CHAPTER – 4

UNIFICATION OF CERTAIN GENERALIZED POLYNOMIALS AND THEIR PROPERTIES

4.1 INTRODUCTION

In 1983, I. Gessel and D. Stanton [1] proved a q-inversion pair in the form:

$$\phi(n) = \sum_{k=0}^{n} \frac{q^{-nk}(Aq^{k+k\beta};q^{\beta})_{n-k}}{[q]_{n-k}} \psi(k) \quad \textbf{(4.1.1)}$$

$\Leftrightarrow$

$$\psi(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k+1)/2+nk} (Aq^{n+(n-1)\beta}, q^{-\beta})_{n-k-1}}{(1-Aq^{k+k\beta})^{-1} [q]_{n-k}}, \quad \textbf{(4.1.2)}$$

with a view to unify several other inverse series relations corresponding to the cases $\beta = \frac{-1}{2}, \frac{-1}{3},\ 0,\ 1,\ \frac{1}{2}\ and\ 2.$

Here in this chapter, a study of the ordinary version of (4.1.1) will be taken up in an extended form, whence a general polynomials set will be defined. For this polynomials set the properties viz.

(i) Integral representations

(ii) Differential equation in θ-form, and

(iii) Inverse series relations

will be obtained in subsequent sections, which will be followed by the particularizations of these properties for special classes of the polynomials. The last section deals with the representation of this general polynomial in a series of the polynomial $S_n(\ell, m, \alpha, \beta; x)$ introduced in Chapter-2.

Now consider an ordinary form of (4.1.1) which may be given by:

$$f(n) = \sum_{k=0}^{n} \frac{\Gamma(A + n\beta + k)}{(n-k)!} g(k), \tag{4.1.3}$$

this can be put into an extended form:

$$F(n) = \sum_{k=0}^{[n/s]} \frac{\Gamma(A + sk + n\beta)}{(n - sk)!} G(k). \tag{4.1.4}$$

From this, we define a class of polynomials $\{M_n(s,A,\beta;x),\ n=0,1,2,\ldots\}$ by

$$M_n(s, A, \beta; x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\, \Gamma(A + sk + n\beta)}{(n - sk)!}\ \psi_k\ \ x^k. \tag{4.1.5}$$

It may be seen that this polynomial contains the following extended versions of (known) polynomials when the parameters involved therein are specialized suitably.

Extended Racah Polynomial

$$R_n^s(x(x+\gamma+\delta+1), \alpha, \beta, \gamma, \delta) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k(x+\gamma+\delta+1)_k}{k!(1+\alpha)_k(1+\gamma)_k(1+\beta+\delta)_k} \tag{4.1.6}$$

Extended Wilson's Polynomial

$$P_n^s(x^2) = (a+b)_n(a+c)_n(a+d)_n \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(a+b+c+d+n-1)_{sk}(a+ix)_k}{k!(a+b)_k(a+c)_k} \cdot \frac{(a-ix)_k}{(a+d)_k}. \tag{4.1.7}$$

These two polynomials include the following ones.

Extended Hahn Polynomial

$$Q_{n,s}(x;\alpha,\beta,N)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k}{k!(1+\alpha)_k(-N)_k},n=0,1,......,N \quad \textbf{(4.1.8)}$$

Extended Jacobi Polynomial

$$P_{n,s}^{(\alpha,\beta)}(x)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(1+\alpha)_n}{k!\ (1+\alpha)_k}\left(\frac{1-x}{2}\right)^k. \quad \textbf{(4.1.9)}$$

Extended Legendre Polynomial

$$P_n^s(x)=\sum_{k=0}^{[n/s]}\frac{(n+sk)!}{(n-sk)!k!k!}\left(\frac{x-1}{2}\right)^k. \quad \textbf{(4.1.10)}$$

Extended Bessel Polynomial

$$Y_n^s(x)=\sum_{k=0}^{[n/s]}\frac{(n+sk)!\ x^k}{(n-sk)!\ k!\ 2^k}. \quad \textbf{(4.1.11)}$$

Extended Laguerre Polynomial

$$L_{n,s}^{(\alpha)}(x)=(1+\alpha)_n\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}\ x^k}{k!\ (1+\alpha)_k}. \quad \textbf{(4.1.12)}$$

The integral representations of the polynomial $M_n(s,A,\beta;x)$ are derived in the following section.

4.2 INTEGRAL FORMS AND TRANSFORMATION FORMULA

In this section, two integral forms of the polynomial (4.1.5) are derived using two fundamental integrals- (i) the beta integral and (ii) integral representation of generalized hypergeometric function ${}_pF_q(z)$. These integrals are:

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 z^{x-1}(1-z)^{y-1}dz, \quad \text{Re}(x)>0, \quad \text{Re}(y)>0 \tag{4.2.1}$$

and

$${}_pF_q\left[\begin{matrix} a_1,\ldots,a_p, \\ b_1,\ldots,b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}.$$

$$\cdot {}_{p-1}F_{q-1}\left[\begin{matrix} a_2,\ldots,a_p; \\ b_2,\ldots,b_q, \end{matrix} zt\right]dt. \tag{4.2.2}$$

A simple transformation formula is also derived by making use of the formula (4.2.1).

Now for deriving the integral representation of the polynomial $M_n(s,A,\beta;x)$, replace this polynomial by $M_n(s,A,\beta;x)\,\Gamma(2A+n\beta)$, then

$$M_n(s,A,\beta;x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\Gamma(A+n\beta+sk)}{(n-sk)!\,\Gamma(2A+n\beta)}\,\psi_k\; x^k \tag{4.2.3}$$

$$= \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\psi_k\; x^k}{(n-sk)!\,\Gamma(A-sk)} \cdot \frac{\Gamma(A+n\beta+sk)\Gamma(A-sk)}{\Gamma(2A+n\beta)}$$

$$= \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\psi_k\; x^k}{(n-sk)!\,\Gamma(A-sk)} \int_0^1 t^{A+n\beta+sk-1}(1-t)^{A-sk-1}dt,$$

where Re (A-sk)>0, Re (A+nβ+sk)>0.

Thus, one obtains

$$M_n(s,A,\beta;x)=\int_0^1 \xi_n(s,A,x)\ t^{A+n\beta-1}(1-t)^{A-1}\ dt, \tag{4.2.4}$$

in which

$$\xi_n(s,A,x)=\sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\ \psi_k\ x^k}{(n-sk)!\,\Gamma(A-sk)}\left(\frac{t}{1-t}\right)^{sk}.$$

The derivation of next integral representation of $M_n(s,A,\beta;x)$ requires the hypergeometric function form of it. For that, let us take

$$\psi_k=\frac{1}{k!\,(a)_k},$$

where 'a' is a complex parameter. With this choice of ψ_k, let us denote the polynomial $M_n(s,A,\beta;x)$ by $F_n(x)$, then after a little simplification one gets

$$F_n(x)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(A+n\beta)_{sk}}{(a)_k\ k!}\ x^k$$

$$={}_{2s}F_1\left[\begin{matrix}\Delta(s;-n),\Delta(s;A+n\beta);\\ a;\end{matrix}\ cx\right], \tag{4.2.5}$$

in which $c=s^{2s}$, is a constant.

In the light of (4.2.2), it is not difficult to obtain

$$F_n(x)=\frac{\Gamma(a)}{\Gamma\left(\frac{A+n\beta}{s}\right)\ \Gamma\left(a-\frac{A+n\beta}{s}\right)}\int_0^1 t^{\frac{A+n\beta}{s}-1}(1-t)^{a-\frac{A+n\beta}{s}-1}.$$

$$\cdot{}_{2s-1}F_0\left[\begin{matrix}\Delta(s;-n),\frac{1+A+n\beta}{s},\ldots,\frac{s+A+n\beta-1}{s};\\ \rule{0pt}{0pt}\overline{\qquad\qquad\qquad};\end{matrix}\ ctx\right]dt, \tag{4.2.6}$$

wherein s=2,3,4,..., $\mathrm{Re}\left(\frac{A+n\beta}{s}\right)>0,\ \mathrm{Re}\left(a-\frac{A+n\beta}{s}\right)>0$.

It is interesting to note that the consideration of the polynomial $M_n(s,A,\beta;x)$ in the integrand of the integral similar to (4.2.1), leads us to a series transformation formula with the aid of the evident fact $e^x.e^{-x}=1$. The procedure is as described below.

Consider $\int_0^1 t^{\lambda+n}(1-t)^{\mu-1}\ M_n(s,A,\beta,xt)\ dt$

$$=\int_0^1 t^{\lambda+n}(1-t)^{\mu-1}\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\Gamma(A+n\beta+sk)}{(n-sk)!}\psi_k\ x^k\ t^k\ dt$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\Gamma(A+n\beta+sk)\,\psi_k\ x^k}{(n-sk)!}\int_0^1 t^{(\lambda+n+k+1)-1}(1-t)^{\mu-1}.dt$$

$$=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\Gamma(A+n\beta+sk)\,\psi_k\,x^k}{(n-sk)!}\frac{\Gamma(\lambda+n+k+1)\Gamma(\mu)}{\Gamma(\mu+\lambda+n+k+1)}. \qquad \textbf{(4.2.7)}$$

In view of the identity $e^t.e^{-t}=1$, the left member of (4.2.7) can be written as:

$$\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}M_n(s,A,\beta;xt)dt=\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}.$$

$$.e^{xt}e^{-xt}\ M_n(s,A,\beta;xt)\ dt. \qquad \textbf{(4.2.8)}$$

But

$$\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1}e^{xt}e^{-xt}M_n(s,A,\beta;xt)\ dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^{i+j}(-1)^j}{i!j!}\int_0^1 t^{(\lambda+n+i+j+1)-1}\ (1-t)^{\mu-1}M_n(s,A,\beta,xt)dt$$

And further, on making an appeal to (4.2.7)

$$\int_0^1 t^{(\lambda+n+1)-1}(1-t)^{\mu-1} e^{xt} e^{-xt} M_n(s,A,\beta,xt)dt$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^{i+j}\Gamma(\mu)}{i!\,j!} \sum_{k=0}^{[n/s]} \frac{(-1)^{sk+j}\psi_k x^k}{(n-sk)!}$$

$$\cdot\frac{\Gamma(A+n\beta+sk)\,\Gamma(\lambda+n+k+i+j+1)}{\Gamma(\mu+\lambda+n+k+i+j+1)}. \qquad \textbf{(4.2.9)}$$

Thus, combining (4.2.7), (4.2.8) and (4.2.9) one arrives at

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{[n/s]} \frac{(-1)^{sk+j}\psi_k x^{k+i+j}\,\Gamma(A+n\beta+sk)\,\Gamma(\lambda+n+k+i+j+1)}{i!\,j!(n-sk)!\,\Gamma(\mu+\lambda+n+k+i+j+1)}$$

$$=\sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\psi_k x^k\,\Gamma(A+n\beta+sk)\,\Gamma(\lambda+n+k+1)}{(n-sk)!\,\Gamma(\mu+\lambda+n+k+1)} \qquad \textbf{(4.2.10)}$$

which is the series transformation formula.

