is a basic factorial function defined as below.

$$(a;q)_n = \begin{cases} (1-a)(1-aq).\ ..(1-aq^{n-1}), n = 1,2,3\ \ldots \\ 1, & n=0 \\ [a]_\infty/[aq^n]_\infty, & n \textit{ is arbitrary}, \end{cases}$$

in which $[a]_\infty \equiv (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, $0<q<1$.

A generalization of (1.3.3) which provides a basic analogue of (1.1.5) is an $_r\phi_s$ function defined as (Askey and Wilson [1], Gasper and Rahman [1]):

$$_r\phi_s\begin{bmatrix} a_1,\ldots\ldots,a_r;q,x \\ b_1,\ldots..b_s; \end{bmatrix} = \sum_{n=0}^{\infty} \frac{[a_1]_n\cdots.[a_r]_n.x^n}{[b_1]_n\ .\ .[b_s]_n[q]_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{s-r+1} . \qquad \textbf{(1.3.4)}$$

The infinite basic series in (1.3.4) converges for all x if $0<|q|<1$ and $r \le s$. If $0<|q|<1$ and $r=s+1$ then it converges for $|x| < 1$. The various specializations of ${}_r\phi_s[x]$ function include the basic exponential functions defined by (W.Hahn [2], Gasper and Rahman [1])

$$e_q(x) = {}_1\phi_0(0,\_\_\_;q,x) = \sum_{k=0}^{\infty} \frac{x^k}{[q]_k} = \frac{1}{[x]_\infty}, (|x|<1)$$

and

$$E_q(x) = {}_0\phi_0(\_\_;\_\_\_;q,x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{(-x^k)}{[q]_k} = [-x]_\infty;$$

the basic sine and cosine functions viz.

$$Sin_q x = \frac{x}{1-q}\, {}_0\phi_1(\_\_;q^3;q^2,-x^2), \quad Sin_q x = \frac{x}{1-q}\, {}_2\phi_1(0,0,q^3,q^2,-x^2)$$

$$Cos_q x = {}_0\phi_1(\_\_;q^3;q^2,-qx^2), \quad Cos_q x = {}_2\phi_1(0,0;q^3;q^2,-x^2),$$

also, the basic Gamma and Beta functions are given by

$\Gamma_q(x) = [q]_\infty (1-q)^{1-x}\, {}_1\phi_0(0;\_\_\_;q^x)$ and $\beta_q(x,y) = (1-q)\, {}_1\phi_0(q^y,\_\_\_;q,q^x)$.

Moreover, the basic Bessel functions are expressible in the particular ${}_r\phi_s$ functions by means of the following relations.

$$J_\upsilon^{(1)}(x;q) = \frac{[q^{\upsilon+1}]_\infty}{[q]_\infty} (x/2)^\upsilon\, {}_2\phi_1(0,0;q^{\upsilon+1};q,-x^2/4) \quad \textbf{(1.3.5)}$$

and

$$J_\upsilon^{(2)}(x;q) = \frac{[q^{\upsilon+1}]_\infty}{[q]_\infty} (x/2)^\upsilon\, {}_0\phi_1(\_\_;q^{\upsilon+1};q,-x^2 q^{\upsilon+1}/4) \quad \textbf{(1.3.6)}$$

(Hahn ([1], [2]), Gasper and Rahman [1], and H.Exton [1]).

Since the choice $a_i(\equiv q^{a_i}) = q^{-n}$, for at least one i $(1 \le i \le r)$, n= 0,1,2..... reduces the infinite series in (1.3.4) to a terminating series, the basic hypergeometric representations of various basic polynomials may be obtained by specializing the parameters in (1.3.4). As an illustration, putting r = 1, s = 1, $a_1 = q^{-n}$, $b_1 = \alpha q(\equiv q^{\alpha+1})$, one gets by replacing x by $-xq^{n+\alpha+1}$, a basic Laguerre polynomial,

$$L_n^{(\alpha)}(x;q) = \frac{[\alpha q]_n}{[q]_n} \, {}_1\phi_1 \, (q^{-n}; \alpha q; q, -xq^{\alpha+n+1}). \qquad \textbf{(1.3.7)}$$

This polynomial was studied by D.S.Moak [1] and also it was taken into account by Al-Salam and Verma [1] who constructed a pair of biorthogonal polynomials: $Z_n^{\alpha}(x;k\,|\,q)$ and $Y_n^{\alpha}(x,k\,|\,q)$ which are known as q-Konhauser polynomials. It is to be noted here that the polynomial

$$Z_n^{(\alpha)}(x;q) = \frac{[\alpha q]_n}{[q^k;q^k]_n} \sum_{j=0}^{n} \frac{(q^{-kn}, q^k)_j}{(q^k;q^k)_j [\alpha q]_{jk}} \, .q^{kj(n+\alpha+1)+kj(kj-1)/2} \, x^{kj} \qquad \textbf{(1.3.8)}$$

reduces to the polynomial (1.3.7) when k = 1.

