4. Transformation Technique

4.1. Introduction:

In this chapter, we focus on transformation technique i.e., Fuzzy Laplace Transform (FLT) to solve a system of fuzzy differential equations. Most of the articles solved the linear differential equation with fuzzy initial condition as $\tilde{x}(0) = \tilde{x}_0$ by fuzzy Laplace transform under generalized Hukuhara derivative as in [18]. We initiated our work [64] using the transform technique for linear homogeneous system i.e., $\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X}$; $\tilde{X}(0) = \tilde{X}_0$ as well as nonhomogeneous system i.e., $\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \oplus \tilde{B}$; $\tilde{X}(0) = \tilde{X}_0$, involving fuzzy parameters and fuzzy initial conditions, as in [18].

And following it, we solved semi-linear dynamical systems with fuzzy parameters and initial conditions involved in the system, using Fuzzy Laplace Transform.

For the rigorous development of solution, for fully fuzzy dynamical system using the Laplace Transform technique, we afresh define Modified Hukuhara derivative (mH-derivative) [65] which is later explained in Section 4.3.1. Under this new derivative, we have redefined the FLT with existence condition and other results. All results are proposed and proved in this chapter.

In the next section, we establish the theory for the fuzzy non-homogeneous linear dynamical system and give result for fuzzy homogeneous linear dynamical system considering a special case of it.

4.2. Fuzzy Nonhomogeneous Linear Dynamical Systems

Consider a fuzzy nonhomogeneous system as follows,

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X}(t) \oplus \tilde{B} \tag{4.1}$$

with fuzzy initial condition, $\tilde{X}(0) = \tilde{X}_0$,

where, $\tilde{A}_{\mathrm{n\times n}} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix}$ and $\tilde{B}_{\mathrm{n\times 1}} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$.

We propose, the following lemma as a result of fuzzy Laplace Transform which is useful in proving further results for equation (4.1).

4.2.1. Lemma

Let $\tilde{F}$ be a continuous fuzzy valued function and its fuzzy Laplace transform, $\mathcal{L}\{\tilde{F}(t)\} = \tilde{F}(s)$ then,

$$\mathcal{L}\{\widetilde{A} \otimes \tilde{F}(t)\} = \widetilde{A} \otimes \tilde{F}(s),$$

if it satisfies two conditions given below,

- $A_l\, \underline{F} \leq A_u\, \overline{F}\ \forall\, \alpha \in (0\ 1]$
- $\alpha < \beta,\, {}^{\alpha}A_l{}^{\alpha}\underline{F} \leq {}^{\beta}A_l{}^{\beta}\underline{F} \leq {}^{\beta}A_u{}^{\beta}\overline{F} \leq {}^{\alpha}A_u{}^{\alpha}\overline{F}$

(4.2)

Proof: By fuzzy Laplace Transform, we have

$$\mathcal{L}\{\widetilde{A} \otimes \tilde{F}(t)\} = \int_0^{\infty} e^{-st}\{\widetilde{A} \otimes \tilde{F}(t)\}dt$$

Taking $\alpha - cut$ of L.H.S,

$${}^{\alpha}\widetilde{A} \otimes {}^{\alpha}\tilde{F}(t) = [\underline{A}\ \overline{A}] \otimes [\underline{F}\ \overline{F}]$$

Using these notations, $A_l = \min(\underline{A}\ \overline{A}),\ A_u = \max(\underline{A}\ \overline{A})$.

$$\mathcal{L}\{{}^{\alpha}\widetilde{A} \otimes {}^{\alpha}\tilde{F}(t)\} = \int_0^{\infty} e^{-st}\,[A_l\underline{F}, A_u\overline{F}]dt$$

$$\mathcal{L}\{{}^{\alpha}\widetilde{A} \otimes {}^{\alpha}\tilde{F}(t)\} = \left[\int_0^{\infty} e^{-st}\ A_l\,\underline{F}\,dt, \int_0^{\infty} e^{-st}\,A_u\,\overline{F}dt\right]$$

$$\mathcal{L}\{{}^{\alpha}\widetilde{A} \otimes {}^{\alpha}\tilde{F}(t)\} = [L\{A_l\,\underline{F}(s)\}, L\{A_u\,\overline{F}(s)\}] \quad (4.3)$$

Now by the first decomposition theorem, as in [70], if equation (4.3) satisfies the two conditions as given in (4.2), then we can write,

$$\mathcal{L}\{\widetilde{A} \otimes \tilde{F}(t)\} = \widetilde{A} \otimes \tilde{F}(s)$$

In the next lemma, we propose and prove results related to the fuzzy solution of fuzzy non-homogeneous linear dynamical system using the decoupling method. In this method, we diagonalize the system represented as in parametric form with diagonal matrix $\widetilde{D}$ such that ${}^{\alpha}\widetilde{D} = [D_l, D_u]$.

4.2.2. Lemma

A nonhomogeneous fuzzy dynamical system $\dot{\tilde{X}} = \widetilde{A} \otimes \tilde{X} \oplus \tilde{B}$ with initial condition $\tilde{X}(0) = \tilde{X}_0$ is diagonalizable.

Proof: Consider a fuzzy nonhomogeneous linear dynamical system as in equation (4.1). The parametric form of equation (4.1), using fuzzy multiplication and addition,

$$\left[\dot{\underline{X}}, \dot{\overline{X}}\right] = \left[A_l\, \underline{X}, A_u\, \overline{X}\right] \oplus [B_l, B_u];\ \left[\underline{X}_0, \overline{X}_0\right] \tag{4.4}$$

Now comparing the components of equation (4.4),

$$\dot{\underline{X}}(t) = A_l\, \underline{X} + B_l$$

$$\dot{\overline{X}}(t) = A_u\, \overline{X} + B_u$$

with initial condition, ${}^{\alpha}\tilde{X}_0 = \left[\underline{X}_0, \overline{X}_0\right]$.

The matrix form of the above equations is given below,

$$\begin{bmatrix} \dot{\underline{X}}(t) \\ \dot{\overline{X}}(t) \end{bmatrix} = \begin{bmatrix} A_l & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} + \begin{bmatrix} B_l \\ B_u \end{bmatrix}; \begin{bmatrix} \underline{X}_0 \\ \overline{X}_0 \end{bmatrix} \tag{4.5}$$

If the system is nonhomogeneous so first, we use the decoupling method to diagonalize equation (4.5). For decoupling, put $\underline{X} = \underline{P}\, \underline{U}$ and $\overline{X} = \overline{P}\, \overline{U}$, here $\underline{P}$ and $\overline{P}$ are corresponding orthogonal matrices of matrix A_l and A_u.

Equation (4.5) gets converted as,

$$\begin{bmatrix} \dot{\underline{U}}(t) \\ \dot{\overline{U}}(t) \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1}\, A_l\, \underline{P} & 0 \\ 0 & \overline{P}^{\,-1} A_u\, \overline{P} \end{bmatrix} \begin{bmatrix} \underline{U} \\ \overline{U} \end{bmatrix} + \begin{bmatrix} \underline{P}^{-1}\, B_l \\ \overline{P}^{\,-1}\, B_u \end{bmatrix}$$

with initial condition $\left[\underline{P}\, \underline{U}_0, \overline{P}\, \overline{U}_0\right]$.

Now using diagonalization, we have,

$$\begin{bmatrix} \dot{\underline{U}}(t) \\ \dot{\overline{U}}(t) \end{bmatrix} = \begin{bmatrix} D_l & 0 \\ 0 & D_u \end{bmatrix} \begin{bmatrix} \underline{U} \\ \overline{U} \end{bmatrix} + \begin{bmatrix} \underline{P}^{-1}\, B_l \\ \overline{P}^{\,-1}\, B_u \end{bmatrix}$$

with initial condition, $\left[\underline{P}\, \underline{U}_0, \overline{P}\, \overline{U}_0\right]$.

If A_l and A_u contains some negative elements then to obtain a fuzzy solution, for such system after taking an alpha cut, interchange parameters $\underline{a}_{ij}$ with $\overline{a}_{ij}$ $\forall\, i, j = 1,2 \ldots \ldots$as in [30] and [72]. Similarly, for D_l and D_u.

In the next section, we give a fuzzy solution of the nonhomogeneous linear system.

4.2.3. Theorem

A fuzzy nonhomogeneous dynamical system given by equation (4.1) is diagonalizable then using FLT, solution of the above system is,

$$\underline{X} = \underline{P}\, \mathcal{L}^{-1}\left\{[sI - D_l\,]^{-1}\left[\underline{P}^{-1}\, \underline{X}(0) + \underline{P}^{-1}\mathcal{L}[B_l]\right]\right\}$$

$$\overline{X} = \overline{P}\mathcal{L}^{-1}\{[sI - D_u]^{-1}[\overline{P}^{\,-1}\, \overline{X}(0) + \overline{P}^{\,-1}\mathcal{L}[B_u]\}$$

Proof: Using lemma 4.2.2, a fuzzy nonhomogeneous linear dynamical system in equation (4.1) gets converted to diagonalized form as,

$$\begin{bmatrix} \dot{\underline{U}}(t) \\ \dot{\overline{U}}(t) \end{bmatrix} = \begin{bmatrix} D_l & 0 \\ 0 & D_u \end{bmatrix} \begin{bmatrix} \underline{U} \\ \overline{U} \end{bmatrix} + \begin{bmatrix} \underline{P}^{-1} B_l \\ \overline{P}^{-1} B_u \end{bmatrix} \tag{4.6}$$

with initial condition, $[\underline{P}\,\underline{U}_0, \overline{P}\,\overline{U}_0]$.

