APPMATH/2020-21/265
FTE/ACED/ 56.

Name of the Ph.D. scholar	:	**Payal Singh**
Registration No	:	**FOTE/ 877**
Registration Date	:	**24/04/2015**
Name of the Guide	:	**Dr. Purnima K. Pandit**
Department	:	**Applied Mathematics**
Faculty	:	**Faculty of Technology & Engineering**
Subject of the Ph.D.		**Mathematical Modelling, Analysis and Applications of Fuzzy Systems**

Forwarded to:

The Registrar (Academics)

The Maharaja Sayajirao University of Baroda

Vadodara- 390002

Payal Singh
Candidate

PKPandit 22/4/2021
Guide

22/04/2021
Head of the Department

Dean

Head
Applied Mathematics Department
Faculty of Technology & Engineering
The Maharaja Sayajirao University of Baroda
Vadodara

Dean
Faculty of Tech. & Engg.
M. S. University of Baroda,
Baroda.

Mathematical Modelling, Analysis and Applications of Fuzzy Systems

Submitted by

Payal Singh

(Registration No:FOTE/877)

Guided By

Dr. Purnima K. Pandit

Towards the Partial Fulfilment for the Degree of

DOCTOR OF PHILOSOPHY

in

APPLIED MATHEMATICS

DEPARTMENT OF APPLIED MATHEMATICS

Faculty of Technology and Engineering

The Maharaja Sayajirao University of Baroda

Vadodara-390001, Gujarat, India

(April, 2021)

1 Introduction

Modelling of various physical phenomenon under the influence of phyasical laws are represented in form of differential equations. In these models various parameters involved are obtained from experiments or observations. These values such obtained may have certain measurement errors or imprecision. These kinds of error in turn leads to the imprecision in the modelling of physical problem. To model such phenonmenon with imprecise parameters, using fuzzy sets give more realistic realizations. Such general dynamical models involving fuzzy parameters are given as,

$$\dot{\tilde{X}} = \tilde{f}(t, \tilde{X}); \tilde{X}(0) = \tilde{X}_0$$

The above equation explains real world problem with imprecision in more realistic way than crisp counter part.

The concept of fuzzy set was first discussed in seminal paper by Zadeh [1] in 1965. The concept of fuzzy derivative was first introduced by Chang and Zadeh [2] in 1972. Dubois and Prade [3] defined derivative based on extension principle in 1982. Puri and Ralescue [4] introduced H-derivative of fuzzy valued function based on Hukuhara difference [5]. Kaleva [6] and Seikkala [7], first simultaneously solved the fuzzy initial value problem with fuzzy initial condition. Kandel and Byatt [8] used this theory for the applications of fuzzy dynamical system. The basic and most popular approach to solve fuzzy differential equation (FDE) is Hukuhara differentiability which is based on H-difference. The drawback in using Hukuhara derivative is that solution does not remain fuzzy as time increases. To overcome this situation, Generalized Hukuhara derivative [9] is proposed, which is very popular among other fuzzy derivatives. When generalized Hukuhara derivative is used to solve the fuzzy differential equation, the solution is obtained as possible set of solutions from which one needs to choose the solution which best satisfies the problem. This is one of the disadvantage of generalized Hukuhara derivative. To overcome this, we have proposed Modified Hukuhra derivative which gives unique fuzzy solution.

Initially, most of the authors have worked on scalar differential equation with fuzzy initial condition using different techniques like analytical, numerical, transformation and semi-analytical. The numerical method for solving FDE is introduced by M. Ma, Friedman and Kandel refer [10]. They used classical Euler method which is followed by complete error analysis . Javed Shokri in [11] solved FDE by modified Euler method . S. Abbasbandy and T. Allahvi-