Amongst the other 'ordinary' polynomials, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is worth noting here for, it possesses two basic analogues. One of which is known as the 'little' q-Jacobi polynomial (Gasper and Rahman [1]) given by:

$$p_n(x;\alpha,\beta;q) = {}_2\phi_1 \, (q^{-n}, \alpha\beta q^{n+1}, \alpha q; q, xq) \qquad \textbf{(1.3.9)}$$

and other is given by

$$P_n(x;a,b,c;q) = {}_3\phi_2(q^{-n}, abq^{n+1}, x, aq, cq, q, q) \quad \textbf{(1.3.10)}$$

known as the 'big' q-Jacobi polynomial.

The little q-Jacobi polynomial (1.3.9) with β = 0 provides two more basic analogues of the Laguerre polynomial $L_n^{\alpha}(x)$; they are the Wall polynomial and the Stieltjes-Wigert polynomial (Gasper and Rahman [1,p.196]):

$$W_n(x;a,q) = (-1)^n [a]_n \, q^{n(n+1)/2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{q^{j(j-1)/2}(-q^{-n}x)^j}{[a]_j} \quad \textbf{(1.3.11)}$$

and

$$S_n(x,p,q) = (-1)^n [p]_n \, q^{-n(2n+1)/2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{q^{j^2}}{[p]_j} \left(-x\sqrt{q}\right)^j . \quad \textbf{(1.3.12)}$$

Besides the 'big' q-polnomial, the basic Hahn polynomial defined as below, also has a ${}_3\phi_2$ representation.

$$Q_n(x;\alpha,\beta,N \mid q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}; q, q \\ \alpha q, q^{-N}; \end{matrix}\right] \quad \textbf{(1.3.13)}$$

is a q-analogue of (1.1.14).

Recently, in the study of general orthogonal q-polynomials, R. Askey and J.A.Wilson ([1]) considered the q-extensions of the Racah polynomial and Wilson polynomial (stated in (1.1.15) and (1.1.16)) in the forms:

$$W_n(x,a,b,c,N \mid q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N}, q, q \\ \alpha q, q^{-N}, bcq, \end{matrix}\right] \quad \textbf{(1.3.14)}$$

and

$$\frac{P_n(x;a,b,c,d\mid q)}{a^{-n}[ab]_n[ac]_n[ad]_n} = {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}, q, q \\ ab, ac, ad, \end{matrix}\right]. \qquad \textbf{(1.3.15)}$$

which they call q-Racah polynomial, and Askey-Wilson polynomial respectively. It can be shown easily that these polynomials contain among the other polynomials, the q-Jacobi polynomials, q-Hahn polynomial and the continuous q-Jacobi polynomial considered by M. Rehman [1] in the form:

$$P_n^{(\alpha,\beta)}(\cos\theta;q) = \frac{[\alpha q]_n[-\beta q]_n}{[q]_n[-q]_n}\ {}_4\phi_3\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}, q, q \\ \alpha q, -\beta q, -q, \end{matrix}\right].$$

1.4 BASIC INVERSE RELATIONS

In the early sixties when the inverse series relations were being discovered by Gould ([3], [4], [5]), Riordan [2] and others (for example Stanton and Sprott [1]), Carlitz studied several inverse series relations and their basic analogues from the point of view of deriving the inverse relations involving certain polynomials. During his study, he was led to several more general inversion pairs. Out of these the following basic pairs are worth mentioning [3. p. 196].

$$U_n = \sum_{k=0}^{[n/2]} \begin{bmatrix} n \\ k \end{bmatrix} V_{n-2k}$$

if, and only if **(1.4.1)**

$$V_n = \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)/2} \frac{1-q^n}{1-q^{n-k}} \begin{bmatrix} n \\ k \end{bmatrix} U_{n-2k}$$

and

$$U_n = \sum_{k=0}^{[n/2]} \left\{ \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right\} V_{n-2k}$$

implies **(1.4.2)**

$$V_n = \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)/2} U_k = \sum_{k=0}^{[n/2]} \left\{ \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right\} U_{n-2k} .$$

It may be observed that (1.4.1) and (1.4.2) provide basic analogues of Chebyshev class of inverse pairs studied by Riordan [2].

