1. Introduction

1.1. Introduction

Mathematical modelling of the various physical phenomenon under the influence of physical laws is represented in form of differential equations. In these models, various parameters involved are obtained from experiments or observations. These values obtained may have certain measurement errors or imprecision. These kinds of error, in turn, leads to imprecision in the modelling of physical problem. The model of such phenomenon with imprecise parameters, using fuzzy sets give more realistic realizations. Such general dynamical models involving fuzzy parameters are given as,

$$\dot{\tilde{X}} = \tilde{f}\left(t, \tilde{X}\right);\ \tilde{X}(0) = \tilde{X}_0.$$

The above equation explains the real-world problem with imprecision in a more realistic way than the its crisp counterpart.

The concept of the fuzzy set was first discussed in a seminal paper by Zadeh [1] in 1965. And the concept of fuzzy derivative was first introduced by Chang and Zadeh [2] in 1972. Dubois and Prade [3], defined derivative based on extension principle in 1982. Puri and Ralescue [4], introduced the H-derivative of fuzzy valued function based on Hukuhara difference [5]. Kaleva [6] and Seikkala [7], first simultaneously solved the fuzzy initial value problem with fuzzy initial condition. Kandel and Byatt [8] used this theory for the applications of fuzzy dynamical systems.

The basic and most popular approach to solve fuzzy differential equation (FDE) is Hukuhara differentiability which is based on H-difference. The drawback in using Hukuhara derivative is that the solution does not remain fuzzy as time increases. To overcome this situation, the Generalized Hukuhara derivative [9] is proposed, which is very popular among other fuzzy derivatives. When a generalized Hukuhara derivative is used to solve the fuzzy differential equation, the solution is obtained as a possible set of solutions from which one needs to choose the solution which best satisfies the problem. This is the disadvantage of generalized Hukuhara derivative. To overcome this in our work, we have proposed the Modified Hukuhra derivative which gives a unique fuzzy solution.

In the beginning, most of the authors have worked on scalar differential equation with fuzzy initial condition using different techniques like analytical, numerical, transformation and semi-analytical. The numerical method for solving FDE was introduced by M. Ma, Friedman and Kandel refer [10]. They used the classical Euler method, which is followed by a complete error analysis. Javed Shokri in [11] solved FDE by the modified Euler method. S. Abbasbandy and T. Allahviranloo refer [12] solved FDE by Taylor's method of order p and proved the order of convergence is $O(h^p)$, which is better than the order of convergence of Euler's method. The FDE is solved by some other numerical methods like Runge-Kutta and Corrector-Predictor method as in [13], [14] respectively. T. Jayakumar, D. Mahesh Kumar and Rangrajan [15] presented solution of FDE by the 5th order of Runge Kutta method. A numerical solution of second-order FDE is given by N. Parandin in [16], using Runge Kutta method. Solution of FDE under generalized differentiability by Improved Euler method is given by K. Kangrajan and R. Suresh refer [17], here only the fuzzy initial condition is considered.

Allahviranloo and Ahmadi [18], proposed fuzzy Laplace transforms (FLT) for solving first-order differential equations under generalized H-differentiability, without giving the existence condition. S. Salahsour and T. Allahviranloo [19], described the existence condition for Laplace transform and its inverse. Many other authors in [20], [21] used FLT to solve FDE. They solved the second-order fuzzy differential equation and n^{th} order fuzzy differential equation with fuzzy initial condition under Generalized Hukuhara derivative.

Some authors have used semi-analytical techniques to solve a nonlinear differential equation like Homotopy perturbation method (HPM), Adomian Decomposition method (ADM). Variational Iteration method (VIM) etc. as in [22], [23], [24], [25], [26], [27]. These techniques are also used in solving fuzzy nonlinear differential equations. Other methods for solving FDE and important results pertaining to continuity, the existence of solution and various applications are given in [28-58].