4.3 DIFFERENTIAL EQUATION(θ-form)

To drive the differential equation (θ-form) satisfied by the polynomial $M_n(s,A,\beta;x)$, it is required to express it in the generalized hypergeometric function form.

In the defining relation (4.1.5), first replace $M_n(s,A,\beta;x)$ by $\frac{M_n(s,A,\beta;x)\Gamma(A+n\beta)}{n!}$ then it takes the form:

$$M_n(s,A,\beta;x)=\sum_{k=0}^{[n/s]} (-n)_{sk}(A+n\beta)_{sk}\ \psi_k\ x^k. \qquad \textbf{(4.3.1)}$$

Now selecting $\psi_k = \frac{1}{k!}$ in (4.3.1), and then denoting this special case by the symbol $G_{n,s}(x)$, one obtains the hypegeometric function form of $G_{n,s}(x)$ as:

$$G_{n,s}(x) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(A+n\beta)_{sk}x^k}{k!}$$

$$= {}_{2s}F_0\left[\begin{matrix}\Delta(s;-n),\Delta(s;A+n\beta); \\ \_\_\_\_\_\_\_\_\_\_ ;\end{matrix} cx\right], \quad \textbf{(4.3.2)}$$

where

$\Delta(s;-n)$ is an array of s parameters: $\frac{-n}{s}, \frac{-n+1}{s}, \ldots, \frac{-n+s-1}{s}$, and $c=s^{2s}$ is a constant.

As mentioned in section-2.3, the θ-form differential equation satisfied by

$$y = {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} z\right]$$

is

$$\left[\theta \prod_{j=1}^{q}(\theta+b_j-1) - z\prod_{i=1}^{p}(\theta+a_i)\right]y = 0, \quad where \quad \theta = z\frac{d}{dz}. \quad \textbf{(4.3.3)}$$

Therefore, the differential equation satisfied by $G_{n,s}(x)$ is:

$$\left[\theta - cx\prod_{i=1}^{2s}(\theta+a_i)\right]G_{n,s}(x) = 0, \quad \textbf{(4.3.4)}$$

in which $a_i = \frac{-n+i-1}{s}$, i=1,...,s and $a_i = \frac{A+n\beta+i-s-1}{s}$, for i=s+1, s+2,......,2s.

4.4 INVERSE SERIES RELATIONS

The inverse series relation of the polynomial $M_n(s,A,\beta;x)$ defined by (4.1.5) will be obtained in this section by means of a general pair of inverse series relations which is proved here in the form of

Theorem-7. For s=1, 2, 3,......, if

$$F(n)=\sum_{k=0}^{[n/s]} a(n,k,s)G(k) \tag{4.4.1}$$

and

$$G(n)=\sum_{k=0}^{sn} b(n,k,s)F(k) \tag{4.4.2}$$

then

$$a(n,k,s)=\frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!} \tag{4.4.3}$$

if and only if

$$b(n,k,s)=\frac{(-1)^{k}(A+k+k\beta)}{\Gamma(A+sn+k\beta+1)(sn-k)!} \tag{4.4.4}$$

and

$$\sum_{k=0}^{n} b(n,k,1)F(k)=0,\quad if \quad n\neq sj, j\in \mathrm{N}. \tag{4.4.5}$$

This theorem is proved in section-4.6. This proof uses an inversion pair which will be proved in section 4.5. It is interesting to note that many special cases arise when the parameters are specialized, some of which are known results (polynomials and combinatorial identities) and some are seemingly new ones. These special cases are discussed in section-4.8.

Another inverse series relation implied by theorem-7 is

Theorem-8. If snβ-srβ-1<sn-sr, where s=1, 2, 3,......, r=0, 1, 2, ..., n, and if

$$f(n)=\sum_{k=0}^{[n/s]} c(n,k,s)\ \ g(k)\,, \tag{4.4.6}$$

$$g(n)=\sum_{k=0}^{sn} d(n,k,s)\ \ f(k) \tag{4.4.7}$$

then

$$c(n,k,s)=\frac{(-1)^{n-sk}(A+sk+sk\beta)}{(n-sk)!\,\Gamma(A+n+sk\beta+1)} \tag{4.4.8}$$

implies and, is implied by

$$d(n,k,s)=\frac{\Gamma(A+k+sn\beta)}{(sn-k)!} \tag{4.4.9}$$

and

$$\sum_{k=0}^{n} d(n,k,1)f(k)=0,\ \ \text{if}\ \ n\neq sj, j\in \mathrm{N}. \tag{4.4.10}$$

This theorem will be proved in section-4.7. However, this theorem does not contain any noteworthy special cases.

4.5 AUXILIARY RESULT

Lemma-1.

If Mβ-jβ-1<M-j, where M=sn-sr, s=0, 1, 2, ..., n, r=0, 1, 2, ..., n, j=0, 1, 2, ..., M then

$$T(M)=\sum_{k=0}^{M}(-1)^k\binom{M}{k}\frac{(A+sr+sr\beta+k\beta+k)}{\Gamma(A+M+sr+sr\beta+k\beta+1)}U(k) \qquad \textbf{(4.5.1)}$$

if and only if

$$U(M)=\sum_{k=0}^{M}(-1)^k\binom{M}{k}\Gamma(A+sr\beta+M\beta+sr+k)T(k). \qquad \textbf{(4.5.2)}$$

Proof In order to prove that (4.5.1)⇒(4.5.2), let us consider

$$\phi(M)=\sum_{k=0}^{M}(-1)^k\binom{M}{k}\ \Gamma(A+sr\beta+M\beta+sr+k)\ \ T(k)$$

Now using (4.5.1), one arrives at

$$\phi(M)=\sum_{k=0}^{M}\sum_{j=0}^{k}(-1)^{k+j}\binom{M}{k}\binom{k}{j}\frac{(A+sr+sr\beta+j\beta+j)\Gamma(A+sr\beta+M\beta+sr+k)}{\Gamma(A+k+sr+sr\beta+j\beta+1)}U(j).$$

Now applying the double series relation

$$\sum_{k=0}^{M}\sum_{j=0}^{k}A(k,j)=\sum_{j=0}^{M}\sum_{k=0}^{M-j}A(k+j,j),$$

one further gets

$$\phi(M)=\sum_{j=0}^{M}\sum_{k=0}^{M-j}(-1)^k\binom{M}{k+j}\binom{k+j}{j}(A+sr+sr\beta+j\beta+j)\cdot$$

$$\cdot\frac{\Gamma(A+M\beta+sr\beta+sr+k+j)}{\Gamma(A+sr\beta+sr+j\beta+j+k+1)}U(j)$$

$$=\frac{(A+sr+sr\beta+M\beta+M)\Gamma(A+M\beta+sr\beta+sr+M)}{\Gamma(A+sr\beta+sr+M\beta+M+1)}U(M)$$

$$+\sum_{j=0}^{M-1}\sum_{k=0}^{M-j}(-1)^k\binom{M}{k+j}\binom{k+j}{j}\frac{(A+sr+sr\beta+j\beta+j)\,\Gamma(A+M\beta+sr\beta+sr+k+j)}{\Gamma(A+sr\beta+sr+j\beta+j+k+1)}U(j)$$

$$= U(M) + \sum_{j=0}^{M-1} \sum_{k=0}^{M-j} (-1)^k \binom{M}{k+j} \binom{k+j}{j} (A + sr + sr\beta + j\beta + j)$$

$$\cdot \frac{\Gamma(A + M\beta + sr\beta + sr + k + j)}{\Gamma(A + sr\beta + sr + j\beta + j + k + 1)} U(j)$$

$$= U(M) + \sum_{j=0}^{M-1} \binom{M}{j} (A + sr + sr\beta + j\beta + j) \; U(j) \sum_{k=0}^{M-j} (-1)^k \binom{M-j}{k}.$$

$$\cdot \frac{\Gamma(A + sr + sr\beta + M\beta + k + j)}{\Gamma(A + sr + sr\beta + j\beta + k + j + 1)}. \qquad \textbf{(4.5.3)}$$

Here the ratio $\frac{\Gamma(A + sr + sr\beta + M\beta + k + j)}{\Gamma(A + sr + sr\beta + j\beta + k + j + 1)}$ is in fact, a polynomial in k of degree Mβ-jβ-1, therefore from (4.5.3) one gets,

$$\phi(M) = U(M) + \sum_{j=0}^{M-1} \binom{M}{j} (A + sr + sr\beta + j\beta + j).$$

$$U(j) \sum_{k=0}^{M-j} (-1)^k \binom{M-j}{k} \sum_{i=0}^{M\beta - j\beta - 1} c_i k^l.$$

Because the inner series in k is the $(M-j)^{th}$ difference of a polynomial in k of degree precisely Mβ-jβ-1, where Mβ-jβ-1<M-j, therefore, ϕ(M)=U(M). This proves that (4.5.1)⇒(4.5.2).