In one of his other papers on q-inverse relations, Carlitz [2] proved a very general as well as useful result in the form of a basic analogue of the pair (1.2.28) due to Gould and Hsu. The result states that if $a_i + q^{-x}b_i \neq 0$, , and $\psi(x,n,q) = \prod_{i=1}^{n} (a_i + q^{-x} b_i)$, then

$$f(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \psi(k,n,q)\ g(k)$$

if and only if **(1.4.3)**

$$g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_{k+1} + q^{-k} b_{k+1})}{\psi(n,k+1,q)} f(k) .$$

With the aid of this pair, he obtained certain particular inverse series relations including the pair (Carlitz [2,p.898]):

$$f(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} a+n+k \\ n \end{bmatrix} g(k)$$

if and only if **(1.4.4)**

$$g(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1-q^{a+2k+1}}{1-q^{a+n+k+1}} \begin{bmatrix} a+n+k \\ k \end{bmatrix}^{-1} f(k)$$

which provides a basic analogue of the pair (1.2.19) when b=1. He also obtained a basic analogue of the pair (1.2.15) in the form:

$$\left.\begin{aligned} f(n) &= \sum_{k=0}^{n} (-1)^k q^{\lambda k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \begin{bmatrix} a+k\lambda \\ n \end{bmatrix} g(k) \\ &\text{if and only if} \\ g(n) &= \sum_{k=0}^{n} (-1)^k q^{\lambda k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{1-q^{a+\lambda k-k}}{1-q^{a+nk-k}} \begin{bmatrix} a+n\lambda \\ k \end{bmatrix}^{-1} f(k) \end{aligned}\right\} \quad \textbf{(1.4.5)}$$

by means of a more general inverse relation (Carlitz [2,p.900]):

$$\left.\begin{aligned} f(n) &= \sum_{k=0}^{n} (-1)^k q^{\lambda k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \psi(-k,n,q^\lambda)\, g(k) \\ &\Leftrightarrow \\ g(n) &= \sum_{k=0}^{n} (-1)^k q^{\lambda k(k-2n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (a_{k+1} + q^{\lambda k} b_{k+1}) \frac{f(k)}{\psi(-n,k+1,q^\lambda)} \end{aligned}\right\} \quad \textbf{(1.4.6)}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_\lambda = \frac{(1-q^{n\lambda})(1-q^{(n-1)\lambda})\ldots\ldots(1-q^{(n-k+1)\lambda})}{(1-q^{k\lambda})(1-q^{(k-1)\lambda})\ldots\ldots(1-q^{2\lambda})(1-q^{\lambda})}, \lambda \neq 0.$$

The other consequences of the pair (1.4.3) although not discussed in (Carlitz [2]), are worth mentioning here. They are the inverse series relations of the basic Jacobi polynomials (1.3.9) and (1.3.10), the q-Hahn polynomial (1.3.13), the q-Racah polynomial and the Askey-Wilson polynomials mentioned in (1.3.14) and (1.3.15) respectively. As an illustration, replacing q by q^{-1} and then setting

$a_i = 1$, $b_i = -q^{\alpha+\beta+i}$ and $g(n)=[\alpha\beta q]_n\, x^n \,/\, [\alpha q]_n$ in (1.4.3), one finds after a little simplification, the following pair of inverse relations involving the little q-Jacobi polynomial.

$$\left.\begin{aligned} p_n(x;\alpha,\beta;q) &= \sum_{k=0}^{n} \frac{[q^{-n}]_k[\alpha\beta q^{n+1}]_k}{[\alpha q]_k[q]_k} x^k q^k \\ &\Leftrightarrow \\ x^n &= [\alpha q]_n \sum_{k=0}^{n} q^{nk} \frac{[q^{-n}]_k(1-\alpha\beta q^{2k+1})}{[\alpha q]_k[\alpha\beta q^{k+1}]_{n+1}} p_k(x;\alpha,\beta;q). \end{aligned}\right\} \quad \textbf{(1.4.7)}$$