Our work [61-67] and [69] comprises of solution of the system of linear fuzzy differential equation with fuzzy parameters as well as fuzzy initial condition that is a fully fuzzy linear dynamical system, followed by some work on a nonlinear fully fuzzy differential equation. To solve fully fuzzy dynamical systems, we have used different techniques like analytical, numerical, transformation and semi-analytical techniques. We have also proposed and proved various existence and uniqueness results under different techniques.

We list the preliminary concepts used in the ensuing sections and details appertain to the development in our work for fuzzy dynamical systems.

1.2. Preliminaries:

1.2.1. Fuzzy Sets

A fuzzy set is collection ordered pair from a set $\widetilde{u}$ and membership function $\widetilde{u}\,(x), \forall x \in \mathbb{X}$, a universal set and $\widetilde{u}\,(x) \in (0,1]$. If a set is constructed by a discrete universe then it is known as a discrete fuzzy set and it is given as $\left\{\frac{\tilde{u}(x_1)}{x_1}, \frac{\tilde{u}(x_2)}{x_2}, \frac{\tilde{u}(x_3)}{x_3}, \ldots\right\}$. If a set is constructed by a continuous universe, then it is known as a continuous fuzzy set.

1.2.2. Fuzzy Number

A fuzzy set $\tilde{u}$ is said to be a fuzzy number if it satisfies the following properties,

- $\tilde{u}$ must be normal i.e membership value should be 1 for at least one point.
- α −cut of a fuzzy set $\tilde{u}$, ${}^{\alpha}\tilde{u} = \{x \in X, \tilde{u}(x) \geq \alpha\}$, which is an ordinary set, should be closed, for all $\alpha \in (0\ 1]$.
- Support ${}^{0}\tilde{u}$ should be bounded.

1.2.3. Fuzzy Number in parametric form

A fuzzy number in parametric form, obtained by performing $\alpha - cut$, is an ordered pair of the form ${}^{\alpha}\tilde{u}$= $[\underline{u}, \overline{u}]$ satisfying the following condition:

- $\underline{u}$ is bounded left continuous increasing function in [0 1].
- $\overline{u}$ is bounded right continuous decreasing function in [0 1].
- $\underline{u} \leq \overline{u}$.

1.2.4. Fuzzy Triangular Number:

The triangular fuzzy number is denoted as a triplet (p, q, r) and its membership function is given as,

$$\tilde{u}(x) = \begin{cases} \dfrac{x-p}{q-p} & p < x \leq q \\ \dfrac{r-x}{r-q} & q < x \leq r \\ 0 & otherwise \end{cases}$$

1.2.5. Fuzzy Arithmetic Operations:

Let $\tilde{u}$, $\tilde{v}$ are fuzzy numbers and k is scalar then the arithmetic operations between $\tilde{u}$ and $\tilde{v}$ are as follows,

$$^{\alpha}\tilde{u} \oplus {}^{\alpha}\tilde{v} = [\underline{u}, \overline{u}] \oplus [\underline{v}, \overline{v}\,] = [\underline{u} + \underline{v}\,, \overline{u} + \overline{v}],$$
$$k \otimes {}^{\alpha}\tilde{u} = k[\underline{u}, \overline{u}] = [k\underline{u}\,, k\overline{u}]\ \forall\, \alpha \in (0\ 1].$$

1.2.6. Fuzzy Lipschitz:

Let $E = \{\tilde{u}: R \to [0\ 1]\}$ is the collection of fuzzy numbers which is convex and compact and d is the Housdorff distance on E and satisfy all properties mentioned in section 1.2.2. For I is $[t_0, t]$ and $\tilde{x}, \tilde{y} \in E$ then $\tilde{f} : I \times E \to E$ is fuzzy Lipschitz [49] if,

$$d\left(\tilde{f}\,(t, \tilde{x}), \tilde{f}\,(t, \tilde{y})\right) \leq d(\tilde{x}, \tilde{y}).$$