Taking FLT on both sides of (4.6),

$$\mathcal{L}[\dot{\underline{U}}(t)] = \mathcal{L}[\,D_l\underline{U}] + \mathcal{L}[\underline{P}^{-1}\,B_l\,]$$

$$\mathcal{L}\left[\dot{\overline{U}}(t)\right] = \mathcal{L}[\,D_l\overline{U}] + \mathcal{L}[\underline{P}^{-1}\,B_u\,]$$

Taking the first equation, we get,

$$s\mathcal{L}[\underline{U}\,(t)] - \underline{U}_0 = D_l\,\mathcal{L}[\underline{U}] + \underline{P}^{-1}\mathcal{L}[B_l]$$

$$[sI - \,D_l\,]\,\mathcal{L}[\underline{U}] = \underline{U}_0 + \underline{P}^{-1}\mathcal{L}[B_l]$$

$$[sI - \,D_l\,]\,\mathcal{L}[\underline{U}] = \underline{U}_0 + \underline{P}^{-1}\mathcal{L}[B_u]$$

$$\mathcal{L}[\underline{U}] = \frac{1}{(sI - D_l)}\left[\underline{U}_0 + \underline{P}^{-1}\mathcal{L}[B_u]\right] \tag{4.7}$$

Taking inverse fuzzy Laplace Transform of equation (4.7),

$$\underline{U} = \mathcal{L}^{-1}\left\{\frac{1}{(sI - D_l)}\left[\underline{U}_0 + \underline{P}^{-1}\mathcal{L}[B_l]\right]\right\}$$

Similarly,

$$[sI - D_u]\mathcal{L}[\overline{U}\,] = \overline{U}_0 + \overline{P}^{\,-1}\mathcal{L}[B_u\,]$$

$$\mathcal{L}[\overline{U}] = \frac{1}{(sI - D_u)}\left[\overline{U}_0 + \underline{P}^{-1}\mathcal{L}[B_u]\right] \tag{4.8}$$

Taking inverse fuzzy Laplace Transform of equation (4.8),

$$\overline{U} = \mathcal{L}^{-1}\{[sI - D_u]^{-1}[\overline{U}_0 + \mathcal{L}[B_u]\}$$

Using the solution $\underline{U}\,, \overline{U}$ of diagonalized system the solution of original system with initial condition $[\underline{P}\,\underline{U}_0, \overline{P}\,\overline{U}_0]$ can be given as,

$$\underline{X} = \underline{P}\,\mathcal{L}^{-1}\left\{[sI - D_l]^{-1}\left[\underline{P}^{-1}\,\underline{X}(0) + \underline{P}^{-1}\mathcal{L}[B_l]\right]\right\}$$

$$\overline{X} = \overline{P}\mathcal{L}^{-1}\left\{[sI - D_u]^{-1}\left[\overline{P}^{\,-1}\,\overline{X}(0) + \overline{P}^{\,-1}\mathcal{L}[B_u]\right]\right\}$$

Similarly, if equation (4.1) does not contain the non-homogeneous term i.e., $\tilde{B} = 0$, then the fuzzy dynamical system is given as follows,

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X}; \tilde{X}_0 \tag{4.9}$$

The fuzzy solution of equation (4.9) in parametric form is given below, using Sections 4.2.1 and 4.2.2.

$$\underline{X}(t) = \underline{X}_0 e^{D_l t},$$
$$\overline{X}(t) = \overline{X}_0 e^{D_u t}.$$

In the next section, we solved one example using proposed theory.

4.2.4. Real life example

There are two countries in an arms race where $x(t)$ is the yearly rate of armament expenditures for country X and $y(t)$ is that expenditure of country Y as in [71]. These expenditures on armament depend on other countries' war strategy so this model is more appropriate in fuzzy setup.

So, the dynamical system in a fuzzy scenario,

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} -\tilde{3} & \tilde{2} \\ \tilde{3} & -\tilde{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{1} \\ \tilde{2} \end{bmatrix} \tag{4.10}$$

with initial conditions, ${}^{\alpha}\tilde{x}_0 = {}^{\alpha}\tilde{y}_0 = {}^{\alpha}\widetilde{100} = [70 + 30\alpha, 130 - 30\alpha]$.

where, $\tilde{3} = (2,3,4), \tilde{2} = (1,2,3), \tilde{4} = (3,4,5), \widetilde{100} = (70,100,130)$, and

${}^{\alpha}\tilde{3} = (2 + \alpha, 4 - \alpha)$, ${}^{\alpha}\tilde{2} = (1 + \alpha, 3 - \alpha)$, ${}^{\alpha}\tilde{4} = (3 + \alpha, 5 - \alpha)$, ${}^{\alpha}\tilde{1} = (\alpha, 2 - \alpha)$.

But in this problem, parameters are negative fuzzy numbers. In this case, to obtain a fuzzy solution, for such system after taking alpha cut interchange parameters $\underline{a}_{ij}$ with $\overline{a}_{ij}$ $\forall\, i, j = 1,2, \ldots$ as in [30] and [72].

Now applying the technique mentioned in Section 4.2.3, that is obtaining the solution by first obtaining the solution of diagonalized system, we get,

$$\underline{x} = [(70 + 30\alpha)]e^{-6t} + \frac{(2\alpha^2 + 4\alpha + 1)(1 - e^{-6t})}{(18 + 12\alpha)} - \frac{(1 + \alpha)}{(9 - 4\alpha^2)}\left(1 - e^{-(2\alpha-3)t}\right),$$

$$\overline{x} = (130 - 30\alpha)e^{-6t} - \frac{(-12\alpha+17+2\alpha^2)}{6(2\alpha-7)}(1 - e^{-6t}) - \frac{(\alpha-3)}{(2\alpha-7)(2\alpha-1)}\left(1 - e^{-(2\alpha-1)t}\right),$$

$$\underline{y} = [(70 + 30\alpha)]e^{-6t} + \frac{(2\alpha^2 + 4\alpha + 1)(1 - e^{-6t})}{(18 + 12\alpha)} + \frac{(2 + \alpha)}{(9 - 4\alpha^2)}\left(1 - e^{-(2\alpha-3)t}\right),$$

$$\overline{y} = (130 - 30\alpha)e^{-6t} - \frac{(-12\alpha+17+2\alpha^2)}{6(2\alpha-7)}(1 - e^{-6t}) - \frac{(4-\alpha)}{(2\alpha-7)(2\alpha-1)}\left(1 - e^{-(2\alpha-1)t}\right).$$

On plotting the solutions graphically with respect to time, we get the evolution as in figures 4.1 and 4.2.