In a similar manner, the inverse relations of the other polynomials may be obtained in the forms as given below:-

$$\left.\begin{aligned} Q_n(x;\alpha,\beta,N\,|\,q) &= \sum_{k=0}^{n} \frac{[q^{-n}]_k[\alpha\beta q^{n+1}]_k[q^{-x}]_k}{[\alpha q]_k[q^{-N}]_k[q]_k} x^k q^k \\ &\Leftrightarrow \\ [q^{-x}]_n &= [\alpha q]_n [q^{-N}]_n \sum_{k=0}^{n} q^{nk} \frac{[q^{-n}]_k[1-\alpha\beta q^{2k+1}]}{[\alpha\beta q^{k+1}]_{n+1}[q]_k} Q_k(x;\alpha,\beta;N\,|\,q). \end{aligned}\right\} \quad \textbf{(1.4.8)}$$

(pair of inverse relations of basic Hahn polynomial)

$$\left.\begin{aligned} R_n(\mu(x);\alpha,\beta;\gamma,\delta;q) &= \sum_{k=0}^{n} \frac{[q^{-n}]_k[\alpha\beta q^{n+1}]_k[q^{-x}]_k[\gamma\delta q^{x+1}]_k}{[\alpha q]_k[\beta\delta q]_k[\gamma q]_k[q]_k q^{-k}} q^k \\ &\Leftrightarrow \\ \frac{[q^{-x}]_n[q^{x+1}\gamma\delta]_n}{[\alpha q]_n[\beta\delta q]_n[\gamma q]_n} &= \sum_{k=0}^{n} q^{nk} \frac{[q^{-n}]_k[1-\alpha\beta q^{2k+1}]_k}{[\alpha\beta q^{k+1}]_{n+1}[q]_k} R_k(\mu(x);\alpha,\beta,\gamma,\delta,q) \end{aligned}\right\} \quad \textbf{(1.4.9)}$$

where $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$.

(pair of inverse relation of basic Racah polynomial)

$$\frac{P_n(\cos\theta,a,b,c,d\,|\,q)}{[ab]_n[ac]_n[ad]_n}=\sum_{k=0}^{n}\frac{[q^{-n}]_k[abcdq^{n-1}]_k[ae^{i\theta}]_k[ae^{-i\theta}]_k}{[ab]_k[ac]_k[ad]_k[q]_k}q^k$$

$\Leftrightarrow$ **(1.4.10)**

$$\frac{[ae^{i\theta}]_n[ae^{-i\theta}]_n}{[ab]_n[ac]_n[ad]_n}=\sum_{k=0}^{n}\frac{[q^{-n}]_k[abcdq^{n-1}]_k q^{nk}}{[abcdq^{k-1}]_{n+1}[ab]_k[ac]_k[ad]_k}P_k(\cos\theta,a,b,c,d\,|\,q).$$

(pair of inverse relation of Askey-Wilson polynomial)

On the other hand the substitutions a_l = 1 and b_l = 0 in (1.4.3) lead us to the inverse relations of the basic Lguerre polynomial: $L_n^{(\alpha)}(x,q)$ defined in (1.3.7), the Wall polynomial $W_n(x;a,q)$given in (1.3.11) and the Stieltjes-Wigert polynomial $S_n(x,p,q)$cited in (1.3.12). In fact, with the aforementioned substitutions, the pair (1.4.3) assumes the simplest type of pair as given below:

$$f(n)=\sum_{k=0}^{n}(-1)^k q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}g(k)\quad\Leftrightarrow\quad g(n)=\sum_{k=0}^{n}(-1)^k q^{k(k-2n+1)/2}\begin{bmatrix}n\\k\end{bmatrix}f(k)$$

(1.4.11)

whence the inverse relation of the above mentioned basic Laguerre polynomials is easily obtainable in the form given below by setting $g(n)=q^{n(n+1)/2+\alpha n}\frac{(1-q)^n x^n}{[\alpha q]_n}$.

$$\left.\begin{aligned} &L_n^{(\alpha)}(x;q) = \sum_{k=0}^{n} q^{k(k+1)/2} \frac{[q^{-n}]_k [\alpha q]_n (1-q)^k (q^{\alpha+n}x)^k}{[\alpha q]_k [q]_k [q]_n} \\ &\Leftrightarrow \\ &\frac{x^n}{[q]_n} = q^{-n\alpha - n(n+1)/2} \sum_{k=0}^{n} \frac{[q^{-n}]_k [\alpha q]_n}{[\alpha q]_k [\alpha]_k [q]_k} L_k^{(\alpha)}(x;q) \end{aligned}\right\} \quad \textbf{(1.4.12)}$$