1.2.7. Fuzzy continuity:

As given in [49], if $\tilde{f} : I \times E \to E$ then $\tilde{f}$ is fuzzy continuous at a point $(t_0, \tilde{x}_0)$ provided that for any fixed number $\alpha \in (0, 1]$ and any $\epsilon > 0, \exists\ \delta(\alpha, \epsilon)$ such that,

$$d\left(\tilde{f}\,(t, \tilde{x}), \tilde{f}\,(t_0, \tilde{x}_0)\right) < \epsilon$$

where, $|t - t_0| < \delta$ and $d(\tilde{x}, \tilde{x}_0\,) < \delta(\alpha, \epsilon)$.

1.2.8. Fuzzy derivative:

1.2.8.1. H difference:

1. The H-difference [5], between two compact intervals the parametric forms of $^{\alpha}\tilde{x} = [\underline{x}\ \overline{x}]$ and $^{\alpha}\tilde{y} = \left[\underline{y}\ \overline{y}\right]$is defined as,

$$^{\alpha}\tilde{x} \ominus {}^{\alpha}\tilde{y} = \left[\underline{x} - \underline{y}, \overline{x} - \overline{y}\,\right]$$

2. Let $\tilde{x}, \tilde{y} \in E$ and if there exists $\tilde{z} \in E$ such that $\tilde{x} = \tilde{y} \oplus \tilde{z}$, then $\tilde{z}$ is called the H-difference of $\tilde{x}$ and $\tilde{y}$ and it is denoted by $\tilde{x} \ominus \tilde{y}$. $\tilde{x} \ominus \tilde{y} \neq \tilde{x} + (-1)\tilde{y}$.

1.2.8.2. Hukuhara Derivative:

Consider, a fuzzy mapping $\tilde{f}: I \to E$ and $t_0 \in I$ then $\tilde{f}$ is said to be differentiable at t_0, as in [4], if there exists an element $\dot{\tilde{f}}(t_0) \in E$, such that for all small $h > 0$, $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0).$$

1.2.8.3. Strongly Generalized Differentiability:

As in [9], consider a fuzzy mapping $\tilde{f}: I \to E$ and $t_0 \in I$ then $\dot{\tilde{f}}$ is said to be strongly generalized differentiable at t_0 in E if,

- For all $h > 0$, sufficiently small $\exists$ $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

- For all $h > 0$, sufficiently small $\exists$ $\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)$, $\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{h} = \dot{\tilde{f}}(t_0)$$

- For all $h > 0$, sufficiently small $\exists$ $\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)$, $\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{-h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{-h} = \dot{\tilde{f}}(t_0)$$

- For all $h > 0$, sufficiently small $\exists$ $\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ exist and the limits,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{-h} = \lim_{h \to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

1.2.8.4. Generalized Hukuhara Difference:

Generalized Hukuhara differences [59] is defined as follows, considering E as space of a convex nonempty set of $\mathbb{X}$, taking $\tilde{A}, \tilde{B} \in E$ then the generalized difference (gH) of $\tilde{A}$ and $\tilde{B}$,

$$\tilde{A} \ominus_g \tilde{B} \Leftrightarrow \begin{cases} \tilde{A} = \tilde{B} \oplus \tilde{C} \\ or\ \tilde{B} = \tilde{A} \oplus (-1)\tilde{C} \end{cases}$$

Based on the above difference, the Generalized Hukuhara derivative is given as follows.