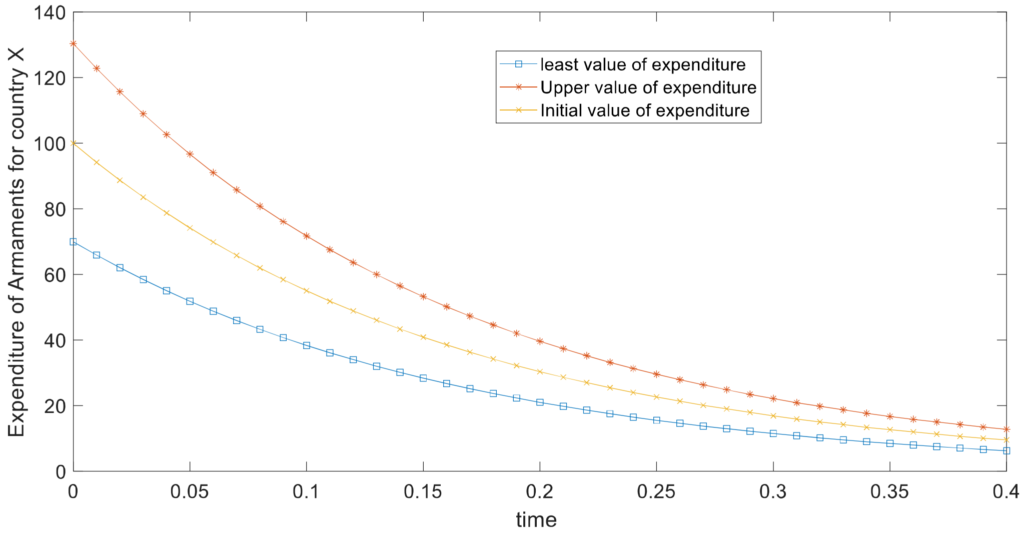

Figure 4.1: Expenditure for country *X*

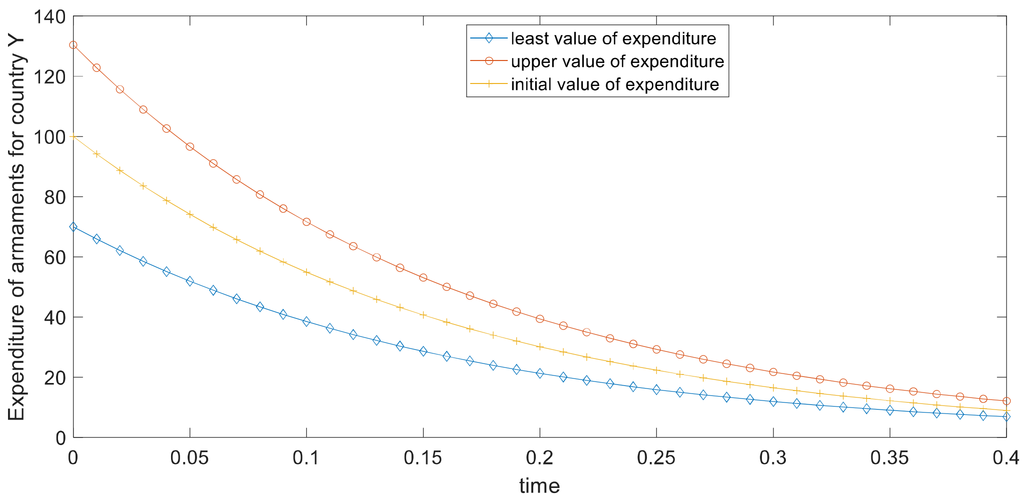

Figure 4.2: Expenditure for country *Y*

It is observed that as time increases, the solution becomes nonfuzzy for large times, so here we have taken evolution on a very small-time interval. From the fig. 4.1 and fig. 4.2, this fuzzy setup gives a range for the yearly armaments for both countries. As time increases, the range shifts left side from the initial condition for both countries.

In next section, we have developed a theory for the fuzzy solution of fully fuzzy Semi-Linear dynamical System.

4.3. Fully Fuzzy Semi-Linear Dynamical System:

This section aims to obtain a fuzzy solution of the fully fuzzy semi-linear dynamical system, given as in (4.11), by redefining fuzzy Laplace Transform under a new fuzzy derivative i.e., Modified Hukuhara derivative (mH-derivative).

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \oplus \tilde{f}\,(t,\tilde{X});\ \tilde{X}(0) = \tilde{X}_0 \tag{4.11}$$

where, $\tilde{f} : I \times E^n \to E^n$ is continuous fuzzy valued mapping.

Initially, people started work on fuzzy differential equations under Hukuhara derivative but this derivative has a drawback that if time increases then support becomes unbounded. This drawback has been taken care of by the generalized Hukuhara derivative. But when one uses generalized Hukuhara derivative to obtain the solution of differential equation, it gives the list of possible solutions from which one is supposed to pick the best possible solution. This limitation of manually selecting the solution is overcome by our proposed method. Our proposed derivative is as shown in [65] has the advantage, that it gives a unique solution with bounded support for fuzzy dynamical system automatically.

In the next section, we have given the existence condition for the modified Hukuhara derivative.

4.3.1. Existence of Modified Hukuhara Derivative:

The motivation to propose the mH- derivative is to get the derivative of the function of the form $\tilde{f}(t) = \tilde{c} \otimes g(t)$ for any $g(t)$. It was shown in [9] that the fuzzy Hukuhara derivative is, $\dot{\tilde{f}}(t) = \tilde{c} \otimes \dot{g}(t)$ if $\dot{g}(t) > 0$ but does not exist if $\dot{g}(t) < 0$. For $\dot{g}(t) < 0$ using Generalized Hukuhara differentiability, the solution was obtained, but as mentioned earlier its limitation is that it is not unique refer [9]. Following gives the definition of proposed mH-derivative.

Modified Hukuhara Derivative:

A function $\tilde{f}: I \to E$ is said to be mH-differentiable if, at $t_0 \in I$, $\exists$ an element $\dot{\tilde{f}}(t_0) \in E$ such that for all $h > 0$ sufficiently small, there exists $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ and the limits,

$$\lim_{h\to 0+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h\to 0-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

The equivalent parametric representation is given as,

$$\lim_{h\to 0+}\frac{{}^{\alpha}\tilde{f}(t_0+h)\ \ominus\ {}^{\alpha}\tilde{f}(t_0)}{h}=$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\},\right.$$
$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\underline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\overline{f}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0+h)-\underline{f}(t_0)\right)}{h}\right\}\right]$$

$$\lim_{h\to 0-}\frac{\tilde{f}(t_0)\ominus\ \tilde{f}(t_0-h)}{h}=$$

$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}\right\}\right.$$
$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{f}(t_0)-\overline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\underline{f}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{f}(t_0)-\overline{f}(t_0-h)\right)}{h}\right\}\right].$$

Now by using the proposed mH-differentiability, we obtain the derivative of $\tilde{f}(t)=\tilde{c}\otimes g(t)$. If, $\dot{g}(t)<0$ and ${}^{\alpha}\tilde{c}=\left[\underline{c},\overline{c}\right]$, then, the equivalent parametric form of right-hand side limit for the derivative is given as,

$$\lim_{h\to 0+}\frac{{}^{\alpha}\tilde{f}(t_0+h)\ominus\ {}^{\alpha}\tilde{f}(t_0)}{h}=$$
$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0+h)-\underline{c}\underline{g}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0+h)-\overline{cg}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0+h)-\overline{cg}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0+h)-\underline{c}\underline{g}(t_0)\right)}{h}\right\},\right.$$
$$\left.\max\left\{\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0+h)-\underline{c}\underline{g}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0+h)-\overline{cg}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0+h)-\overline{cg}(t_0)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0+h)-\underline{c}\underline{g}(t_0)\right)}{h}\right\}\right]$$

Since, $\dot{g}(t)<0=g(t_0+h)-g(t_0)$ is a negative quantity so min $(-\underline{c}\ \dot{g}(t),-\overline{c}\ \dot{g}(t)\ )$ and max $(-\underline{c}\ \dot{g}(t),-\overline{c}\ \dot{g}(t))$ value of the right-hand side limit is $-\overline{c}\ \dot{g}(t)$ and $-\underline{c}\ \dot{g}(t)$ respectively. Similarly, for the left-hand side limit,

$$\lim_{h\to 0-}\frac{{}^{\alpha}\tilde{f}(t_0)\ominus\ {}^{\alpha}\tilde{f}(t_0-h)}{h}=$$
$$\left[\min\left\{\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0)-\underline{c}\underline{g}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0)-\overline{cg}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0)-\overline{cg}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{cg}(t_0)-\underline{c}\underline{g}(t_0-h)\right)}{h}\right\},\right.$$

$$\max\left\{\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0)-\underline{c}\underline{g}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\underline{c}\,\underline{g}(t_0)-\overline{c}\overline{g}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{c}\overline{g}(t_0)-\overline{c}\overline{g}(t_0-h)\right)}{h},\lim_{h\to 0}\frac{\left(\overline{c}\overline{g}(t_0)-\underline{c}\underline{g}(t_0-h)\right)}{h}\right\}\Bigg]$$

Now, $g(t_0)-g(t_0-h)<0$ is a negative quantity so again min $(-\underline{c}\;\dot{g}(t),-\overline{c}\;\dot{g}(t))$ and max $(-\underline{c}\;\dot{g}(t),-\overline{c}\;\dot{g}(t))$ value of the right-hand side limit is $-\overline{c}\;\dot{g}(t)$ and $-\underline{c}\;\dot{g}(t)$ respectively.

Hence, the left-hand and right-hand limit exists and are equal.

So, $\dot{\tilde{f}}(t)=\tilde{c}\otimes\dot{g}(t)$ differentiable function under modified Hukuhara derivative.

Under the proposed new derivative, we solve the following fuzzy initial value problem in example.

4.3.2. Example:

Consider the fuzzy initial value problem (FIVP).

$$\dot{\tilde{y}}=-\tilde{y}\,;\;\tilde{y}(0)=(0.96,1,1.01),\;{}^{\alpha}\tilde{y}(0)=(0.96+0.04\alpha,1.01-0.01\alpha)$$

An attempt to solve such an example was done by [14], which was corrected by [50] using gH-derivative involving human intervene to pick the best solution. Using mH-derivative, it can be solved as follows,

Taking $\alpha-cut$ on both sides of the given example,

i.e., $${}^{\alpha}\dot{\tilde{y}}=-\,{}^{\alpha}\tilde{y}$$

$$\left[\underline{\dot{y}},\overline{\dot{y}}\right]=-\left[\underline{y},\overline{y}\right],\forall\,t>0$$

Comparing both the sides, we get,

$$\underline{\dot{y}}=\min\left(-\overline{y},-\underline{y}\right)$$

$$\overline{\dot{y}}=\max\left(-\overline{y},-\underline{y}\right)$$

i.e., $$\underline{\dot{y}}=-\overline{y}$$

and $$\overline{\dot{y}}=-\underline{y}$$

Solving these equations, we get,

$$\underline{y}(t)=c_1e^{t}+c_2e^{-t}$$

$$\overline{y}(t)=-c_1e^{t}+c_2e^{-t}$$

After putting initial condition, they become,

$$\underline{y}(t)=(0.025\alpha-0.025)e^{t}+(0.985+0.015\alpha)e^{-t}$$

$$\overline{y}(t)=-(0.025\alpha-0.025)e^{t}+(0.985+0.015\alpha)e^{-t}$$

which is the same for $\alpha=0$ as in [50].