In order to prove the converse part it is sufficient to show that the diagonal elements of the coefficient matrices of (4.5.1) and (4.5.2) are non zero.

Let the diagonal elements of (4.5.1) and (4.5.2) be denoted by t_{MM} and u_{MM} respectively. Then it is clear that

$$t_{MM} = \frac{(-1)^M}{\Gamma(A + sr + sr\beta + M\beta + M)} \neq 0$$

and

$$u_{MM} = (-1)^M \Gamma(A + sr + sr\beta + M\beta + M) \neq 0.$$

Therefore, the inverse is unique and thus, (4.5.2)⇒(4.5.1) which completes the proof of the lemma.

4.6 PROOF OF THEOREM-7

It will be first proved that (4.4.3)⇒(4.4.4).

Let the right hand side of (4.4.2) be denoted by t_n, then in view of (4.4.4)

$$t_n = \sum_{k=0}^{sn} (-1)^k \frac{(A + k + k\beta) \quad F(k)}{(sn - k)!\,\Gamma(A + sn + k\beta + 1)}.$$

Now using (4.4.1), t_n becomes

$$t_n = \sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} \frac{(-1)^{k+sr}(A + k + k\beta)\Gamma(A + sr + k\beta)}{(sn - k)!\,(k - sr)!\,\Gamma(A + sn + k\beta + 1)} \; G(r).$$

Since,

$$\sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} A(k,r) = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} A(k + sr, r),$$

$$t_n = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} \frac{(-1)^k (A + k + sr + k\beta + sr\beta)\,\Gamma(A + sr + k\beta + sr\beta)\,G(r)}{(sn - sr - k)!\,k!\,\Gamma(A + sn + k\beta + sr\beta + 1)}$$

$$= G(n) + \sum_{r=0}^{n-1} \sum_{k=0}^{sn-sr} \frac{(-1)^k (A + k + sr + k\beta + sr\beta)\,\Gamma(A + sr + k\beta + sr\beta) G(r)}{(sn - sr - k)!\,k!\,\Gamma(A + sn + k\beta + sr\beta + 1)}.$$

Denoting n-r by N one obtains,

$$t_n = G(n) + \sum_{r=0}^{n-1} \frac{G(n)}{(sN)!} \sum_{k=0}^{sN} (-1)^k \binom{sN}{k} \frac{(A+k+k\beta+sr+sr\beta)\,\Gamma(A+sr+k\beta+sr\beta)}{\Gamma(A+sn+k\beta+sr\beta+1)}$$

(4.6.1)

On comparing (4.6.1) with Lemma-1, it is clear that the inner series in k above, is nothing but (4.5.1) with the choice $U(k) = \Gamma(A+sr+sr\beta+k\beta)$.

Taking

$$T(k) = \binom{0}{k} = \begin{cases} 1, & if \quad k=0 \\ 0, & if \quad k \neq 0 \end{cases}$$

in (4.5.2), one gets

$$U(M) = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \Gamma(A+sr\beta+M\beta+sr+k) \binom{0}{k}$$

$$= \Gamma(A+sr+sr\beta+M\beta).$$

Hence, with $T(k) = \binom{0}{k}$, the choice $U(k) = \Gamma(A+sr+sr\beta+k\beta)$ is restored.

Therefore, the sum of the inner series in k in (4.6.1) is

$$\sum_{k=0}^{sN} (-1)^k \binom{sN}{k} \frac{(A+k+k\beta+sr+sr\beta)\,\Gamma(A+sr+k\beta+sr\beta)}{\Gamma(A+sn+k\beta+sr\beta+1)} = T(sN) = \binom{0}{sN}.$$

But since r=0, 1, 2,......, n-1, therefore sN=s(n-r)$\neq$0, and hence the sum is zero. Thus, from (4.6.1), t_n=G(n).

This proves that (4.4.3)$\Rightarrow$(4.4.4).

Now to show that (4.4.3)⇒(4.4.5), let us denote the left hand side of (4.4.5) by ψ_n, then in the light of (4.4.1)

$$\psi_n = \sum_{k=0}^{n} \sum_{j=0}^{[k/s]} \frac{(-1)^{k+sj}(A+k+k\beta)\,\Gamma(A+sj+k\beta)}{\Gamma(A+n+k\beta+1)(n-k)!(k-sj)!} G(j)$$

Now using the double series

$$\sum_{k=0}^{n} \sum_{j=0}^{[k/s]} A(k,j) = \sum_{j=0}^{[n/s]} \sum_{k=0}^{n-sj} A(k+sj,j),$$

and denoting n-sj by N, ψ_n takes the form

$$\psi_n = \sum_{j=0}^{[n/s]} \sum_{k=0}^{N} \frac{(-1)^k (A+sj\beta+sj+k\beta+k)\Gamma(A+sj\beta+sj+k\beta)}{(N-k)!k!\Gamma(A+N+sj\beta+sj+k\beta+1)} G(j),$$

$$= \sum_{j=0}^{[n/s]} \frac{G(j)}{N!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{(A+sj\beta+sj+k\beta+k)\Gamma(A+sj\beta+sj+k\beta)}{\Gamma(A+sj\beta+sj+N+k\beta+1)}. \quad \textbf{(4.6.2)}$$

Now in Lemma-1, if M(=sn-sr) is replaced by N(=n-sj) then it can be written as:

$$T(N) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{(A+sj\beta+sj+k\beta+k)}{\Gamma(A+sj\beta+sj+N+k\beta+1)} U(k) \quad \textbf{(4.6.3)}$$

⇔

$$U(N) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma(A+N\beta+sj\beta+sj+k)T(k). \quad \textbf{(4.6.4)}$$

In (4.6.2) above, if one replaces Γ(A+sjβ+sj+kβ) by U(k), then it coincides with (4.6.3) whose inverse series is (4.6.4) by lemma-1.

Here choosing

$$T(k) = \binom{0}{k} = \begin{cases} 0, & if\ k \neq 0 \\ 1, & if\ k = 0, \end{cases}$$

in (4.6.4),

$$U(N) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \Gamma(A + N\beta + sj\beta + sj + k) \binom{0}{k}$$

$$= \Gamma(A + sj\beta + sj + N\beta).$$

Thus the choice for U(k) is restored with $T(k) = \binom{0}{k}$. Hence from (4.6.2), one now arrives at

$$\psi_n = \sum_{j=0}^{[n/s]} \frac{G(j)}{N!} T(N)$$

$$= \sum_{j=0}^{[n/s]} \frac{G(j)}{N!} \binom{0}{N}$$

=0 , if N≠0, that is n ≠ sj, j=1, 2, ...

This proves that (4.4.3)⇒(4.4.5), completing the proof of the first part.

For proving the converse part, it will be shown that (4.4.4) and (4.4.5) together imply (4.4.3).

Suppose (4.4.4) and (4.4.5) hold true.

Consider

$$\overline{G}_n = \sum_{k=0}^{n} \frac{(-1)^k (A + k + k\beta) F(k)}{(n-k)!\, \Gamma(A + n + k\beta + 1)}, \qquad \textbf{(4.6.5)}$$

then in view of (4.4.5)

$\overline{G}_n$ =0 if n≠sj, j∈N, and **(4.6.6)**

$\overline{G}_{sn}$ =G(n). **(4.6.7)**

Further, (4.6.5) implies that

$$F(n) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(A+k+n\beta)}{(n-k)!} \overline{G}_k \quad \textbf{(4.6.8)}$$

but $\overline{G}_n$ =0 as shown in (4.6.6) hence in view of (4.6.7), one gets

$$G(n) = \sum_{k=0}^{sn} (-1)^k \frac{(A+k+k\beta)F(k)}{(sn-k)!\,\Gamma(A+sn+k\beta+1)}$$

⇒

$$F(n) = \sum_{sk=0}^{n} (-1)^{sk} \frac{\Gamma(A+sk+n\beta)}{(n-sk)!} \overline{G}_{sk}$$

$$= \sum_{sk=0}^{n} (-1)^{sk} \frac{\Gamma(A+sk+n\beta)}{(n-sk)!} G(k).$$

This shows that (4.4.4)⇒(4.4.3) subject to (4.4.5), which completes the proof of theorem-7.