1.2.8.5. Generalized Hukuhara Derivative:

Let fuzzy mapping $\tilde{f}: I \to E$ and $t_0 \in I$ then $\tilde{f}$ is said to be generalized Hukuhara differentiable at $t_0 \in I, \exists$ an element $\dot{\tilde{f}}(t_0) \in E$ and given as,

$$\lim_{h \to 0+} \frac{\tilde{f}(t_0 + h) \ominus_g \tilde{f}(t_0)}{h} = \dot{\tilde{f}}(t_0).$$

1.2.8.6. Seikkala Derivative:

The definition of Seikkala derivative [7] is as follows if these $\dot{x}_1(t, \alpha), \dot{x}_2(t, \alpha)$ are $\alpha - cuts$ of fuzzy number for $t \in I$, then Seikkala derivative denoted by $SDX(t)$ exist and it is given as,

$$SDX(t)(\alpha) = [\dot{x}_1(t, \alpha), \dot{x}_2(t, \alpha)]$$

We have carried out the major work in direction of a fully fuzzy dynamical system. Since, our work is for the system of fully fuzzy differential equations, we have extended results for continuity, differentiability, integrability etc. in the generalized n vector space E^n as given in the research articles [53], [60].

1.3. Our Work

Research in the area of fuzzy dynamical systems, till the start of our work majorly included the solution of scalar differential equation with only initial condition of fuzzy. To extend it, we initially started with solving, the fuzzy Prey-Predator model analytically. Before us, most of the people have solved Prey- Predator model using numerical techniques. We solved the Prey-Predator model using an analytical technique and got the closed-form fuzzy solution.

1.3.1. Analytical Method (Eigenvector and Eigenvalue):

In chapter 2, we have considered following the Prey-Predator model,

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{aligned} \tag{1.1}$$

with fuzzy initial conditions, $x(0) = \tilde{x}_0$ and $y(0) = \tilde{y}_0$.
The general form of equation (1.1) is,

$$\dot{\tilde{X}} = A \otimes \tilde{X} \oplus f(t, \tilde{X})\,; \tilde{X}_0 \tag{1.2}$$

Now, using fundamental matrix $\psi(t)$, transition matrix $\phi(t, 0)$ and B the nonhomogeneous part then the solution of (1.2) is given as,

$$\tilde{X}(t) = \phi(t, 0)\, \tilde{X}_0 + \int_0^t \phi(t, \tau) B dt.$$

After this work, we solved the system of differential equations involving fuzzy parameters as well as fuzzy initial conditions. In the beginning, we solved these systems using numerical techniques, then using transform technique like Laplace Transform, and finally using a semi-analytical technique, the Adomian decomposition method.

1.3.2. Numerical techniques:

In chapter 3, we have solved the system having fuzzy parameters with fuzzy initial conditions i.e., fully fuzzy systems using two numerical techniques. The first one numerical technique is based on the discretization of Hukuhara derivative and the other one is the Improved Euler Method.
We have considered the following system of differential equations,

$$\dot{\tilde{X}} = \tilde{f}\big(t, \tilde{X}\big) \tag{1.3}$$

with initial condition, $\tilde{X}(0) = \tilde{X}_0$.
Here, $\tilde{f} : I \times E^n \to E^n$ and

$$\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}, \tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$$

where, each $\tilde{f}_i$ is Hukuhara differentiable, $\forall\, i = 1,2,3, \dots n$ and $\tilde{f}$ to be linear or nonlinear.
We consider $\tilde{f}$, given as,

where, $\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix}$ is $n \times n$ fuzzy matrix and $\tilde{B} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$ is $n \times 1$ column vector.

The parametric numerical scheme for equation (1.3), with $\tilde{f}$ as in equation (1.4) is given by,

$$\begin{bmatrix} \underline{X}_{k+1} \\ \overline{X}_{k+1} \end{bmatrix} = \begin{bmatrix} A_l & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} \underline{X}_k \\ \overline{X}_k \end{bmatrix} + \begin{bmatrix} B_l \\ B_u \end{bmatrix}, \tag{1.5}$$

where, $k = 0, 1, 2, \dots K$ with $\underline{X}(0) = \underline{X}_0$ and $\overline{X}(0) = \overline{X}_0$.

For equation (1.3), results for the existence of solution and convergence analysis are proposed and proved.

Further, for a system such as equation (1.3) a numerical scheme on the line of Improved Euler method is proposed and established along with error analysis.