In the next section, we redefine fuzzy Laplace transform under Modified Hukuhara derivative. We also, give the results for the existence condition of FLT, FLT of derivative and convolution theorem.

4.3.3. Fuzzy Laplace Transform under mH-derivative

4.3.3.1. Fuzzy Laplace Transform

The fuzzy Laplace Transform technique is very useful in solving FDEs and their corresponding initial and boundary value problems. In this section, we redefine the Fuzzy Laplace Transform (FLT) under the new proposed fuzzy derivative along with other properties.

Definition:

Consider a bounded and piecewise continuous fuzzy valued function whose parametric form is ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}, \overline{f}\right]$ and let $\tilde{f}(t) \otimes e^{-st}$ is improper fuzzy Riemann integrable, then $\tilde{F}(s)$ denotes fuzzy Laplace Transform and it is defined as,

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \int_0^{\infty} e^{-st} \otimes \tilde{f}(t)dt$$

$$\tilde{F}(s) = \mathcal{L}\left(\tilde{f}(t)\right) = \lim_{t\to\infty} \int_0^{t} e^{-st} \otimes \tilde{f}(t)dt$$

Taking alpha cut on both sides,

$$\mathcal{L}\left({}^{\alpha}\tilde{f}(t)\right) = \lim_{t\to\infty} \int_0^{t} e^{-st} \otimes \left[\underline{f}(t), \overline{f}(t)\right] dt$$

$$\mathcal{L}\left(\left[\underline{f}(t), \overline{f}(t)\right]\right) = \lim_{t\to\infty} \left[\int_0^{t} e^{-st}\underline{f}(t)\, dt, \int_0^{t} e^{-st}\overline{f}(t)dt\right]$$

$$\therefore \left[\mathcal{L}\underline{f}(t), \mathcal{L}\overline{f}(t)\right] = \left[\underline{F}(s), \overline{F}(s)\right]$$

where,

$$\underline{F}(s) = \mathcal{L}\left[\underline{f}(t)\right] = \min\left\{\lim_{t\to\infty}\left[\int_0^{t} e^{-st}\underline{f}(t)\, dt, \int_0^{t} e^{-st}\overline{f}(t)dt\right]\right\}$$

$$\overline{F}(s) = \mathcal{L}\left[\overline{f}(t)\right] = \max\left\{\lim_{t\to\infty}\left[\int_0^{t} e^{-st}\underline{f}(t)\, dt, \int_0^{t} e^{-st}\overline{f}(t)dt\right]\right\}$$

And the Fuzzy inverse Laplace Transform is denoted by $\mathcal{L}^{-1}[\tilde{F}(s)]$ and its parametric form is defined as,

$$\mathcal{L}^{-1}\left[\underline{F}(s)\right] = \min\left[\underline{f}(t), \overline{f}(t)\right]$$

$$\mathcal{L}^{-1}\left[\overline{F}(s)\right] = \max\left[\underline{f}(t), \overline{f}(t)\right].$$

4.3.3.2. Existence of Fuzzy Laplace Transform:

We know that for crisp function if $f(t)$ is piecewise continuous in a given closed interval and is of exponential order then its Laplace transform exist. The existence condition for fuzzy Laplace transform in parametric form is defined as,

${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ should be exponential order p i.e., $\lim_{t\to\infty} \frac{\underline{f}(t)}{e^{pt}} = 0$ and $\lim_{t\to\infty} \frac{\overline{f}(t)}{e^{pt}} = 0$

for some constants $\underline{M}$ and $\overline{M}$, $\left|\underline{f}(t)\right| \leq \underline{M}e^{pt}$, $\left|\overline{f}(t)\right| \leq \overline{M}\, e^{pt}$.

Let $\tilde{f}(t)$ be fuzzy valued function and exponential bounded with constant $M = \min(\underline{M}, \overline{M})$.

Now, in the next section, we prove the existence condition for the fuzzy Laplace transform.

4.3.3.3. Theorem

Let ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be piecewise continuous on every finite interval $t \geq 0$ and satisfies the condition as in definition 4.3.3.2 then $\mathcal{L}\left({}^{\alpha}\tilde{f}(t)\right) = \left(\mathcal{L}\left[\underline{f}(t)\right], \mathcal{L}\left[\overline{f}(t)\right]\right)$ exist for $s > p$, $\lim_{s\to\infty} \mathcal{L}\left[\underline{f}(t)\right] = 0$ and $\lim_{s\to\infty} \mathcal{L}\left[\overline{f}(t)\right] = 0$.

Proof: From this inequality,

$$\left|\underline{f}(t)\right| \leq \underline{M}e^{pt}$$

$$\int_0^{\infty} e^{-st}\left|\underline{f}(t)\right| dt \leq \int_0^{\infty} e^{-st}\,\underline{M}e^{pt} dt$$

$$\mathcal{L}\left[\underline{f}(t)\right] \leq \frac{\underline{M}}{s-p}$$

As $s \to \infty, \mathcal{L}\left[\underline{f}(t)\right] \to 0$

Similarly,

$$\left|\overline{f}(t)\right| \leq \overline{M}\, e^{pt}$$

$$\int_0^{\infty} e^{-st}\left|\overline{f}(t)\right| dt \leq \int_0^{\infty} e^{-st}\,\overline{M}e^{pt} dt$$

$$\mathcal{L}\left[\overline{f}(t)\right] \leq \frac{\overline{M}}{s-p}$$

As, $s \to \infty$, $\mathcal{L}\left[\overline{f}(t)\right] \to 0$.

Hence, $\mathcal{L}\left(\tilde{f}(t)\right)$ exist when it is of exponential order p with constant $M = \min(\underline{M}, \overline{M})$.

In the next section, the result for the Laplace transform of fuzzy derivative is given in theorem form.

4.3.3.4. Theorem

If ${}^{\alpha}\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be continuous fuzzy valued function, $\lim_{t\to\infty} e^{-st}\underline{f}(t) \to 0$ and $\lim_{t\to\infty} e^{-st}\overline{f}(t) \to 0$ for large value of s and $\dot{\tilde{f}}(t)$ is piecewise continuous then$\mathcal{L}\left(\dot{\tilde{f}}(t)\right)$ exist, and is given by,

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = s\mathcal{L}\left(\tilde{f}(t)\right) \ominus \tilde{f}_0$$

Proof: In theorem 4.3.3.3, we already proved $\mathcal{L}\left(\tilde{f}(t)\right)$ exist because $\tilde{f}(t)$ is of exponential order p and continuous and $\dot{\tilde{f}}(t)$ is piecewise continuous. So fuzzy Laplace derivative is given as,

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = \int_0^\infty e^{-st}\dot{\tilde{f}}(t)dt$$

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = \lim_{t\to\infty} \int_0^t e^{-st}\dot{\tilde{f}}(t)dt$$

Taking alpha cut on both sides,

$$\mathcal{L}\left(\left[\dot{\underline{f}}(t), \dot{\overline{f}}(t)\right]\right) = \lim_{t\to\infty} \int_0^t e^{-st}\left[\dot{\underline{f}}(t), \dot{\overline{f}}(t)\right] dt$$

$$\mathcal{L}\left(\left[\dot{\underline{f}}(t), \dot{\overline{f}}(t)\right]\right) = \lim_{t\to\infty} \left[\int_0^t e^{-st}\dot{\underline{f}}(t)dt, \int_0^t e^{-st}\dot{\overline{f}}(t)dt\right]$$

Now using integration by parts, we get,

$$\mathcal{L}\left(\dot{\underline{f}}(t)\right) = \min\left\{s\mathcal{L}\left(\underline{f}(t)\right) - \underline{f}(0), s\mathcal{L}\left(\underline{f}(t)\right) - \overline{f}(0), s\mathcal{L}\left(\overline{f}(t)\right) - \underline{f}(0), s\mathcal{L}\left(\overline{f}(t)\right) - \overline{f}(0)\right\}$$

$$\mathcal{L}\left(\dot{\overline{f}}(t)\right) = \max\left\{s\mathcal{L}\left(\underline{f}(t)\right) - \underline{f}(0), s\mathcal{L}\left(\underline{f}(t)\right) - \overline{f}(0), s\mathcal{L}\left(\overline{f}(t)\right) - \underline{f}(0), s\mathcal{L}\left(\overline{f}(t)\right) - \overline{f}(0)\right\}$$

Then, by the first decomposition theorem as in [70].

$$\mathcal{L}\left(\dot{\tilde{f}}(t)\right) = s\mathcal{L}\left(\tilde{f}(t)\right) \ominus \tilde{f}_0.$$

4.3.3.5. Example

We solve the illustrative example,

$$\dot{\tilde{y}} = -\tilde{y}, 0 \le t \le T\,;\ {}^{\alpha}\tilde{y}_0 = [-a(1-\alpha), a(1-\alpha)]$$