4.7 PROOF OF THEOREM-8

With a view to prove that (4.4.8)⇒(4.4.9), let the right member of (4.4.7) be denoted by h_n, then

$$h_n = \sum_{k=0}^{sn} \frac{\Gamma(A+k+sn\beta)}{(sn-k)!} f(k),$$

which in the light of (4.4.6) becomes

$$h_n = \sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} \frac{(-1)^{k-sr}(A+sr+sr\beta)\Gamma(A+k+sn\beta)}{(k-sr)!(sn-k)!\,\Gamma(A+k+sr\beta+1)} g(r).$$

Now applying the double series relation

$$\sum_{k=0}^{sn} \sum_{r=0}^{[k/s]} A(k,r) = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} A(k+sr,r),$$

it takes the form

$$h_n = \sum_{r=0}^{n} \sum_{k=0}^{sn-sr} (-1)^k \frac{(A+sr+sr\beta)\,\Gamma(A+k+sr+sn\beta)}{k!(sn-sr-k)!\,\Gamma(A+k+sr+sr\beta+1)} g(r).$$

Putting n-r=N, this can be rewritten as

$$h_n = g(n) + \sum_{r=0}^{n-1} \sum_{k=0}^{sN} \frac{(-1)^k (A+sr+sr\beta)\,\Gamma(A+sN\beta+sr\beta+sr+k)}{k!\,(sN-k)!\,\Gamma(A+sr\beta+sr+k+1)} g(r)$$

$$= g(n) + \sum_{r=0}^{n-1} \frac{(A+sr+sr\beta)g(r)}{(sN)!} \sum_{k=0}^{sN} (-1)^k \binom{sN}{k} \frac{\Gamma(A+sN\beta+sr\beta+sr+k)}{\Gamma(A+sr\beta+sr+k+1)}.$$

(4.7.1)

Here the ratio of the gamma functions is infact,

$$\frac{\Gamma(A+sr\beta+sr+k+sN\beta)}{\Gamma(A+sr\beta+sr+k+1)} = \sum_{i=0}^{sN\beta-1} c_i\, k^i,$$

hence from (4.7.1),

$$h_n = g(n) + \sum_{r=0}^{n-1} \frac{(A+sr+sr\beta)g(r)}{(sN)!} \sum_{k=0}^{sN} (-1)^k \binom{sN}{k} \sum_{i=0}^{sN\beta-1} c_i\, k^i$$

$$= g(n) + \sum_{r=0}^{n-1} \frac{(A+sr+sr\beta)g(r)}{(sN)!} \sum_{i=0}^{sN\beta-1} c_i \sum_{k=0}^{sN} (-1)^k \binom{sN}{k} k^i$$

The inner series in k here is zero as it is the $(sN)^{th}$ difference of a polynomial in k of degree less than sN, precisely of degree sNβ-1.

Therefore, $h_n=g(n)$. Thus (4.4.8)⇒(4.4.9).

Now to prove that (4.4.8) also implies (4.4.10), consider the left hand side of (4.4.10) and denote it by θ_n, then

$$\theta_n = \sum_{k=0}^{n} \frac{\Gamma(A+k+n\beta)}{(n-k)!} f(k)$$

$$= \sum_{k=0}^{n} \sum_{r=0}^{[k/s]} \frac{(-1)^{k-sr}(A+sr+sr\beta)\,\Gamma(A+k+sn\beta) g(r)}{(k-sr)!(n-k)!\,\Gamma(A+k+sr\beta+1)}.$$

Now using the double series relation,

$$\sum_{k=0}^{n} \sum_{r=0}^{[k/s]} A(k,r) = \sum_{r=0}^{[n/s]} \sum_{k=0}^{n-sr} A(k+sr,r),$$

θ_n becomes

$$\theta_n = \sum_{r=0}^{[n/s]} \sum_{k=0}^{n-sr} \frac{(-1)^k (A+sr+sr\beta)\,\Gamma(A+k+sr+n\beta) g(r)}{k!(n-sr-k)!\,\Gamma(A+k+sr+sr\beta+1)}.$$

Taking n-sr=N, one gets

$$\theta_n = \sum_{r=0}^{[n/s]} \frac{(A+sr+sr\beta)g(r)}{N!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \frac{\Gamma(A+k+sr+n\beta)}{\Gamma(A+k+sr+sr\beta+1)}.$$

Here

$$\frac{\Gamma(A+k+sr+n\beta)}{\Gamma(A+k+sr+sr\beta+1)} = \sum_{j=0}^{N\beta-1} c_j\, k^j,$$

thus

$$\theta_n = \sum_{r=0}^{[n/s]} \frac{(A+sr+sr\beta)\, g(r)}{N!} \sum_{j=0}^{N\beta-1} c_j \sum_{k=0}^{N} (-1)^k \binom{N}{k} k^j = 0,$$

as the inner series in k is the N^{th} difference of a polynomial in k of degree Nβ-1, where Nβ-1<N, which proves that (4.4.8)⇒(4.4.10), completing the proof of the first part.

In proving the converse part it is assumed that (4.4.9) and (4.4.10) hold true. Denoting the left hand side of (4.4.10) by $\bar{g}_n$, one gets

$$\bar{g}_n = \sum_{k=0}^{n} \frac{\Gamma(A+k+n\beta)}{(n-k)!} f(k). \qquad \textbf{(4.7.2)}$$

But as (4.4.10) holds

$$\bar{g}_n = 0 \quad if \quad n \neq sj, j \in N, \qquad \textbf{(4.7.3)}$$

and also, because (4.4.9) holds true

$$\bar{g}_{sn} = g(n). \qquad \textbf{(4.7.4)}$$

Also,

(4.7.2) implies that

$$f(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(A+k+k\beta)}{(n-k)!\,\Gamma(A+n+k\beta+1)} g(k). \qquad \textbf{(4.7.5)}$$

In view of (4.7.3) and (4.7.4), (4.7.5) and (4.7.2) give

$$\bar{g}_{sn} = g(n) = \sum_{k=0}^{sn} \frac{\Gamma(A+k+sn\beta)}{(sn-k)!} f(k)$$

⇒

$$f(n) = \sum_{sk=0}^{n} (-1)^{n-sk} \frac{(A+sk+sk\beta)}{(n-sk)!\,\Gamma(A+n+sk\beta+1)},$$

with this, the proof of the converse part is completed and hence the proof of the theorem.

4.8 PARTICULAR CASES

As mentioned in the introduction of this chapter, the special cases of the general class of polynomials $M_n(s,A,\beta;x)$ (given in (4.1.5)) and those of the properties discussed in sections 4.2 to 4.7 will be taken up in this section.

First, the extended versions of the various known polynomials occurring as the special cases of the general polynomial $M_n(s,A,\beta;x)$ will be illustrated.

The polynomial

$$M_n(s,A,\beta;x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!}\,\psi_k\,x^k \tag{4.8.1}$$

is reconsidered in a slightly modified form (given in (4.3.1)):

$$M_n(s,A,\beta;x) = \sum_{k=0}^{[n/s]} (-n)_{sk}\,(A+n\beta)_{sk}\,\psi_k\,x^k. \tag{4.8.2}$$

This form will be used to obtain various extended polynomials as follows:

First the Racah polynomial in an extended form denoted by $R_n^s(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)$, may be obtained from (4.8.2) by setting β=1, and then choosing A=1+α+β, x=1 and $\psi_k = \frac{(-x)_k(x+\gamma+\delta+1)_k}{k!(1+\alpha)_k(\beta+\delta+1)_k(1+\gamma)_k}$.

Thus

$$R_n^s(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k(x+\gamma+\delta+1)_k}{k!(1+\alpha)_k(1+\gamma)_k(\beta+\delta+1)_k}. \tag{4.8.3}$$

Next, if β=1, x=1, and A=a+b+c+d-1 then by selecting $\psi_k = \frac{(a+ix)_k (a-ix)_k}{k!(a+b)_k (a+c)_k (a+d)_k}$ one gets from (4.8.2) an extended form of the Wilson's polynomial denoted here by $P_n^s(x^2)$.

Thus one finds

$$P_n^s(x^2)(a+b)_n(a+c)_n(a+d)_n = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(a+b+c+d+n-1)_{sk}(a+ix)_k(a-ix)_k}{k!(a+b)_k(a+c)_k(a+d)_k}$$

or

$$P_n^s(x^2) = \frac{1}{(a+b)_n(a+c)_n(a+d)_n} \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(a+b+c+d+n-1)_{sk}(a+ix)_k(a-ix)_k}{k!(a+b)_k(a+c)_k(a+d)_k}.$$

(4.8.4)

An extended form of Hahn polynomial $Q_{n,s}(x;\alpha,\beta,N)$ can now be obtained by setting β=1, A=1+α+β, and x=1 and then $\psi_k = \frac{(-x)_k}{k!(1+\alpha)_k(-N)_k}$, where n=0,1,2,..., N.

With this one gets

$$Q_{n,s}(x;\alpha,\beta,N) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k}{k!(1+\alpha)_k(-N)_k}, \text{ where } n=0,1,2,\ldots,N.$$

(4.8.5)

An extension of the well known Jocobi polynomial denoted here by $P_{n,s}^{(a,\beta)}(x)$, is obvious. In fact, from (4.8.2) by taking β=1 and, then A=1+α+β, $\psi_k = \frac{1}{k!(1+\alpha)_k}$ and replacing x by $\left(\frac{1-x}{2}\right)$, we get

$$(1+\alpha)_n \; P_{n,s}^{(\alpha,\beta)}(x) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}}{k!(1+\alpha)_k} \left(\frac{1-x}{2}\right)^k$$

or

$$P_{n,s}^{(\alpha,\beta)}(x)=\frac{1}{(1+\alpha)_n}\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}}{k!(1+\alpha)_k}\left(\frac{1-x}{2}\right)^k. \qquad \textbf{(4.8.6)}$$

The Legendre polynomial, in an extended form, carrying the notation $P_n^s(x)$ can be obtained by choosing A=β=1, $\psi_k=\frac{1}{k!k!}$ and replacing x by $\left(\frac{1-x}{2}\right)$.

Hence,

$$P_n^s(x)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(n+1)_{sk}}{k!k!}\left(\frac{1-x}{2}\right)^k. \qquad \textbf{(4.8.7)}$$

The Bessel Polynomial $Y_n(x)$ (Riordan [1], p. 77) is also occurring as a special case of (4.8.2). An extension of $Y_n(x)$, denoted by $Y_n^s(x)$ is obtained, when A=β=1, $\psi_k=\frac{1}{k!}$ and when x is replaced by $\left(\frac{-x}{2}\right)$.