After numerical technique, we solved fully fuzzy dynamical system by transformation technique, Fuzzy Laplace Transform.

1.3.3. Fuzzy Laplace Transform

We developed the theory in which we apply Laplace transformation initially for linear homogeneous then for the non-homogeneous system in chapter 4. For the solution of both kinds of systems, we used the diagonalization concept before applying the fuzzy Laplace transform.

We consider the following model for a homogeneous fuzzy dynamical system.

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \,;\; \tilde{X}(0) = \tilde{X}_0 \tag{1.6}$$

where, $\tilde{A}$ is as defined in equation (1.4).

The parametric form of equation (1.6) is,

$$\left[\underline{\dot{X}}\ \dot{\overline{X}}\right] = \left[\min\left(\underline{A}\,\underline{X}, \underline{A}\,\overline{X}, \underline{X}\,\overline{A}, \overline{A}\,\overline{X}\right), \max\left(\underline{A}\,\underline{X}, \underline{A}\,\overline{X}, \underline{X}\,\overline{A}, \overline{A}\,\overline{X}\right)\right];\ \left[\underline{X}_0\,, \overline{X}_0\right] \tag{1.7}$$

The solution of equation (1.7), is given as,

$$\underline{X} = \underline{X}_0 e^{\underline{D}t}, \overline{X} = \overline{X}_0 e^{\overline{D}t}$$

where, D_l and D_u both are diagonal matrices, obtained from A_l and A_u.

For equivalent non-homogeneous linear systems i.e., $\tilde{f}(t, \tilde{X}) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B}$ and the solution of such system is given as,

$$\underline{X} = \underline{P}\,\mathcal{L}^{-1}\left\{\left[sI - \underline{D}\,\right]^{-1}\left[\underline{P}^{-1}\,\underline{X}(0) + \underline{P}^{-1}\mathcal{L}[\underline{B}]\right]\right\},$$

$$\overline{X} = \overline{P}\mathcal{L}^{-1}\{\left[sI - \overline{D}\right]^{-1}\left[\overline{P}^{\,-1}\,\overline{X}(0) + \overline{P}^{\,-1}\mathcal{L}[\overline{B}\,]\right\},$$

where, $\underline{P}$ and $\overline{P}$ are orthogonal matrices. The result for the existence of a solution using fuzzy Laplace Transform is also proved.

For the rigorous development of theory to obtain the solution of a fully fuzzy dynamical system using the Laplace Transform technique, we afresh define the Modified Hukuhara derivative in chapter 4, for the first time. Our proposed derivative has the advantage that it gives a unique solution of fuzzy dynamical system automatically without the author to select the suitable one as in the case of Generalized Hukuhara derivative shown in [59]. The proposed Modified Hukuhare derivative is defined as follows, for the proposal we took difference as in [59].

1.3.3.1. Modified Hukuhare derivative (mH-derivative)

Let $f: I \to E^n$ is said to be modified Hukuhara differentiable at t_0 if there exists an element, $\dot{\tilde{f}}(t_0) \in E^n$, such that for all $h > 0$ sufficiently small, $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ should exist and the limits,

$$\lim_{h\to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h\to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

Its equivalent parametric form is given below,

$$\lim_{h\to 0+} \frac{{}^{\alpha}\tilde{f}\,(t_0+h) - {}^{\alpha}\tilde{f}\,(t_0)}{h} =$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\},\right.$$

$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\}\right]$$

And,

$$\lim_{h\to 0-} \frac{{}^{\alpha}\tilde{f}\,(t_0) - {}^{\alpha}\tilde{f}\,(t_0-h)}{h} =$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}\right\},\right.$$

$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}, \lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h}\right\}\right].$$

Along with the proposed mH-derivative, various results pertaining to the existence of fuzzy Laplace transform for the function, Laplace transform of the derivative and fuzzy convolution theorem are proposed and proved.