The solution of above problem [18] is given by FLT under generalized differentiability in such a manner if $\tilde{y}$ is (i)- differentiable then support $\tilde{y}(t) = 2ae^{t}$ becomes unbounded as $t \to \infty$ and $\tilde{y}$ is (ii)- differentiable then support $\tilde{y}(t) = 2ae^{-t}$ becomes bounded as $t \to \infty$. Now, we solve this problem using FLT under modified Hukuhara differentiability and obtain a unique solution with bounded support.

$$\mathcal{L}[\dot{\tilde{y}}(t)] = -\mathcal{L}[\tilde{y}(t)]$$

Taking $\alpha - cut$ on both sides of the above equation, we get,

$$\mathcal{L}[\,{}^{\alpha}\dot{\tilde{y}}(t)] = -\mathcal{L}[\,{}^{\alpha}\tilde{y}(t)],$$

which gives,

$$\mathcal{L}\left[\dot{\underline{y}}, \dot{\overline{y}}\right] = -\mathcal{L}\left[\underline{y}, \overline{y}\right].$$

Comparing component-wise,

$$\mathcal{L}\left[\dot{\underline{y}}(t)\right] = -\mathcal{L}[\overline{y}(t)]$$

$$\mathcal{L}[\dot{\overline{y}}(t)] = -\mathcal{L}\left[\underline{y}(t)\right].$$

Now, using Laplace transform of a fuzzy derivative under mH-derivative as in 4.3.3.4, on above equations, we have,

$$\mathcal{L}\left(\dot{\underline{y}}(t)\right) = \min\left\{s\mathcal{L}\left(\underline{y}(t)\right) - \underline{y}(0), s\mathcal{L}\left(\underline{y}(t)\right) - \overline{y}(0), s\mathcal{L}(\overline{y}(t)) - \underline{y}(0), s\mathcal{L}(\overline{y}(t)) - \overline{y}(0)\right\} = -\mathcal{L}(\overline{y}(t)),$$

$$\mathcal{L}\left(\dot{\overline{y}}(t)\right) = \max\left\{s\mathcal{L}\left(\underline{y}(t)\right) - \underline{y}(0), s\mathcal{L}\left(\underline{y}(t)\right) - \overline{y}(0), s\mathcal{L}(\overline{y}(t)) - \underline{y}(0), s\mathcal{L}(\overline{y}(t)) - \overline{y}(0)\right\} = -\mathcal{L}\left(\underline{y}(t)\right).$$

After solving above equations using fuzzy Laplace transform and its inverse, fuzzy solution of a problem is,

$$\underline{y}(t) = -a(1-\alpha)e^{-t},$$

$$\overline{y}(t) = a(1-\alpha)e^{-t}.$$

In the following section, we propose and prove the fuzzy Laplace transform for the product of two fuzzy functions.

4.3.3.6. Fuzzy Convolution Theorem

Let $\tilde{f}(s)$ and $\tilde{g}(s)$ denote the fuzzy inverse Laplace transforms of $\tilde{f}(t)$ and $\tilde{g}(t)$ respectively. Then the Laplace transform of $\tilde{f}(t) * \tilde{g}(t)$, is given by,

$$\mathcal{L}\left(\tilde{f}(t) * \tilde{g}(t)\right) = \tilde{f}(s) * \tilde{g}(s)$$

Proof:

$$\mathcal{L}\left(\tilde{f}(t) * \tilde{g}(t)\right) = \int_0^t e^{-st}[\tilde{f}(t) \otimes \tilde{g}(t)]dt$$

$$\mathcal{L}\left(\left[\underline{f}(t), \overline{f}(t)\right] * \left[\underline{g}(t), \overline{g}(t)\right]\right) = \int_0^t e^{-st}\left[\underline{f}(t), \overline{f}(t)\right] \otimes \left[\underline{g}(t), \overline{g}(t)\right] dt$$

From fuzzy multiplication, we can write,

$$\mathcal{L}\left[\underline{f}(t) * \underline{g}(t)\right] = \min \int_0^t e^{-st}\left[\underline{f}(\tau)\underline{g}(t-\tau), \underline{f}(\tau)\overline{g}\,(t-\tau), \overline{f}(\tau)\underline{g}(t-\tau), \overline{f}(\tau)\overline{g}\,(t-\tau)\right] dtd\tau$$

$$\mathcal{L}\left[\overline{f}(t) * \overline{g}(t)\right] = \max \int_0^t e^{-st}\left[\underline{f}(\tau)\underline{g}(t-\tau), \underline{f}(\tau)\overline{g}\,(t-\tau), \overline{f}(\tau)\underline{g}(t-\tau), \overline{f}(\tau)\overline{g}\,(t-\tau)\right] dtd\tau$$

For solving the above integration, we use substitution, $t - \tau = u, \tau = v$

$$\mathcal{L}\left[\underline{f}(t) * \underline{g}(t)\right] = \min \int_0^t e^{-st}\left[\underline{f}(v)\underline{g}(u), \underline{f}(v)\overline{g}\,(u), \overline{f}(v)\underline{g}(u), \overline{f}(v)\overline{g}\,(u)\right] dudv$$

$$\mathcal{L}\left[\overline{f}(t) * \overline{g}(t)\right] = \max \int_0^t e^{-st}\left[\underline{f}(v)\underline{g}(u), \underline{f}(v)\overline{g}\,(u), \overline{f}(v)\underline{g}(u), \overline{f}(v)\overline{g}\,(u)\right] dudv$$

Then, by the first decomposition theorem as in Klir [70],

$$\mathcal{L}\left(\tilde{f}(t) * \tilde{g}(t)\right) = \tilde{f}(s) * \tilde{g}(s).$$

Now we give the main result pertaining to the existence and uniqueness solution for the semi-linear system such as in equation (4.11) by using Fuzzy Laplace Transform under mH-derivative.

4.3.4. Main Theorem

The solution of the system $\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \oplus \tilde{f}\left(t, \tilde{X}\right);\ \tilde{X}(0) = \tilde{X}_0$ as in equation (4.11) exists and is unique if $\tilde{f} : I \times E^n \rightarrow E^n$ is a fuzzy valued continuous function and Lipschitz. Before proving the above theorem, in the most generalized form, we take up particular cases and prove them as lemmas.

4.3.4.1. Lemma

Suppose $\tilde{f}\left(t, \tilde{X}\right) = 0$ in equation (4.11), then the solution of equation (4.11) is given by

$$\underline{X} = \underline{X}_0 e^{A_l t}$$
$$\overline{X} = \overline{X}_0\, e^{A_u t}$$

where, $A_l = \min(\underline{A}, \overline{A})$, $A_u = \max(\underline{A}, \overline{A}\,)$, and $\underline{X} \leq \overline{X}$.

Proof: If $\tilde{f}\,(t, \tilde{X}) = 0$ in equation (4.11) then the parametric form is obtained as,

$$\underline{\dot{X}} = \min\,(\underline{A}\;\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\;\overline{X})$$
$$\dot{\overline{X}} = \max\,(\underline{A}\;\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\;\overline{X})$$

with initial conditions ${}^{\alpha}\tilde{X}_0 = [\underline{X}_0, \overline{X}_0]$.

Now using, $A_l = \min(\underline{A}, \overline{A})$, $A_u = \max(\underline{A}, \overline{A}\,)$,

$$\underline{\dot{X}} = A_l\, \underline{X}$$
$$\dot{\overline{X}} = A_u\, \overline{X}$$

with initial condition, ${}^{\alpha}\tilde{X}_0 = [\underline{X}_0, \overline{X}_0]$.

Now using Laplace Transform of fuzzy derivative on $\underline{\dot{X}} = A_l\, \underline{X}$ and $\dot{\overline{X}} = A_u\, \overline{X}$ as in 4.3.3.4, we get,

$$s\mathcal{L}[\underline{X}] - \underline{X}_0 = A_l\, \mathcal{L}[\,\underline{X}]$$
$$s\mathcal{L}[\overline{X}\,] - \overline{X}_0 = A_u \mathcal{L}[\overline{X}]$$

Thus, applying inverse fuzzy Laplace Transform and we obtain the solution as,

$$\underline{X} = \underline{X}_0 e^{A_l t}$$
$$\overline{X} = \overline{X}_0\, e^{A_u t}.$$

In the next section, we linearize the nonlinear term involved in equation (4.11) around the equilibrium point by using Taylor's theorem and find the solution for the system. The result for the same is as follows.

4.3.4.2. Lemma

$\tilde{f}\,(t, \tilde{X}) \neq 0$ and $\tilde{f}$ in the system (4.11) is n times differentiable at a point then it can be linearized around equilibrium point by Taylor's expansion.