Thus,

$$Y_n^s(x)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(n+1)_{sk}}{k!}\left(\frac{-x}{2}\right)^k. \qquad \textbf{(4.8.8)}$$

And lastly with β=0 and $\psi_k=\frac{1}{k!(1+\alpha)_k(A)_{sk}}$, (4.8.2) yields an extended version of the Laguerre polynomial denoted here by $L_{n,s}^{(\alpha)}(x)$:

$$L_{n,s}^{(\alpha)}(x)\,(1+\alpha)_n=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}\;x^k}{k!\,(1+\alpha)_k}$$

or

$$L_{n,s}^{(\alpha)}(x)=\frac{1}{(1+\alpha)_n}\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}}{k!}\frac{x^k}{(1+\alpha)_k}. \qquad \textbf{(4.8.9)}$$

The above polynomials (4.8.3) to (4.8.9) are extensions in the sense that when s=1, all of them reduce to the original known polynomials.

Now the particularizations of the properties which are discussed in sections 4.2 to 4.7 will be taken up as follows.

(A) Special Cases of Integrals.

(B) Special Cases of θ-form differential equation.

(C) Special Cases of inverse series relations.

(C-1) Ordinary extended forms of inversion pairs of Gessel and Stanton.

(C-2) Pairs of inverse series relations of extended forms of polynomials.

(C-3) Combinatorial identities.

(A) Special Cases of Integrals

The integral forms of the aforementioned polynomials in view of the integral (4.2.4) are listed below.

$$R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)=\int_0^1 t^{\alpha+\beta+n+k}(1-t)^{\alpha+\beta-k}\xi_n(\alpha,\beta,\gamma,\delta;x)dt, \qquad \textbf{(4.8.10)}$$

where

$$\xi_n(\alpha,\beta,\gamma,\delta;x)=\sum_{k=0}^{n}\frac{(-n)_k(-x)_k\,(x+\gamma+\delta+1)_k}{n!\,\Gamma(1+\alpha+\beta-k)\,(1+\alpha)_k\,(1+\gamma)_k\,(\beta+\delta+1)_k\,k!}$$

and Re(1+α+β-k)>0, Re(1+α+β+n+k)>0.

$$P_n(x^2)=\int_0^1 t^{a+b+c+d+n+k-2}\,(1-t)^{a+b+c+d-k-2}\,\lambda_n(a,b,c,d;x)dt, \qquad \textbf{(4.8.11)}$$

wherein

$$\lambda_n(a,b,c,d;x)=\sum_{k=0}^{n}\frac{(-1)^k\,(a+ix)_k\,(a-ix)_k}{(n-k)!\;k!\;\Gamma(a+b+c+d-k-1)(a+b)_k\,(a+c)_k\,(a+d)_k},$$

and Re(a+b+c+d-k-1)>0, Re(a+b+c+d+n+k-1)>0.

$$Q_n(x;\alpha,\beta,N)=\int_0^1 t^{\alpha+\beta+n+k}(1-t)^{\alpha+\beta-k}\,\upsilon_n(\alpha,\beta;x)dt, \qquad \textbf{(4.8.12)}$$

in which

$$\upsilon_n(\alpha,\beta;x)=\sum_{k=0}^{n}\frac{(-1)^k(-x)_k}{(n-k)!\,k!\,\Gamma(1+\alpha+\beta-k)\,\Gamma(1+\alpha+\beta+n)(1+\alpha)_k(-N)_k},$$

and Re(1+α+β-k)>0, Re(1+α+β+n+k)>0.

$$P_n^{(\alpha,\beta)}(x)=\int_0^1 t^{\alpha+\beta+n+k}(1-t)^{\alpha+\beta-k}\,\vartheta_n(\alpha,\beta;x)dt, \qquad \textbf{(4.8.13)}$$

with

$$\vartheta_n(\alpha,\beta;x)=\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!\;k!\;\Gamma(1+\alpha+\beta-k)(1+\alpha)_k}\left(\frac{1-x}{2}\right)^k,$$

and Re(1+α+β-k)>0, Re(1+α+β+n+k)>0.

In the second integral (4.2.6) the parameter s is a natural number greater than 1. Because of this, the integral forms of the extended versions of the polynomials are derived from (4.2.6).

In the beginning the integral forms of an extension of the Jacobi polynomial denoted here by $P_{n,s}^{(\alpha,\beta)}(x)$ and an extension of the Legendre polynomial denoted here by $P_{n,s}(x)$ are derived from (4.2.6) by setting $\beta=1$ and replacing x by $\frac{1-x}{2}$.

In particular, for $P_{n,s}^{(\alpha,\beta)}(x)$ putting $a=1+\alpha$ and $A=1+\alpha+\beta$, one gets from the general integral (4.2.6),

$$P_{n,s}^{(\alpha,\beta)}(x)=\frac{\Gamma(1+\alpha)}{\Gamma\left(\frac{1+\alpha+\beta+n}{s}\right)\Gamma\left(1+\alpha-\frac{1+\alpha+\beta+n}{s}\right)}\int_0^1 t^{\frac{1+\alpha+\beta+n}{s}-1}(1-t)^{\alpha-\frac{1+\alpha+\beta+n}{s}}.$$

$$\cdot {}_{2s-1}F_0\left[\begin{matrix}\Delta(-n,s),\frac{2+\alpha+\beta+n}{s},\ldots,\frac{s+\alpha+\beta+n}{s}, \\ \rule{20em}{0.4pt}\ ;\end{matrix}\ tc\left(\frac{1-x}{2}\right)\right]dt, \quad \textbf{(4.8.14)}$$

where $s=2,3,4,\ldots$, $Re(1+\alpha)>0$.

The integral forms of the extended Racah, Wilson and Hahn polynomials are obtained separately in the light of the integral (4.2.2) by taking $\beta=1$, $x=1$ in (4.8.2). Because of their peculiar explicit forms, it is not straight forward to obtain their integrals directly from (4.2.6).

With the values $\beta=1$ and $x=1$ the polynomial (4.8.2) reduces to

$$M_n(s,A,1;1)=\sum_{k=0}^{[n/s]}(-n)_{sk}\ (A+n)_{sk}\ \psi_k\ . \quad \textbf{(4.8.15)}$$

Setting $A=1+\alpha+\beta$ and choosing

$$\psi_k = \frac{(-x)_k (x+\gamma+\delta+1)_k}{k!(1+\alpha)_k (1+\gamma)_k (\beta+\delta+1)_k}$$

in (4.8.15), one gets the hypergeometric function form of the extended Racah polynomial

$$R_n^s(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) =$$

$$_{2s+2}F_3\left[\begin{matrix}\Delta(s;-n),\Delta(s;1+\alpha+\beta+n),-x,x+\gamma+\delta+1; \\ 1+\alpha,1+\gamma,\beta+\delta+1;\end{matrix}\; c\right]$$

where $c=s^{2s}$.

Now using the integral (4.2.2) for a $_pF_q(x)$, one gets from above:

$$R_n^s(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = \frac{\Gamma(1+\alpha)}{\Gamma\left(\frac{1+\alpha+\beta+n}{s}\right)\Gamma\left(1+\alpha-\frac{1+\alpha+\beta+n}{s}\right)}$$

$$\int_0^1 t^{\frac{1+\alpha+\beta+n}{s}-1}(1-t)^{\alpha-\frac{1+\alpha+\beta+n}{s}}$$

$$_{2s+1}F_2\left[\begin{matrix}\Delta(s;-n),\frac{2+\alpha+\beta+n}{s},\ldots\ldots\frac{\alpha+\beta+n+s}{s},-x,x+\gamma+\delta+1; \\ 1+\gamma,\beta+\delta+1;\end{matrix}\; ct\right]dt \qquad \textbf{(4.8.16)}$$

The substitutions A=a+b+c+d-1 and $\psi_k = \frac{(a+ix)_k (a-ix)_k}{k!(a+b)_k (a+c)_k (a+d)_k}$

in (4.8.15) yields

$$P_n^s(x^2) = \frac{1}{(a+b)_n (a+c)_n (a+d)_n}.$$

$$\cdot\, _{2s+2}F_3\left[\begin{matrix}\Delta(s;-n);\Delta(s;a+b+c+d+n-1),a-ix,a+ix, \\ a+b,a+c,a+d,\end{matrix}\; c\right],$$

where as above $c=s^{2s}$.

This in view of the integral (4.2.2) gives

$$P_n^s(x^2) = \frac{\Gamma(a+b)}{(a+b)_n (a+c)_n (a+d)_n \Gamma\left(\frac{a+b+c+d+n-1}{s}\right)\Gamma\left(a+b-\frac{a+b+c+d+n-1}{s}\right)} \cdot$$

$$\int_0^1 t^{\frac{a+b+c+d+n-1}{s}-1} (1-t)^{a+b-\frac{a+b+c+d+n-1}{s}-1}$$

$$ {}_{2s+1}F_2\left[\begin{matrix} \Delta(s;-n), \frac{a+b+c+d+n}{s}, \ldots, \frac{a+b+c+d+n+s-2}{s}, a-ix, a+ix; \\ a+c, a+d; \end{matrix} \; ct\right] dt,$$

(4.8.17)

where $c=s^{2s}$.

The integral form of the extended Hahn polynomial can be obtained when $A=1+\alpha+\beta$ and $\psi_k = \frac{(-x)_k}{k!(1+\alpha)_k(-N)_k}$, n=0,1,2,...,N.