1.3.3.2. Fuzzy Laplace Transform under mH-derivative

Consider a fuzzy valued function ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ in parametric form, is bounded and piecewise continuous on the interval $[0, \infty)$ and suppose that $\tilde{f}(t) \otimes e^{-st}$ is improper fuzzy Riemann integrable [48], then fuzzy Laplace Transform of $\tilde{f}(t)$ defined as,

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \int_0^{\infty} e^{-st} \otimes \tilde{f}(t)dt,$$

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \lim_{t\to\infty} \int_0^{t} e^{-st} \otimes \tilde{f}(t)dt.$$

Taking alpha cut on both sides,

$$\mathcal{L}\left({}^{\alpha}\tilde{f}(t)\right) = \lim_{t\to\infty} \int_0^{t} e^{-st} \otimes \left[\underline{f}(t), \overline{f}(t)\right] dt$$

$$\mathcal{L}\left(\left[\underline{f}(t), \overline{f}(t)\right]\right) = \lim_{t\to\infty} \left[\int_0^{t} e^{-st}\underline{f}(t)\, dt, \int_0^{t} e^{-st}\overline{f}(t)dt\right]$$

$$\underline{F}(s) = \min\left\{\lim_{t\to\infty} \left[\int_0^t e^{-st}\underline{f}(t)\, dt, \int_0^t e^{-st}\overline{f}(t)dt\right]\right\},$$

$$\overline{F}(s) = \max\left\{\lim_{t\to\infty} \left[\int_0^t e^{-st}\underline{f}(t)\, dt, \int_0^t e^{-st}\overline{f}(t)dt\right]\right\}.$$

1.3.3.3. Fuzzy Laplace of Derivative

If ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be continuous fuzzy valued function, $\lim_{t\to\infty} e^{-st}\underline{f}(t) \to 0$ and $\lim_{t\to\infty} e^{-st}\overline{f}(t) \to 0$ for large value of s and $\dot{\tilde{f}}(t)$ is piecewise continuous then $\mathcal{L}\left(\dot{\tilde{f}}(t)\right)$ exist, and is given by,

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = s\mathcal{L}\left(\tilde{f}(t)\right) \ominus \tilde{f}_0.$$

1.3.3.4. Fuzzy convolution Theorem

Let $\tilde{f}(s)$ and $\tilde{g}(s)$ denote the fuzzy inverse Laplace transforms of $\tilde{f}(t)$ and $\tilde{g}(t)$ respectively. Then the product, $\tilde{f}(s)\tilde{g}(s)$ is the fuzzy inverse Laplace transform of the convolution of $\tilde{f}$ and $\tilde{g}$, is given by,

$$\mathcal{L}\left(\tilde{f}(t) * \tilde{g}(t)\right) = \tilde{f}(s) * \tilde{g}(s).$$

After that, we have solved the fully fuzzy system by semi-analytical technique.

1.3.4. Semi-analytical Technique

Following semi-analytical techniques are used.

- Fuzzy Adomian Decomposition Method in parametric form (FADMP)
- Fuzzy Adomian Decomposition Method (FADM)

1.3.4.1. Fuzzy Adomian Decomposition Method in Parametric form (FADMP)

In chapter 5, we investigate the solution of a fully fuzzy nonlinear dynamical system by FADM in parametric form. In this method, we decompose the nonlinear part of a fully fuzzy system into a series of fuzzy Adomian polynomials and obtain the solution of the system in series form.