Proof: System (4.11) is given as,

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \oplus \tilde{f}\,(t, \tilde{X})$$

with initial condition, $\tilde{X}(0) = \tilde{X}_0$

First taking α −cut of the above system,

$$\underline{\dot{X}} = \min\,(\underline{A}\;\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\;\overline{X}) + \underline{f}(t, \underline{X}, \overline{X})$$

$$\dot{\overline{X}} = \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\,\overline{X}\right) + \overline{f}\left(t, \underline{X}, \overline{X}\right)$$

where, $\underline{f}\left(t, \underline{X}, \overline{X}\right) = \min f\left(t, \underline{X}, \overline{X}\right)$ and $\overline{f}\left(t, \underline{X}, \overline{X}\right) = \max f\left(t, \underline{X}, \overline{X}\right)$

Put $\dot{\underline{X}} = 0, \dot{\overline{X}} = 0$ and we get $\left(\underline{X}_e, \overline{X}_e\right)$ equilibrium point then applying Taylor's expansion and we have,

$$\dot{\underline{X}} = \min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\,\overline{X}\right) + \underline{f}\left(\underline{X}_e, \overline{X}_e\right) + \frac{\partial \underline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \underline{X}}\left(\underline{X} - \underline{X}_e\right) + \left(\overline{X} - \overline{X}e\right)\frac{\partial \underline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \overline{X}}$$

$$\dot{\overline{X}} = \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\,\overline{X}\right) + \overline{f}\left(\underline{X}_e, \overline{X}_e\right) + \left(\overline{X} - \overline{X}e\right)\frac{\partial \overline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \overline{X}} + \left(\underline{X} - \underline{X}_e\right)\frac{\partial \overline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \underline{X}}$$

Since, using these notations,

$$A_l\,\underline{X} = \min\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\,\overline{X}\right) \text{ and } A_u\overline{X} = \max\left(\underline{A}\,\underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\,\overline{X}\right).$$

We get,

$$\dot{\underline{X}} = A_l\,\underline{X} + \underline{f}\left(\underline{X}_e, \overline{X}_e\right) + \frac{\partial \underline{f}}{\partial \underline{X}}\left(\underline{X} - \underline{X}_e\right) + \left(\overline{X} - \overline{X}e\right)\frac{\partial \underline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \overline{X}}$$

$$\dot{\overline{X}} = A_u\overline{X} + \overline{f}\left(\underline{X}_e, \overline{X}_e\right) + \frac{\partial \overline{f}}{\partial \underline{X}}\left(\underline{X} - \underline{X}_e\right) + \left(\overline{X} - \overline{X}e\right)\frac{\partial \overline{f}\left(t, \underline{X}, \overline{X}\right)}{\partial \underline{X}}$$

Now, using the following notations,

$$A_{Ll}\,X_{Ll} = A_l\,\underline{X} + \frac{\partial \underline{f}}{\partial \underline{X}}\,\underline{X} + \overline{X}\,\frac{\partial \underline{f}}{\partial \overline{X}}, \quad B_{Ll} = \underline{f}\left(\underline{X}_e, \overline{X}_e\right) - \frac{\partial \underline{f}}{\partial \underline{X}}\,\underline{X}_e - \frac{\partial \overline{f}}{\partial \overline{X}}\overline{X}_e,$$

$$A_{Lu}\,X_{Lu} = A_u\,\overline{X} + \frac{\partial \overline{f}}{\partial \overline{X}}\,\overline{X} + \frac{\partial \overline{f}}{\partial \underline{X}}\underline{X} \text{ and } B_{Lu} = \overline{f}\left(\underline{X}_e, \overline{X}_e\right) - \frac{\partial \overline{f}}{\partial \overline{X}}\,\overline{X}_e - \frac{\partial \overline{f}}{\partial \underline{X}}\underline{X}_e.$$

We have,

$$\dot{\underline{X}} = A_{Ll}X_{Ll} + B_{Ll} \quad \text{and} \quad \dot{\overline{X}} = A_{Lu}X_{Lu} + B_{Lu}$$

Thus, by first decomposition theorem as in [70], using the parametric form of ${}^{\alpha}\tilde{A}_L = [A_{Ll}, A_{Lu}]$ and ${}^{\alpha}\tilde{B}_L = [B_{Ll}, B_{Lu}]$, we get,

$$\dot{\tilde{X}} = \tilde{A}_L \otimes \tilde{X}_L \oplus \tilde{B}_L.$$

The solution of the above linearized system can be obtained as given in Section 4.3.4.1.

In the next section, we give result that is useful for the fuzzy solution of equation (4.11) involving nonlinear term. For this, we extend the result refer [25] in a fuzzy environment. We apply fuzzy Laplace transform to the system (4.11) and converted this system into the Volterra Integral equation.

4.3.4.3. Lemma

System (4.11) can be converted into Volterra integral equation as given below,

$$\underline{X} = e^{A_l t}\underline{X}_0 + e^{A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(t, \underline{X}, \overline{X}\right)d\tau,$$

$$\overline{X} = e^{A_u t}\overline{X}_0 + e^{A_u t}\int_0^t e^{-A_u \tau}\overline{f}\left(\tau, \underline{X}, \overline{X}\right) d\tau.$$

And,

$$\underline{X}(t) = \lim_{i\to\infty} \underline{X_i}$$

$$\overline{X}(t) = \lim_{i\to\infty} \overline{X}_i$$

where,

$$\underline{X_i} = \underline{X}(0)e^{A_l t} + e^{-A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(\tau, \underline{X_{i-1}}, \overline{X}_{i-1}\right) d\tau,$$

$$\overline{X}_i = \bar{X}(0)e^{A_u t} + e^{-A_u t}\int_0^t e^{-A_u \tau}\overline{f}\left(\tau, \underline{X_{i-1}}, \overline{X}_{i-1}\right) d\tau.$$

Then equation (4.11) has a fuzzy solution if its integral equations have a fuzzy solution i.e., $\underline{X_i} \leq \overline{X}_i$.

Proof: Taking $\alpha -$ cut and apply Fuzzy Laplace Transform on both sides of equation (4.11), we get,

$$\dot{\underline{X}} = \min\left(\underline{A}\ \underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\ \overline{X}\right) + \underline{f}\left(t, \underline{X}, \overline{X}\right)$$

$$\dot{\overline{X}} = \max\left(\underline{A}\ \underline{X}, \underline{A}\overline{X}, \overline{A}\underline{X}, \overline{A}\ \overline{X} + \overline{f}\left(t, \underline{X}, \overline{X}\right)\right.$$

where,

$$\underline{f} = \min f\left(t, \underline{X}, \overline{X}\right), \overline{f} = \max f\left(t, \underline{X}, \overline{X}\right)$$

Now the equation (4.11) becomes,

$$\dot{\underline{X}} = A_l\ \underline{X} + \underline{f}\left(t, \underline{X}, \overline{X}\right) \tag{4.12-a}$$

$$\dot{\overline{X}} = A_u\ \overline{X} + \overline{f}\left(t, \underline{X}, \overline{X}\right) \tag{4.12-b}$$

Taking Fuzzy Laplace Transform on equation (4.12-a), we get

$$s\mathcal{L}[\underline{X}] = \underline{X}_0 + A_l\mathcal{L}[\underline{X}] + \mathcal{L}\left[\underline{f}\left(t, \underline{X}, \overline{X}\right)\right]$$

Using derivative of fuzzy Laplace transform, we get

$$s\mathcal{L}[\underline{X}] - A_l\mathcal{L}[\underline{X}] = \underline{X}_0 + \mathcal{L}\left[\underline{f}\left(t, \underline{X}, \overline{X}\right)\right]$$

$$(s - IA_l)\mathcal{L}[\underline{X}] = \underline{X}_0 + \mathcal{L}\left[\underline{f}\left(t, \underline{X}, \overline{X}\right)\right]$$

$$\mathcal{L}[\underline{X}] = \frac{\underline{X}_0}{(s - IA_l)} + \frac{\mathcal{L}\left[\underline{f}\left(t, \underline{X}, \overline{X}\right)\right]}{(s - IA_l)}$$

Now by fuzzy convolution theorem and inverse fuzzy Laplace transform, we get,

$$\underline{X} = \mathcal{L}^{-1}\frac{\underline{X}_0}{(s - IA_l)} + \mathcal{L}^{-1}\frac{L\left[\underline{f}(t,\underline{X},\overline{X})\right]}{(s - IA_l)}$$

$$\underline{X} = \underline{X}_0 e^{A_l t} + e^{A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(\tau,\underline{X},\overline{X}\right)d\tau$$

Similarly,

$$\overline{X} = e^{A_u t}\overline{X}_0 + e^{A_u t}\int_0^t e^{-A_u \tau}\overline{f}\left(\tau,\underline{X},\overline{X}\right)d\tau$$

Now applying iterative scheme, we get,

$$\underline{X_i} = \underline{X}(0)e^{A_l t} + e^{A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right)d\tau \quad \forall\, i = 0,1,2,\ldots$$

$$\overline{X}_i = \overline{X}(0)e^{A_u t} + e^{A_u t}\int_0^t e^{-A_u \tau}\overline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right)d\tau \;\; \forall\, i = 0,1,2,\ldots$$

Thus, $\underline{X_i} \leq \overline{X}_i$, equation (4.11) has a fuzzy solution.

We prove the convergence result for the proposed scheme in 4.3.4.3 as follows.

4.3.4.4. Theorem

If ${}^{\alpha}\tilde{f} = \left[\underline{f},\overline{f}\right]$ is Lipschitz then proposed numerical technique in 4.3.4.3 is convergent.

Proof: Let ${}^{\alpha}\tilde{X} = \left[\underline{X},\overline{X}\right]$ is the exact solution of system (4.11) and ${}^{\alpha}\tilde{X}_i = \left[\underline{X_i},\overline{X}_i\right]$ be the numerical solution of the system (4.11).