Likewise, with $f(k) = \dfrac{x^k}{[a]_k [q]_k}$ and $(-1)^n [a]_n [q]_n \, q^{n(n+1)/2} \, g(n) = W_n(x;a,q)$

the above pair (1.4.11) gives

$$\left.\begin{aligned} &W_n(x;a,q) = (-1)^n [a]_n \, q^{n(n+1)/2} \sum_{k=0}^{n} (-1)^k \, q^{k(k-2n-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k}{[a]_k} \\ &\Leftrightarrow \\ &x^k = [a]_n \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{W_k(x;a,q)}{[q]_k} \end{aligned}\right\} \quad \textbf{(1.4.13)}$$

and similarly putting $g(k) = q^{k+(k^2/2)} x^k$ in (1.4.11) and comparing it with (1.3.12), one arrives at the inverse relations involving stieltjes-Wigert polynomial as mentioned below.

$$\left.\begin{aligned} &S_n(x;p,q) = (-1)^n q^{-n(2n+1)/2} [p]_n \sum_{k=0}^{n} (-1)^k \, q^{k^2+k/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k}{[p]_k} \\ &\Leftrightarrow \\ &q^{n^2+n/2} x^n = [p]_n \sum_{k=0}^{n} q^{k-nk+3k^2/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{S_k(x;p,q)}{[p]_k}. \end{aligned}\right\} \quad \textbf{(1.4.14)}$$

Finally, the following interesting results due to Gessel and Stanton [1] needs worth mentioning. They proved that

$$\text{if } A_n = \sum_{k=0}^{n} p(n,k)B_k \quad \text{and} \quad B_n = \sum_{k=0}^{n} r(n,k)A_k \qquad \textbf{(1.4.15)}$$

then

Theorem-1.

$$p(n,k)=\frac{q^{-nk}(Aq^{3k/2},q^{1/2})_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^{1/2};q^{1/2})_{3n-1}(1-Aq^{3k/2})}{[q]_{n-k}(Aq^{1/2};q^{1/2})_{2n+k}}.$$

Theorem-2.

$$p(n,k)=q^{-nk}\frac{(Aq^{2k};q)_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq;q)_{2n-1}(1-Aq^{2k})}{[q]_{n-k}(Aq;q)_{n+k}}.$$

Theorem-3.

$$p(n,k)=q^{-nk}\frac{(Aq^{k};q)_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^{k};q)_{n-k}}{[q]_{n-k}}.$$

Theorem-4.

$$p(n,k)=q^{-nk}\frac{(Aq^{k/2};q^{-1/2})_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^{(n+1)/2};q^{1/2})_{n-k-1}(1-Aq^{1/2})}{[q]_{n-k}}$$

Theorem-5.

$$p(n,k)=q^{-nk}\frac{(Aq^{2k/3};q^{-1/3})_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^{(2n+1)/3};q^{1/3})_{n-k-1}(1-Aq^{2k/3})}{[q]_{n-k}}$$

Theorem-6.

$$p(n,k)=q^{-nk/2}\frac{(Aq^{3k/2},q)_{n-k}}{(q^{1/2};q^{1/2})_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{((n-k)(n-k+1)+nk)/2}\frac{(Aq^{(3n-2)/2},q^{-1})_{n-k-1}(1-Aq^{3k/2})}{(q^{1/2};q^{1/2})_{n-k}}.$$

These inverse series relations were then unified in the form which is stated here as

Theorem-7.

$$p(n,k)=q^{-nk}\frac{(Aq^k h^k;h)_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^k h^{k-1},h^{-1})_{n-k-1}(1-Ah^k q^k)}{[q]_{n-k}}.$$

With h = $q^{\frac{1}{b}}$ in theorem 7, one gets back to theorems 1 to 6 by specializing b suitably.

To see this, take b=2. Then theorem-7 reduces to

$$p(n,k)=q^{-nk}\frac{(Aq^{3k/2},q^{1/2})_{n-k}}{[q]_{n-k}}$$

$\Leftrightarrow$

$$r(n,k)=(-1)^{n-k}q^{(n-k)(n-k+1)/2+nk}\frac{(Aq^{(3k-1)/2};q^{-1/2})_{n-k-1}(1-Aq^{3k/2})}{[q]_{n-k}}.$$

which is theorem-1.

Similarly, putting b=-2 theorem-7 yields theorem-4.

It is although not mentioned in their work (Gessel and Stanton[1]), the q-Racah polynomial (1.3.14) and Askey-Wilson polynomial (1.3.15) are contained in (1.4.15) wherein p(n,k) is given by theorem-7.