Consider nonlinear fuzzy differential equation given as,

$$\mathbb{L}\tilde{u} \oplus N\tilde{u} \oplus \mathbb{R}\tilde{u} = \tilde{g}$$

where, $\mathbb{L}$ is linear highest order differential operator, N is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of $\mathbb{L}$ and $\tilde{g}$ is source term. Then applying, $\mathbb{L}^{-1}$ i.e., fuzzy integration operator on both sides we get,

$$\tilde{u} = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}(N\tilde{u}) \oplus \mathbb{L}^{-1}(\mathbb{R}\tilde{u}).$$

The parametric form of the above equation is as follows,

$$[\underline{u}\ \overline{u}] = \mathbb{L}^{-1}\left[\min\left(\underline{g}, \overline{g}\right), \max\left(\underline{g}, \overline{g}\right)\right] \ominus \mathbb{L}^{-1}[\min(N\underline{u}, N\overline{u}), \max(N\underline{u}, N\overline{u})] \oplus \mathbb{L}^{-1} [\min(\mathbb{R}\underline{u}, \mathbb{R}\overline{u}), \max(\mathbb{R}\underline{u}, \mathbb{R}\overline{u})] \tag{1.8}$$

Now, we decompose the nonlinear term, $[\min(N\underline{u}, N\overline{u}), \max(N\underline{u}, N\overline{u})]$ into the series of Adomian polynomials $\underline{A}_1, \underline{A}_2, \underline{A}_3, \ldots$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots$ with decomposition factor λ, the general polynomial is given as,

$$\underline{A}_n = \min\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{\infty}\underline{u}_k\lambda^k\right), \frac{1}{n!}\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{\infty}\overline{u}_k\lambda^k\right)\right],$$

$$\overline{A}_n = \max\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{\infty}\underline{u}_k\lambda^k\right), \frac{1}{n!}\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{\infty}\overline{u}_k\lambda^k\right)\right].$$

So, substituting these Adomian polynomials in equation (1.8), the desirable solution obtained as,

$$[\underline{u}, \overline{u}] = [\underline{u}_0, \overline{u}_0] + [\underline{u}_1, \overline{u}_1] + [\underline{u}_2, \overline{u}_2] + [\underline{u}_3, \overline{u}_3] + \cdots$$

Using FADMP, we have solved the real-life application, which is given in [67].

1.3.4.2. Application: Mathematical Modelling of Air Heating Solar Collectors with Fuzzy Parameters

The problem of solar collectors is to convert solar radiation into energy, which can be used for many purposes such as electricity generation, blow-drying process, heating of water in many industrial applications. Various physical factors play a vital role in the mathematical formulation [68], [76]. The ambient temperature depends on weather and hence varies from morning to evening, so taking a range of such value is more realistic than the fixed value of the same. Similarly, the rate of mass of airflow also varies. Due to this reason, we have proposed the mathematical model of solar air collector involving fuzzy initial temperature and fuzzy rate of mass of airflow. Thus, the fuzzy model is proposed,

$$\left(\left(\frac{\tilde{\dot{m}}}{W}\right)\otimes a\otimes\left(\frac{d\tilde{T}}{dx}\right)\right)\oplus\left(\left(\frac{\tilde{\dot{m}}}{W}\right)\otimes b\otimes\left(\frac{d\tilde{T}}{dx}\right)\otimes\tilde{T}(x)\right)\oplus\left(\tilde{F}\otimes U_l\otimes\tilde{T}(x)\right)=\tilde{F}\otimes(S+T_aU_l)\ ;\ \tilde{T}(0)=\tilde{T}_o.$$

The above model is solved by FADMP techniques in chapter 6. The solution technique in detail is explained in a further chapter.

Till now the solution technique that we adopted, required the fuzzy systems to be converted to the crisp by considering its equivalent parametric form. This motivated us to develop the results for fuzzy calculus so that we can solve fuzzy systems directly. This required the extension of the mH-derivative for a function $\tilde{f}: E \rightarrow E$ and further.

1.3.4.3. Fuzzy Adomian Decomposition Method (FADM)

To construct FADM, we need to redefine the Modified Hukuhara (mH) derivative (later, we name it Modified Generalized Hukuhara derivative (mgH)) for such fuzzy valued function which deals with the complexity involved in fuzzy mapping as well as both input and output are fuzzy. To evaluate this derivative, we need to take the parametric form of input and the fuzzy mapping which is involved in it. Our definition gives a more realistic explanation of fuzzy derivatives and under this derivative, we also develop fuzzy Taylor's series along with its convergence. With the help of the above-mentioned results, we proposed and proved the Fuzzy Adomian Decomposition method (FADM) [69]. The advantage of this technique is that we solve a problem completely in a fuzzy environment instead of bringing it to the crisp for solving and going back in fuzzy as done conventionally.