Consider error in i^{th} term,

$$\underline{e_i} = \underline{X_i} - \underline{X}\,,\;\; \overline{e}_i = \overline{X}_i - \overline{X}$$

Considering the first term,

$$\left|\underline{e_i}\right| = \left|\underline{X_i} - \underline{X}\right|$$

$$\left|\underline{e_i}\right| = \left|\underline{X}(0)e^{A_l t} + e^{A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right)d\tau - \underline{X}_0 e^{A_l t} + e^{A_l t}\int_0^t e^{-A_l \tau}\underline{f}\left(\tau,\underline{X},\overline{X}\right)d\tau\right|$$

$$\left|\underline{e_i}\right| \leq \left|\underline{X}(0)e^{A_l t} - \underline{X}_0 e^{A_l t}\right| + \left|e^{A_l t}\int_0^t (e^{-A_l \tau}\,\underline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right) - e^{-A_l \tau}\underline{f}\,(\tau,\underline{X},\overline{X}))d\tau\right|$$

Since, $\underline{X}(0) = \underline{X}_0$.

$$\left|\underline{e_i}\right| \leq e^{A_l t}\int_0^t \left|e^{-A_l \tau}\underline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right) - e^{-A_l \tau}\underline{f}\left(\tau,\underline{X},\overline{X}\right)\right|d\tau,$$

$$\left|\underline{e_i}\right| \leq e^{A_l t}\int_0^t e^{A_l \tau}\left|\underline{f}\left(\tau,\underline{X_{i-1}},\overline{X}_{i-1}\right) - \underline{f}\left(\tau,\underline{X},\overline{X}\right)\right|d\tau.$$

$\therefore\ \underline{f}$ is Lipschitz with Lipschitz constant $\underline{L}$, which gives,

$$|\underline{e_i}| \leq e^{A_l t} \int_0^t e^{A_l \tau} \underline{L}\, d\tau$$

$$= e^{A_l t} \underline{L} \left(\frac{e^{A_l t} - 1}{A_l} \right)$$

$$= \underline{L} \left(\frac{e^{2A_l t} - e^{A_l t}}{A_l} \right)$$

Now, using the Ratio test, the right-hand side expression converges for all t.

Thus, $|\underline{e_i}|$ is bounded.

Similarly, using Lipschitz constant $\overline{L}$ for function $\overline{f}$, $|\overline{e}_i|$ is bounded.

Hence, ${}^{\alpha}\tilde{f} = \left[\underline{f}, \overline{f}\right]$ is Lipschitz continuous with constant $L = \min(\underline{L}, \overline{L})$.

So, the proposed technique in Section 4.3.4.3 is convergent.

4.3.4.5. Proof of Main Theorem:

The solution of the system (4.11) is obtained using Sections 4.3.4.1, 4.3.4.2 and 4.3.4.3.

4.4. Example:

Now we apply these 3 cases to solve the fuzzy Prey- Predator model [73],

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{0.1} \otimes \tilde{x} \ominus \widetilde{0.005} \otimes \tilde{x} \otimes \tilde{y} \\ \dot{\tilde{y}} &= \ominus\, \widetilde{0.4} \otimes \tilde{y} \oplus \widetilde{0.008} \otimes \tilde{x} \otimes \tilde{y} \end{aligned} \qquad (4.13)$$

with initial conditions, $\tilde{x}_0 = \widetilde{130}$ and $\tilde{y}_0 = \widetilde{40}$

where, the value of parameters is given as,

$\widetilde{0.1} = (0.05, 0.1, 0.15), \widetilde{0.005} = (0.004, 0.005, 0.006),$

$\widetilde{0.4} = (0.3, 0.4, 0.5), \widetilde{0.008} = (0.007, 0.008, 0.009),$

$\widetilde{130} = (120, 130, 150), \widetilde{40} = (20, 40, 50)$

${}^{\alpha}\widetilde{0.1} = [0.05 + 0.05\,\alpha, 0.15 - 0.05\,\alpha],$

${}^{\alpha}\widetilde{0.005} = [0.004 + 0.001\alpha, 0.006 - 0.001\alpha],$

${}^{\alpha}\widetilde{0.4} = [0.3 + 0.1\alpha, 0.5 - 0.1\alpha],$

${}^{\alpha}\widetilde{0.008} = [0.007 + 0.001\alpha, 0.009 - 0.001\alpha],$

${}^{\alpha}\widetilde{130} = [120 + 10\alpha, 150 - 20\alpha]$ and ${}^{\alpha}\widetilde{40} = [20 + 20\alpha, 50 - 10\alpha]$

Solution:

The solution of above problem is given by considering the following three cases.

Case 1:

Neglecting the nonlinear term in equation (4.13) that is considering the homogeneous system only.

$$\dot{\tilde{x}} = \widetilde{0.1} \otimes \tilde{x}$$
$$\dot{\tilde{y}} = \ominus\, \widetilde{0.4} \otimes \tilde{y} \tag{4.14}$$

with initial condition, $\tilde{x}_0 = \widetilde{130}$ and $\tilde{y}_0 = \widetilde{40}$

After applying the proposed scheme as, in Section 4.3.4.1, graphs are obtained as follows,

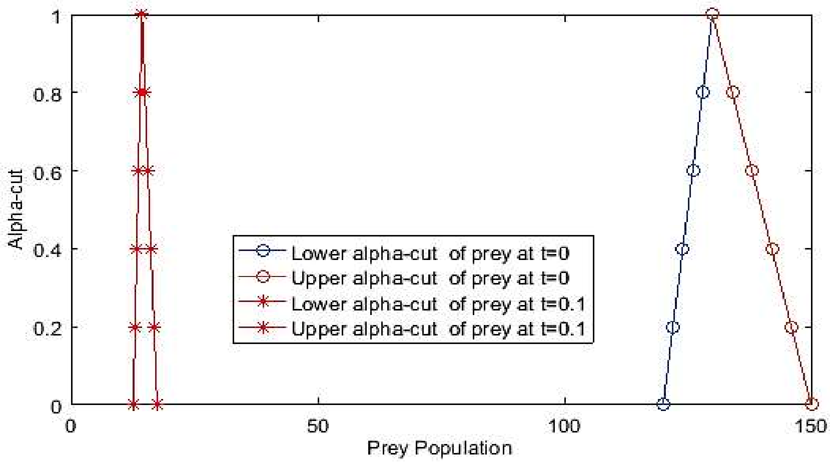

Figure 4.3: Fuzzy Number representation of Prey population at $t = 0$ & 0.1

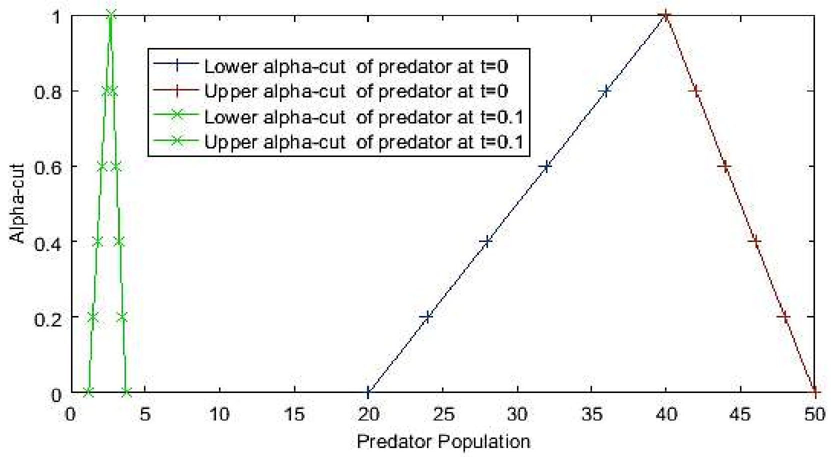

Figure 4.4: Fuzzy number representation of Predator population at $t = 0$ & 0.1

The number of Prey-Predator at different time is given in Table 4.1.

Table -4.1: Number of prey and predator in first approach

Time	Number of Prey	Number of Predator
0	(120,130,150)	(20,40,50)
0.1	(12.61,14.36,17.42)	(1.21,2.68,3.704)
0.2	(25.23,28.73,34.85)	(2.426,5.36, 7.408)
0.3	(37.84,43.10,52.28)	(3.63,8.04,11.11)
0.4	(50.46,57.46,69.71)	(4.85,10.72,14.81)
0.5	(63.07, 71.83, 87.13)	(6.06, 13.4, 18.52)
1	(126.15,143.67,174.27)	(12.13,26.81,37.04)

Case 2:

First, linearize the eq. (4.13) around equilibrium point by Taylor's expansion, we obtain the linearized form as below,

$$\dot{\tilde{x}} = \ominus \ \widetilde{0.25} \otimes \tilde{x} \ \oplus \tilde{5} \qquad (4.15)$$

$$\dot{\tilde{y}} = \ \widetilde{0.16} \otimes \ \tilde{y} \ominus \tilde{8}$$

with initial conditions, $\tilde{x}(0) = \ \widetilde{130}, \tilde{y}(0) = \ \widetilde{40}$

where, the value of parameters in fuzzy triangular form is given as,

$\widetilde{0.25} = (0.20, 0.25, 0.30), \tilde{5} = (4, 5, 6),$

$\widetilde{0.16} = (0.15, 0.16, 0.17), \tilde{8} = (6, 8,10).$

The parametric form of parameters is given below,

${}^{\alpha}\widetilde{0.25} = [0.20 + 0.05\alpha, 0.30 - 0.05\alpha],$

${}^{\alpha}\widetilde{0.16} = [0.15 + 0.01\alpha, 0.17 - 0.01\alpha],$

${}^{\alpha}\tilde{5} = [4 + \alpha, 6 - \alpha]$, ${}^{\alpha}\tilde{8} = [6 + 2\alpha, 10 - 2\alpha]$.