With these substitutions, (4.8.15) reduces to

$$Q_{n,s}(x;\alpha,\beta,N) = {}_{2s+1}F_2\left[\begin{matrix} \Delta(s;-n), \Delta(s;1+\alpha+\beta+n), -x; \\ 1+\alpha, -N; \end{matrix} \; c\right]$$

whose integral form is

$$Q_{n,s}(x;\alpha,\beta,N)=\frac{\Gamma(1+\alpha)}{\Gamma\left(\frac{1+\alpha+\beta+n}{s}\right)\Gamma\left(1+\alpha-\frac{1+\alpha+\beta+n}{s}\right)},$$

$$\int_0^1 t^{\frac{1+\alpha+\beta+n}{s}-1}(1-t)^{\alpha-\frac{1+\alpha+\beta+n}{s}}.$$

$$_{2s}F_1\left[\begin{matrix}\Delta(s;-n),\frac{2+\alpha+\beta+n}{s},\ldots,\frac{\alpha+\beta+n+s}{s},-x, \\ -N;\end{matrix} ct\right]dt. \quad \textbf{(4.8.18)}$$

(B) <u>Special Cases of θ-form differential equation</u>

The polynomial $M_n(s,A,\beta,x)$ reduces to various known polynomials as shown in the beginning of this section, when the parameters involved are particularized suitably. In view of these particular values of the parameters, the differential equation (4.3.3) derived earlier gives the differential equations for the specialized polynomials, which are listed below.

$$[\theta(\theta+\alpha)(\theta+\gamma)(\theta+\beta+\delta)-(\theta-n)(\theta+1+\alpha+\beta+n)(\theta-x)(\theta+x+\gamma+\delta+1)]$$

$$R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta)=(\,. \quad \textbf{(4.8.19)}$$

$$[\theta(\theta+a+b-1)(\theta+a+c-1)(\theta+a+d-1)$$

$$-(\theta-n)(\theta+a+b+c+d+n-1)(\theta+a-ix)(\theta+a+ix)]P_n(x^2)=0. \quad \textbf{(4.8.20)}$$

$$[\theta(\theta+\alpha)(\theta-N-1)-(\theta-n)(\theta+1+\alpha+\beta+n)(\theta-x)]Q_n(x;\alpha,\beta,N)=(\,. \quad \textbf{(4.8.21)}$$

$$\left[\theta(\theta+\alpha)-\left(\frac{1-x}{2}\right)(\theta-n)(\theta+1+\alpha+\beta+n)\right]P_n^{(\alpha,\beta)}(x)=(\,. \quad \textbf{(4.8.22)}$$

$$\left[\theta^2 - \left(\frac{1-x}{2}\right)(\theta - n)(\theta + n - 1)\right]P_n(x) = 0. \qquad \textbf{(4.8.23)}$$

$$\left[\frac{x}{2}(\theta - n)(\theta + n + 1)\right]Y_n(x) = 0. \qquad \textbf{(4.8.24)}$$

and

$$[\theta(\theta + \alpha) - x(\theta - n)]L_n^{(\alpha)}(x) = 0. \qquad \textbf{(4.8.25)}$$

(C) Special Cases of Inverse Series Relation

The special cases of the general inversion pair proved in theorem-7 can be classified as follows.

(C-1) Ordinary extended forms of inversion pairs of Gessel and Stanton.

(C-2) Pairs of inverse series relations of extended forms of polynomials.

(C-3) Combinatorial identities.

(C-1)Ordinary extended forms of inversion pairs of Gessel and Stanton

It is interesting to see that the various q-inversion pairs taken up by Gessel and Stanton [1] when considered in ordinary form, that means in the limiting case $q \to 1$, admit extensions through theorem-7.

The pairs that are obtained from theorem-7 by taking $\beta=1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}$, and 2 are as stated below.

(1) $\beta=1$

$$A_n = \sum_{k=0}^{[n/s]} \frac{\Gamma(A + sk + n)}{(n - sk)!} B_k$$

$\Leftrightarrow$

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{(A+2k)}{(sn-k)!\,\Gamma(A+sn+k+1)} A_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{A+2k}{(n-k)!\,\Gamma(A+n+k+1)} A_k = 0, \text{if } n \neq sj,\ j=0,1,2,\ldots$$

(2) <u>β=1/2</u>

$$A_n = \sum_{k=0}^{[n/s]} \frac{\Gamma(A+sk+n/2)}{(n-sk)!} B_k$$

⇔

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{(A+3k/2)}{(sn-k)!\,\Gamma(A+sn+\frac{k}{2}+1)} A_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(A+3k/2)}{(n-k)!\,\Gamma(A+n+\frac{k}{2}+1)} A_k = 0, \text{if } n \neq sj,\ j=0,1,2,\ldots$$

(3) <u>β=-1/2</u>

$$A_n = \sum_{k=0}^{[n/s]} \frac{\Gamma(A+sk-n/2)}{(n-sk)!} B_k$$

⇔

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{(A+k/2)}{(sn-k)!\,\Gamma(A+sn-\frac{k}{2}+1)} A_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(A+\frac{k}{2})}{(n-k)!\,\Gamma(A+n-\frac{k}{2}+1)} A_k = 0, \text{if } n \neq sj,\ j=0,1,2,\ldots$$

(4) β=-1/3

$$A_n = \sum_{k=0}^{[n/s]} \frac{\Gamma(A+sk-n/3)}{(n-sk)!} B_k$$

⇔

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{(A+2k/3)}{\Gamma(A+sn-\frac{k}{3}+1)(sn-k)!} A_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(A+2k/3)}{(n-k)!\,\Gamma(A+n-\frac{k}{3}+1)} A_k = 0, \text{if } n \neq sj,\ j=0,1,2,...$$

(5) β=2

$$A_n = \sum_{k=0}^{[n/s]} \frac{\Gamma(A+sk-2n)}{(n-sk)!} B_k$$

⇔

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{(A+3k)}{(sn-k)!\,\Gamma(A+sn+2k+1)} A_k$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(A+3k)}{(n-k)!\,\Gamma(A+n+2k+1)} A_k = 0, \text{if } n \neq sj,\ j=0,1,2,...$$

(C-2)Pairs of inverse series relations of extended forms of polynomials

This set of special cases contains some well known polynomials in their further extended form, which are occurring as particular cases of theorem-7.

As shown in the starting of this section, the first series in theorem-7, namely (4.4.1) yields the polynomial

$$M_n(s,A,\beta,x)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\,\Gamma(A+sk+n\beta)}{(n-sk)!}\psi_k\,x^k,$$

when G(k) is taken as $\psi_k\,x^k$ and F(n) is then denoted by $M_n(s,A,\beta;x)$.

In order to extract the polynomial special cases of this polynomial, its elegent form is obtained, given by (4.8.2), by replacing $M_n(s,A,\beta,x)$ by $\frac{M_n(s,A,\beta;x)\Gamma(A+n\beta)}{n!}$. Making corresponding replacements in (4.4.2) it yields the inverse series of $M_n(s,A,\beta,x)$ after some simplifications in the form:

$$\psi_n\,x^n=\sum_{k=0}^{sn}\frac{(-1)^k(A+k+k\beta)\,M_k(s,A,\beta;x)}{(sn-k)!\,k!\,(A+k\beta)_{sn+1}}. \qquad \textbf{(4.8.26)}$$

It is already shown that the extensions of the Racah, Wilson, Hahn, Jacobi, Legendre, Bessel and Laguerre polymomials are special instances of the general class of polynomials $\{M_n(s,A,\beta;x)$, n=0,1,2,...} with the help of (4.8.2), while starting this section of particular cases. There the respective specializations are also mentioned. Using the same set of particular values of the parameters in (4.8.26) one immediately gets the inverse series relations of the respective polynomials. In the following these pairs of inverse series relations are enlisted.

Inversion pair of extended Racah Polynomial

$$R_n^s\left(x(x+\gamma+\delta+1),\alpha,\beta,\gamma,\delta\right)$$

$$=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k(x+\gamma+\delta+1)_k}{(1+\alpha)_k(1+\gamma)_k(\beta+\delta+1)_k}$$

$$\Leftrightarrow \qquad \textbf{(4.8.27)}$$

$$(-x)_n(x+\gamma+\delta+1)_n=n!(1+\alpha)_n(1+\gamma)_n(\beta+\delta+1)_n$$

$$\sum_{k=0}^{sn}\frac{(-1)^k(1+\alpha+\beta+2k)}{(sn-k)!k!(1+\alpha+\beta+k)_{sn+1}}.R_k^s\left(x(x+\gamma+\delta+1),\alpha,\beta,\gamma,\delta\right)$$

Inversion pair of extended Wilson polynomial

$$P_n^s(x^2)=(a+b)_n(a+c)_n(a+d)_n$$

$$\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(a+b+c+d+n-1)_{sk}(a+ix)_k(a-ix)_k}{(a+b)_k(a+c)_k(a+d)_k\ k!}$$

$$\Leftrightarrow \qquad \textbf{(4.8.28)}$$

$$\frac{(a+ix)_n(a-ix)_n}{(a+b)_n(a+c)_n(a+d)_n n!}=$$

$$\sum_{k=0}^{sn}\frac{(-1)^k(a+b+c+d+2k-1)}{k!(sn-k)!(a+b+c+d+k-1)_{sn+1}(a+b)_k(a+c)_k(a+d)_k}P_k^s(x^2)$$

Inversion pair of extended Hahn Polynomial

$$\left.\begin{aligned}&Q_{n,s}(x;\alpha,\beta,N)=\sum_{k=0}^{[n/s]}\frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}(-x)_k}{k!(1+\alpha)_k(-N)_k}\\&\Leftrightarrow\\&(-x)_n=(1+\alpha)_n(-N)_n\,n!\sum_{k=0}^{sn}\frac{(-1)^k(1+\alpha+\beta+2k)\ Q_{k,s}(x;\alpha,\beta,N)}{k!(sn-k)!(1+\alpha+\beta+k)_{sn+1}}\end{aligned}\right\}\qquad\textbf{(4.8.29)}$$