All the results are given below to support this fuzzy technique.

1.3.4.4. Fuzzy Taylor's Theorem:

If a fuzzy valued function, $\tilde{f}(x): E^n \rightarrow E^n$ is, *n* times modified generalized Hukuhara (mgH) differentiable, then fuzzy Taylor's expansion is given as,

$$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}_0) \oplus \dot{\tilde{f}}(\tilde{x}_0) \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \frac{\ddot{\tilde{f}}(\tilde{x}_0)}{2!} \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \frac{\dddot{\tilde{f}}(\tilde{x}_0)}{3!} \otimes (\tilde{x} \ominus \tilde{x}_0)^3 \oplus \frac{\tilde{f}^4(\tilde{x}_0)}{4!} \otimes (\tilde{x} \ominus \tilde{x}_0)^4 \oplus \frac{\tilde{f}^5(\tilde{x}_0)}{5!} \otimes (\tilde{x} \ominus \tilde{x}_0)^5 \oplus \dots \dots \oplus \frac{\tilde{f}^n(\tilde{x}_0)}{n!} \otimes (\tilde{x} \ominus \tilde{x}_0)^n \oplus \dots$$

1.3.4.5. Fuzzy Power series and its radius of convergence:

A power series of fuzzy valued functions around the point $\tilde{x}_0$ can be given as,

$\sum \tilde{a}_n \otimes (\tilde{x} \ominus \tilde{x}_0)^n = \tilde{a}_0 \oplus \tilde{a}_1 \otimes (\tilde{x} \ominus \tilde{x}_0) \oplus \tilde{a}_2 \otimes (\tilde{x} \ominus \tilde{x}_0)^2 \oplus \dots$ where, $\tilde{a}_n$ are any fuzzy coefficients and *n* is a positive integer. The radius of convergence for this power series is given by,

$$\tilde{R} = \lim_{n\to\infty} \left| \frac{\widetilde{a_n}}{\widetilde{a_{n+1}}} \right|.$$

In the next section, we give brief introduction of Fuzzy Adomian Decomposition method in a complete fuzzy environment.

1.3.4.6. FADM in fuzzy environment:

Consider a nonlinear fuzzy differential equation,

$$\mathbb{L}\tilde{u} \oplus N\tilde{u} \oplus \mathbb{R}\tilde{u} = \tilde{g}$$

where, $\mathbb{L}$ is highest order linear (mgH) differentiable operator, N is nonlinear operator, $\mathbb{R}$ is the operator of less order than that of L and $\tilde{g}$ is source term.

$$\tilde{u} = \mathbb{L}^{-1}\tilde{g} \ominus \mathbb{L}^{-1}(N\tilde{u} \oplus \mathbb{R}\tilde{u})$$

This method gives that unknown fuzzy function can be expressed as a series of, $\tilde{u} = \sum_{n=0}^{\infty} \tilde{u}_n$. In this method, the nonlinear fuzzy term can be decomposed into the series of Adomian polynomials, and it is defined as,

$$N\tilde{u} = \sum_{n=0}^{\infty} \tilde{A}_n\,(\tilde{u}_0, \tilde{u}_1, \dots \tilde{u}_n),$$

We have calculated the value of Adomian polynomials using the fuzzy Taylor series and also proved the convergence of the fuzzy Adomian Decomposition method. We have solved examples by FADM, which is explained in chapter 5.

1.4. Conclusion

This chapter contains preliminary required for research work and gives glimpse for different techniques, like analytic, numerical, transform and semi-analytic, proposed and proved in later chapters, to obtain the solution of the fuzzy systems.