Applying the proposed scheme as, in Section 4.3.4.2, graphs are obtained as follows,

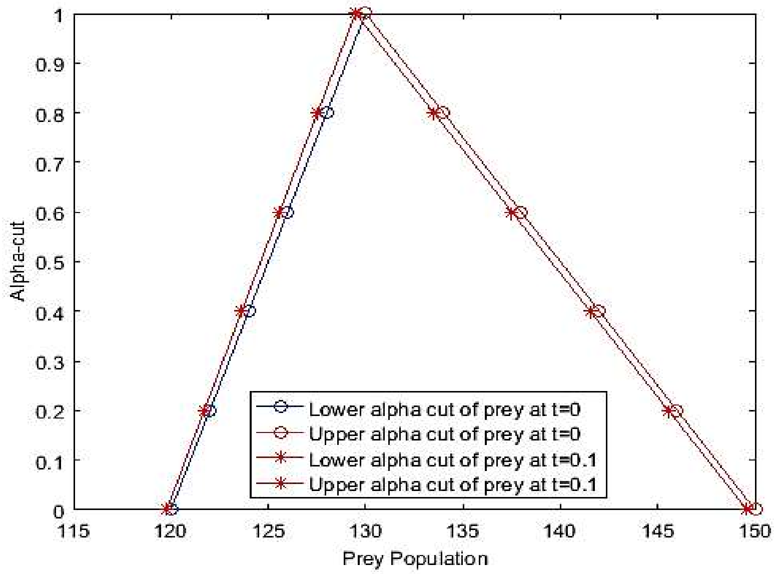

Figure 4.5: Fuzzy Number representation of Prey population at *t* = 0 & 0.1

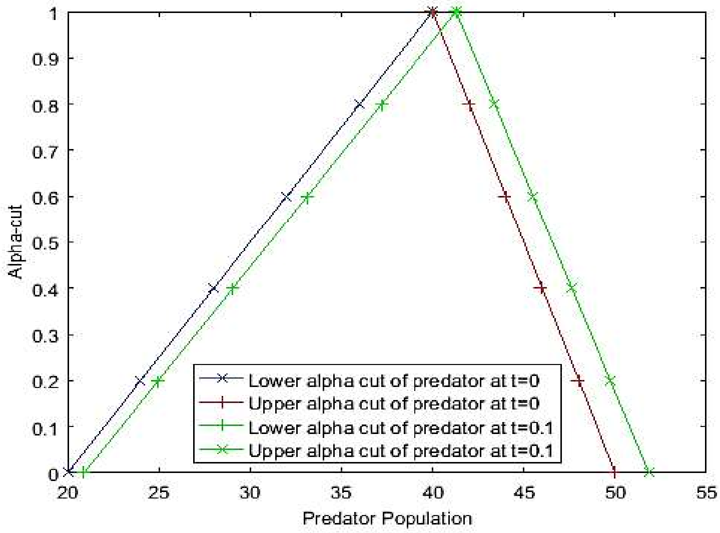

Figure 4.6: Fuzzy Number representation of Predator population at *t* = 0 & 0.1

The number of Prey-Predator at a different time is given in Table 4.2.

Table 4.2. Number of prey and predator in second approach

Time	Number of Prey	Number of Predator
0	(120,130,150)	(20,40,50)
0.1	(118.66,127.10,147.54)	(24.45,46.28,59.152)
0.2	(119.54,128.93,149.12)	(21.79,42.54,53.68)
0.3	(119.27,128.35,148.63)	(22.68,43.80,55.51)
0.4	(118.98,127.74,148.10)	(23.57,45.05,57.33)
0.5	(118.66,127.10,147.54)	(24.45,46.28,59.56)
1.0	(116.67,123.43,144.177)	(28.78,53.31,68.05)

Case 3:

Applying technique as given in Section 4.3.4.3, we obtain following graphs for fuzzy solution,

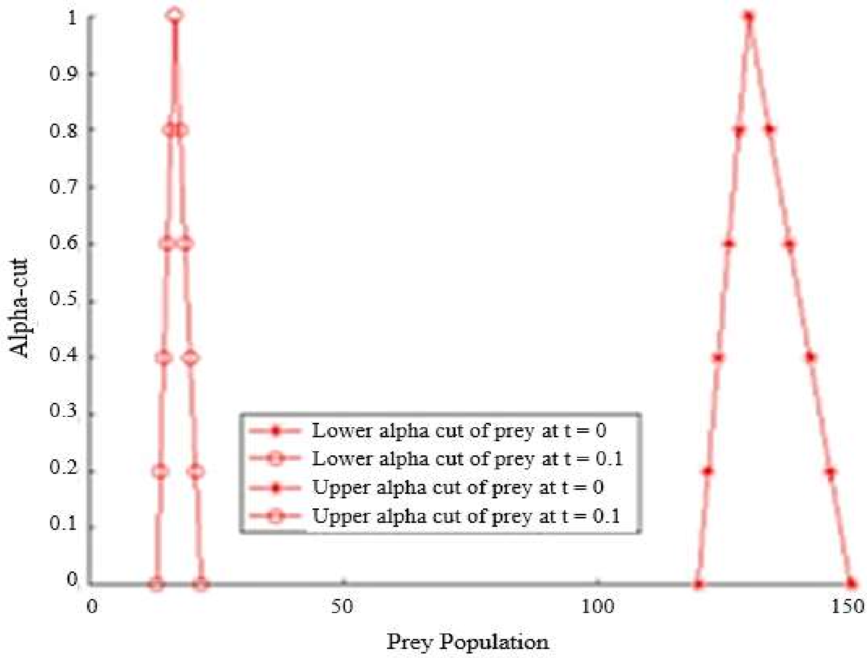

Figure 4.7: Fuzzy Number representation of Prey population at *t* = 0 & 0.1

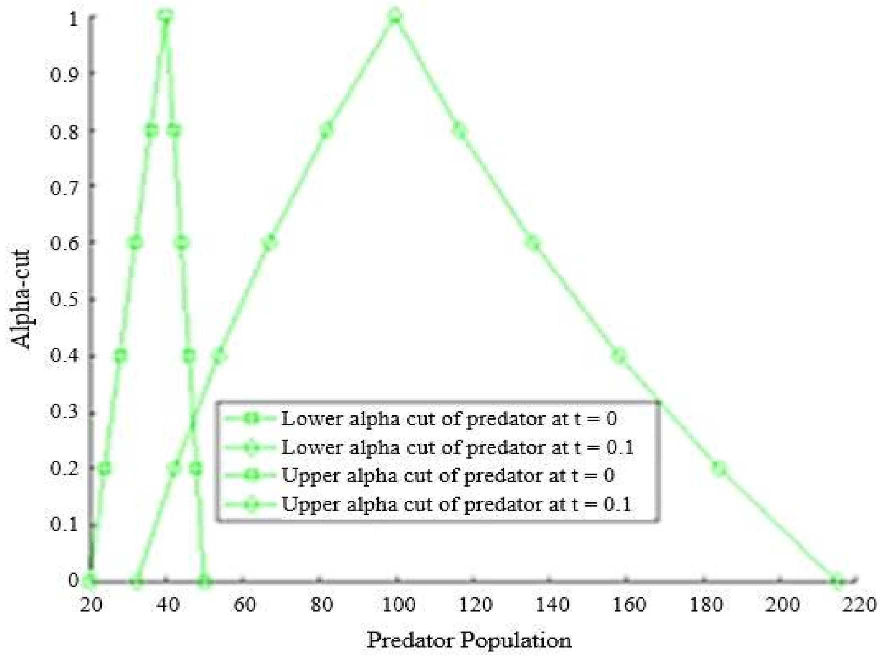

Figure 4.8: Fuzzy Number representation of Predator population at $t = 0$ & 0.1

The number of Prey and Predator for different time is given in below Table 4.3.

Table 4.3. Number of prey and predator in third approach

Time	Number of Prey	Number of Predator
0	130	40
0.1	118.49	60.56
0.2	114.774	66.2506
0.3	114.19	65.18
0.4	116.41	65.229
0.5	117.199	60.27

4.5. Conclusion

In this chapter, first we have solved fuzzy linear dynamical system using the existing fuzzy Laplace transform. We then redefined Fuzzy Laplace Transform under new derivative i.e., Modified Hukuhara derivative along with its existence condition. We also revised all results related to Fuzzy Laplace Transform under this new derivative. Lastly, we have solved fully fuzzy Prey-Predator model by considering three cases and results in all cases are compared at the core.