Inversion pair of extended form of Jacobi Polynomial

$$\left.\begin{aligned}
&P_{n,s}^{(\alpha,\beta)}(x) = (1+\alpha)_n \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(1+\alpha+\beta+n)_{sk}}{k!(1+\alpha)_k}\left(\frac{1-x}{2}\right)^k \\
&\Leftrightarrow \\
&\left(\frac{1-x}{2}\right)^n = n!(1+\alpha)_n \sum_{k=0}^{sn} \frac{(-1)^k(1+\alpha+\beta+2k)}{k!(sn-k)!(1+\alpha+\beta+k)_{sn+1}(1+\alpha)_k} P_{k,s}^{(\alpha,\beta)}(x)
\end{aligned}\right\} \quad \textbf{(4.8.30)}$$

Inversion pair of extended Legendre Polynomial

$$\left.\begin{aligned}
&P_n^s(x) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(n+1)_{sk}}{k!k!}\left(\frac{1-x}{2}\right)^k \\
&\Leftrightarrow \\
&\left(\frac{1-x}{2}\right)^n = n!\,n! \sum_{k=0}^{sn} \frac{(-1)^k(2k+1)}{(sn-k)!\,k!\,(k+1)_{sn+1}} P_k^s(x)
\end{aligned}\right\} \quad \textbf{(4.8.31)}$$

Inversion pair of extended Bessel Polynomial

$$\left.\begin{aligned}
&Y_n^s(x) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(n+1)_{sk}}{k!}\left(\frac{-x}{2}\right)^k \\
&\Leftrightarrow \\
&\left(\frac{x}{2}\right)^n = n! \sum_{k=0}^{sn} \frac{(-1)^k(2k+1)}{(sn-k)!\,k!\,(k+1)_{sn+1}} Y_k^s(x)
\end{aligned}\right\} \quad \textbf{(4.8.32)}$$

Inversion pair of extended Laguerre Polynomial

$$\left.\begin{aligned} L_{n,s}^{(\alpha)}(x) &= (1+\alpha)_n \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}\, x^k}{k!\,(1+\alpha)_k} \\ \frac{x^n}{(1+\alpha)_n} &= \sum_{k=0}^{sn} \frac{(-sn)_k\, L_{k,s}^{(\alpha)}(x)}{(1+\alpha)_k} \end{aligned}\right\} \qquad \textbf{(4.8.33)}$$

(C-3) Combinatorial Identities

Besides yielding extended forms of polynomials together with their inverse series relations and the extensions to the ordinary forms of the inversion pairs due to Gessel and Stanton, theorem-7 also possesses the potential to give rise to some seemingly new combinatorial identities. Some known combinatorial identities are also contained in the theorem & some of them are inverted through it.

The combinatorial Identity (J. Riodan[1], p. 57)

$$x^{2n} = \sum_{k=0}^{n} \binom{2n}{n-k} b_{2k}(x)$$

and its inverse are special cases of theorem-7. The above identity in an equivalent form with its inverse is given by

$$x^{2n} = \sum_{k=0}^{n} \frac{(2n)!\, b_{2k}(x)}{(n+k)!\,(n-k)!}$$

$\Leftrightarrow$ **(4.8.34)**

$$\frac{b_{2n}(x)}{(2n)!} = \sum_{k=0}^{n} \frac{(-1)^{n-k}(k+1)\, x^{2k}}{(n+k)!\,(n+1)\,(2k)!}$$

In fact taking s=1 in the theorem, one gets

$$T(n) = \sum_{k=0}^{n} \frac{\Gamma(A+k+n\beta)}{(n-k)!} R(k)$$

$\Leftrightarrow$ **(4.8.35)**

$$R(n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(A+k+k\beta)T(k)}{\Gamma(A+n+k\beta+1)(n-k)!}.$$

Putting A=1 and β=0 in this pair, it becomes

$$T(n) = \sum_{k=0}^{n} \frac{\Gamma(1+k)}{(n-k)!} R(k)$$

$\Leftrightarrow$

$$R(n) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(k+1)T(k)}{\Gamma(n+2)(n-k)!}.$$

Now choosing $R(k) = \frac{b_{2k}(x)}{(n+k)!\,k!}$ one obtains from the first series above $T(n) = \frac{x^{2n}}{(2n)!}$. Thus the first series of (4.8.34) is obtained. Its inverse follows with the same choices from the second series in the above pair.

Consider now the pair of inverse series relations which is of the simpler Legendre Class [J. Riordan 1, p. 68] namely,

$$a_n = \sum_{k=0}^{n} \binom{n+p+k}{n-k} b_k$$

$\Leftrightarrow$

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} \left[\binom{2n+p}{n-k} - \binom{2n+p}{n-k-1} \right] a_k$$

or equivalently

$$a_n = \sum_{k=0}^{n} \frac{(n+p+k)!\ b_k}{(p+2k)!(n-k)!}$$

$\Leftrightarrow$ **(4.8.36)**

$$b_n = \sum_{k=0}^{n} \frac{(-1)^{n+k}(p+2k+1)(p+2n)!}{(n+p+k+1)!(n-k)!}\ a_k .$$

When β=1, A=p+1, and $R(k) = \dfrac{b_k}{(p+2k)!}$ then from the first series of (4.8.35) one gets T(n)=a_n; which gives the first series of the pair (4.8.36). With these substitutions the inverse series follows from the second series of (4.8.35). Thus the pair (4.8.36) can be obtained.

The second pair in the Table 2.5 (Simpler Legendre Class) of J. Riordan[1], (p. 68) also occurs as a special case.

For obtaining this, put β=1, A=p then the pair (4.8.35) reduces to

$$T(n) = \sum_{k=0}^{n} \frac{\Gamma(p+n+k)}{(n-k)!}\ R(k)$$

$\Leftrightarrow$

$$R(n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(p+2k)\ T(k)}{\Gamma(p+n+k+1)(n-k)!} .$$

Here

$$R(n) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(p+2k)\ T(k)}{\Gamma(p+n+k)(n-k)!} .$$

Now setting $T(k) = \dfrac{(-1)^k b_k}{p+2k}$ and replacing R(n) by $(-1)^n$ R(n), one arrives

at

$$\frac{a_n}{(2n+p)!} = \sum_{k=0}^{n} \frac{b_k}{(p+n+k)!(n-k)!}$$

with $R(n) = \dfrac{a_n}{(2n+p)!}$, which is known [J. Riordan].

Using the same substitutions in the second series in (4.8.35) the inverse is obtained, thus one arrives at the inversion pair:

$$b_n = \sum_{k=0}^{n} \frac{(-1)^{n+k}(p+2n)(n+p+k-1)!}{(p+2k)!(n-k)!} a_k$$

$$\Leftrightarrow \tag{4.8.37}$$

$$a_n = \sum_{k=0}^{n} \frac{(p+2n)! \ b_k}{(n+p+k)!(n-k)!}.$$

The inverse pair in J. Riordan[1, p. 79] is:

$$(-1)^n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} (-1)^k$$

$$\Leftrightarrow$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1}\right]$$

or equivalently,

$$(-1)^n = \sum_{k=0}^{n} \frac{(n+k)!\,(-1)^k}{(n-k)!\,k!\,k!}$$

$$\Leftrightarrow \tag{4.8.38}$$

$$\frac{1}{n!\,n!} = \sum_{k=0}^{n} \frac{(2k+1)}{(n+k+1)!(n-k)!}.$$

This pair is easily obtained from (4.8.35) by setting A=β=1 and choosing $R(k)=\frac{(-1)^k}{k!\,k!}$. Then one gets from the first series of (4.8.35), T(n)=$(-1)^n$, and thus the first identity of the above pair is obtained. The inverse series is obtained similarly.

The combinatorial identity in J. Riordan[1, p. 38] is:

$$(-1)^{n+1}n=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k$$

or

$$(-1)^{n+1}\,n^2=\sum_{k=0}^{n-1}\frac{(n+k)!(-1)^k}{(n-k-1)!\,k!\,k!}. \qquad \textbf{(4.8.39)}$$

Taking β=1, A=2 in (4.8.35), it takes the form:

$$T(n)=\sum_{k=0}^{n}\frac{(n+k+1)!}{(n-k)!}\;R(k)$$

$$\Leftrightarrow$$

$$R(n)=\sum_{k=0}^{n}(-1)^{n-k}\frac{2(k+1)\;\;T(k)}{(n+k+2)!(n-k)!}.$$

On replacing n by n-1, this pair becomes

$$T^*(n)=\sum_{k=0}^{n-1}\frac{(n+k)!}{(n-k-1)!}\;R^*(k)$$

$$\Leftrightarrow \qquad \textbf{(4.8.40)}$$

$$R^*(n)=\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{2(k+1)\ T^*(k)}{(n+k+1)!(n-k-1)!}.$$

On comparing the series in (4.8.39) with the first series in (4.8.40) suggests the particularizations $R^*(m)=\frac{(-1)^m}{m!\,m!}$ and $T^*(m)=(-1)^{m+1}m^2$.

The inverse series identity for (4.8.39) is thus obtained from the second series in (4.8.40) which is given by

$$\frac{1}{n!\,n!}=\sum_{k=0}^{n-1}\frac{2(k+1)\,k^2}{(n+k+1)!\,(n-k-1)!}. \qquad \textbf{(4.8.41)}$$

Another identity (J. Riordan [1, p.83])

$$(-1)^{n+1}=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k+1}(-1)^k$$

or equivalently

$$(-1)^{n+1}=\sum_{k=0}^{n-1}\frac{(n+k)!\,(-1)^k}{(n-k-1)!\,(k+1)!\,k!} \qquad \textbf{(4.8.42)}$$

is also contained in the pair (4.8.35) for β=1 and A=2.

Once again following the process, one gets with β=1, A=2 and with n is replaced by n-1 in (4.8.35), the particular pair

$$T^*(n)=\sum_{k=0}^{n-1}\frac{(n+k)!\ R^*(k)}{(n-k-1)!}$$

$\Leftrightarrow$

$$R^*(n)=\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}\,2(k+1)\ T^*(k)}{(n+k+1)!\,(n-k-1)!},$$

which is in (4.8.42).

Choosing $R^*(k)=\frac{(-1)^k}{k!(k+1)!}$ in the first series above (that is, in (4.8.40)) suggests that,

$T^*(k)=(-1)^{k+1}$.

Hence, one finds the inverse series of (4.8.42) in the form

$$\frac{(-1)^n}{n!(n+1)!}=\sum_{k=0}^{n-1}\frac{(-1)^n 2(k+1)}{(n+k+1)!(n-k-1)!}.$$

There is one more series identity whose inverse is also constructed through the theorem.

It is (J. Riordan[1, p.84])

$$f_n=\sum_{k=1}^{n}\binom{2n}{n-k}\frac{1}{k}.$$

When this series is compared with the second series in (4.8.35) it suggests the substitutions A=2, β=1, and further,

$$R^*(m)=\frac{(-1)^m}{(2m)!}f_m,\ T^*(m)=\frac{(-1)^{m+1}}{2(m+1)^2}.$$

Thus the above identity in its equivalent form, together with its inverse (from the theorem) reads as:

$$f_n=\sum_{k=0}^{n-1}\frac{(2n)!}{(n+k+1)!(n-k-1)!(k+1)}$$

$\Leftrightarrow$ **(4.8.43)**

$$\frac{(-1)^{n+1}}{2(n+1)^2}=\sum_{k=0}^{n-1}\frac{(n+k)!(-1)^k f_k}{(n-k-1)!(2k)!}.$$

Apart from the above series identities there is one more namely:

$$\bar{2}^{2n}(2n+1)\binom{2n}{n} = \sum_{k=0}^{2n} (-1)^k \binom{2n+1}{k+1} \binom{2k}{k} \frac{1}{2^k},$$

that is

$$\frac{1}{2^{2n}\, n!\, n!} = \sum_{k=0}^{2n} \frac{(-1)^k (2k)!}{(2n-k)!(k+1)!\, k!\, k!\, 2^k}. \qquad \textbf{(4.8.44)}$$

Setting s=2, β=0 and A=1, theorem-7 takes the form:

$$F(n) = \sum_{k=0}^{[n/2]} \frac{(2k)!G(k)}{(n-2k)!}$$

$\Leftrightarrow$

$$G(n) = \sum_{k=0}^{2n} \frac{(-1)^k (k+1)F(k)}{(2n+1)!(2n-k)!}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}(k+1)F(k)}{(n+1)!(n-k)!} = 0, \text{ if } n \neq 2m,\ m=0, 1, 2, \ldots\ .$$

Here choose

$$F(k) = \frac{(2k)!}{(k+1)\,(k+1)!\, k!\, k!\, 2^k} \text{ then } G_k = \frac{1}{2^{2k}\, k!\, k!}.$$

Thus the inverse of (4.8.44) follows in the form:

$$\frac{(2n)!}{(n+1)\,(n+1)!\, n!\, n!\, 2^n} = \sum_{k=0}^{[n/2]} \frac{(2k)!}{(n-2k)!\, 2^{2k}\, k!\, k!} \qquad \textbf{(4.8.45)}$$

with

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}(2k)!}{(n+1)!(n-k)!(k+1)!\, k!\, k!\, 2^k} = 0 \text{ if } n \neq 2m,\ m=0, 1, 2 \ldots\ .$$

4.9 INTERRELATION OF $S_n(\ell,m,\alpha,\beta;x)$ AND $M_n(s,A,B;x)$

In this section, one of the polynomials $S_n(\ell,m,\alpha,\beta;x)$ and $M_n(s,A,H,x)$ is expressed in a series of the other using the inverse series relation proved earlier.

The inverse series relations of these two polynomials are given by (Chapter-2)

$$S_n(\ell,m,\alpha,\beta;x)=\sum_{r=0}^{[n/m]}\frac{(-1)^{mr}\sigma_r\ x^r}{\Gamma(1+\beta-n\alpha+\ell r)\ (n-mr)!} \tag{4.9.1}$$

if and only if

$$\sigma_n x^n=\sum_{r=0}^{[mn]}\frac{(-1)^r\beta\,\Gamma(\beta+\ell n-r\alpha)}{(n-mr)!}\ S_r(\ell,m,\alpha,\beta;x), \tag{4.9.2}$$

and

$$M_n(s,A,H;x)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\Gamma(A+sk+nH)}{(n-sk)!}\psi_k\ x^k \tag{4.9.3}$$

if and only if

$$\psi_n\ x^n=\sum_{k=0}^{sn}\frac{(-1)^k(A+k+kH)}{\Gamma(A+sn+kH+1)(sn-k)!}M_k(s,A,H,x). \tag{4.9.4}$$

For instance, from (4.9.4)

$$x^r=\frac{1}{\psi_r}\sum_{j=0}^{sr}\frac{(-1)^j(A+j+jH)}{\Gamma(A+sr+jH+1)(sr-j)!}M_j(s,A,H;x).$$

Now using the expansion of x^n given in (4.9.4) in the explicit form (4.9.1), one gets

$$S_n(\ell,m,\alpha,\beta;x)=\sum_{r=0}^{[n/m]}\frac{(-1)^{mr}\sigma_r}{\Gamma(1+\beta-n\alpha+\ell r)\ (n-mr)!\ \psi_r}.$$

$$.\sum_{j=0}^{sr}\frac{(-1)^j(A+j+jH)}{\Gamma(A+sr+jH+1)\ (sr-j)!}M_j(s,A,H;x)$$

$$=\sum_{r=0}^{[n/m]}\sum_{j=0}^{sr}\frac{(-1)^{mr+j}(A+j+jH)\sigma_r}{(n-mr)!(sr-j)!\Gamma(1+\beta-n\alpha+\ell r)\psi_r}$$

$$\frac{M_j(s,A,H;x)}{\Gamma(A+sr+jH+1)}. \qquad \textbf{(4.9.5)}$$

Thus, $S_n(\ell,m,\alpha,\beta;x)$ is expressed in a series of $M_n(s,A,H,x)$.

In a similar manner using (4.9.2) in (4.9.3), one can express $M_n(s,A,H;x)$ in a series of $S_n(\ell,m,\alpha,\beta;x)$ as follows.

$$M_n(s,A,H;x)=\sum_{k=0}^{[n/s]}\frac{(-1)^{sk}\Gamma(A+sk+nH)}{(n-sk)!}\frac{\psi_k}{\sigma_k}.$$

$$.\sum_{r=0}^{mk}(-1)^r\beta.\frac{\Gamma(\beta+\ell k-r\alpha)\ S_r(\ell,m,\alpha,\beta,x)}{(mk-r)!}.$$

Thus,

$$M_n(s,A,H;x)=\sum_{k=0}^{[n/s]}\sum_{r=0}^{mk}\frac{(-1)^{r+sk}\beta\,\Gamma(A+sk+nH)\,\Gamma(\beta+\ell k-r\alpha)}{(n-sk)!(mk-r)!}.$$

$$.\frac{\psi_k}{\sigma_k}S_r(\ell,m,\alpha,\beta;x). \qquad \textbf{(4.9.6)}$$

A particular case of (4.9.5) may be illustrated by putting H=1, s=1, A=1+α+β, $\psi_k=\frac{1}{k!(1+\alpha)_k}$, with this the polynomial $M_n(s,A,H;x)$

yields the Jacobi polynomial $p_n^{(\alpha,\beta)}(x)\frac{n!}{(1+\alpha)_n}$, and on substituting

$\sigma_k = \frac{(a_1)_k \cdot (a_p)_k c^k}{(b_1)_k \cdots (b_q)_k k!}$ and replacing $S_n(l,m,\alpha,\beta;x)$ by $\frac{S_n(l,m,\alpha,\beta;x)}{\Gamma(1+\beta-n\alpha)n!}$, one

gets $S_n(l,m,\alpha,\beta;x) = \mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b);x]$. Hence from (4.9.5), the expansion

formula

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \sum_{r=0}^{mk} \frac{(-1)^{r+k}\beta\Gamma(1+\alpha+\beta+n+k)\Gamma(\beta+mk\alpha-r\alpha)(b_1)_k \cdots (b_q)_k}{(n-sk)!(mk-r)!(1+\alpha)_k \Gamma(1+\beta-r\alpha)r!\, c^k (a_1)_k (a_p)_k}.$$

$$.\ \mathcal{H}_{r,l,m}^{(\alpha,\beta)}[(a);(b),x]. \qquad \textbf{(4.9.7)}$$

On the other hand, taking $\sigma_k = \frac{(a_1)_k \cdot (a_p)_k}{(b_1)_k \cdots (b_q)_k k!}$ and replacing the

polynomial $S_n(l,m,\alpha,\beta;x)$ by $\frac{S_n(l,m,\alpha,\beta;x)}{\Gamma(1+\beta-n\alpha)n!}$, it reduces to the extended

Jacobi polynomial $\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a),(b);x]$ and putting H=1, s=1, A=1,

$\psi_k = \frac{1}{k!k!}$ in (4.9.3), one gets the Legendre polynomial

$$P_n(2x+1) = {}_2F_1\left[\begin{matrix}-n, n+1; x\\ 1,\end{matrix}\right].$$

Hence the expansion formula (4.9.6) yields

$$\mathcal{H}_{n,l,m}^{(\alpha,\beta)}[(a);(b);x] = \sum_{r=0}^{[n/m]} \sum_{j=0}^{r} \frac{(-1)^{mr+j}(2j+1)r! \prod_{i=1}^{p}(a_i)_r\, n!\, P_j(2x+1)}{(n-mr)!(r-j)! \prod_{j=1}^{q}(b_j)_r \Gamma(1+\beta-n\alpha+lr)\Gamma(r+j+r)}.$$