ranloo refer [12] solved FDE by Taylor's method of order p and proved the order of convergence is $O(h^p)$ which is better than the order of convergence of Euler's method. FDE is solved by some other numerical methods like Runge-Kutta and Corrector-Predictor method as in [13], [14] respectively. T. Jayakumar, D. Mahesh Kumar and Rangrajan [15] presented solution of FDE by 5th order of Runge Kutta method. Numerical solution of second order FDE is given by N. Parandin in [16] by using Runge Kutta method. Solution of FDE under generalized differentiability by Improved Euler method under strongly generalized differentiability is given by K. Kangrajan and R. Suresh refer [17], here only the fuzzy initial condition is considered.
Allahviranloo and Ahmadi [18] proposed fuzzy Laplace transforms (FLT) for solving first order differential equations under generalized H-differentiability without giving the existence condition. S. Salahsour and T. Allahviranloo [19] described the existence condition for Laplace transform and its inverse. Many other authors in [20], [21] used FLT to solve FDE. They solve second order fuzzy differential equantion and n^{th} order fuzzy differential equation with fuzzy initial condition under Generalized Hukuhara derivative.
Some authors have used semi-analytical techniques to solve nonlinear differential equation like Homotopy perturbation method (HPM), Adomian Decomposition method (ADM) , Variational Iteration method (VIM) etc. as in [22], [23], [24], [25], [26], [27]. These techniques are also used in solving fuzzy nonlinear differential equations. Other methods for solving FDE and important results pertaining to continuity, existence of solution and various applications are given in [28-58].
Our work comprises of solution of system of linear fuzzy differential equation with fuzzy parameters as well as fuzzy initial condition that is fully fuzzy linear dynamical system, followed by some work on nonlinear fully fuzzy differential equation. To solve fully fuzzy dynamical systems, we have used different techniques like analytical, numerical, transformation and semi-analytical techniques. We have also proposed and proved various existence and uniqueness results. We list the preliminary concepts used in the ensuing sections and details appertain to the development in our work for fuzzy dynamical systems.

2 Preliminary

2.1 Fuzzy Sets

A fuzzy set is orderd pair of a set $\tilde{A}$ and membership function $\tilde{A}(x)$. If set is constructed by discrete universe then it is known as discrete fuzzy set and it is given as, $\left\{\frac{\tilde{A}(x_1)}{x_1}, \frac{\tilde{A}(x_2)}{x_2}...\right\}$. If set is constructed by continuous universe then it is known as continuous fuzzy set.

2.2 Fuzzy Number

A fuzzy set is said to be fuzzy number if it satisfies following properties,

1. $\tilde{u}$ must be normal i.e membership value should be 1 for atleast one point.

2. All α-cut must be closed. Where an α-cut of a fuzzy set $\tilde{u}$ is an ordinary set such that, ${}^{\alpha}\tilde{u} = \{x \in X, \tilde{u}(x) \geq \alpha\}$, $\alpha \in (0\ 1]$.

3. Support should be bounded.

2.3 Fuzzy Number in parametric form

A fuzzy number in parametric form is an order pair of the form ${}^{\alpha}\tilde{u} = [\underline{u}, \overline{u}]$ satisfying following condition:

1. $\underline{u}$ is bounded left continuous increasing function in [01].

2. $\overline{u}$ is bounded right continuous decreasing function in [01].

3. $\underline{u} \leq \overline{u}$

2.4 A fuzzy triangular number

Triangular fuzzy number is denoted as triplet (d, e, f), and its membership function, $\tilde{u}(x) = \begin{cases} \frac{(x-d)}{d-e} & d < x \leq e \\ \frac{(f-x)}{f-e} & e < x \leq f \\ 0 & otherwise \end{cases}$

2.5 Fuzzy arithematic operations

Let $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers, then the arithmetic operations between $\tilde{A}$ and $\tilde{B}$ are as follows,

$$^{\alpha}\tilde{A} \oplus^{\alpha} \tilde{B}=[\underline{A}, \overline{A}] + [\underline{B}, \overline{B}]=[\underline{A} + \underline{B}, \overline{A} + \overline{B}]$$

and

$$\lambda\ ^{\alpha}\tilde{A}=\lambda\ [\underline{A}, \overline{A}]=[\lambda\ \underline{A}, \lambda\ \overline{A}]$$

2.6 Fuzzy Lipschitz

Let $E = \{\tilde{u} : R \rightarrow [0\ 1]\}$ is the collection of fuzzy numbers which satisfies all property mentioned in section 2.2 and I is $[t_0, T]$ then $\tilde{f} : I \times E \rightarrow E$ is fuzzy lipschitz [49] if,

$$d(\tilde{f}(t, \tilde{X})), d(\tilde{f}(t, \tilde{Y})) \leq d(\tilde{X}, \tilde{Y})$$

Where, $\tilde{X}, \tilde{Y} \in E$

2.7 Fuzzy continuity

As given in [49], if $\tilde{f} : I \times E \rightarrow E$ then $\tilde{f}$ is fuzzy continuous at point (t_0, x_0) provided that for any fixed number $\alpha \in (0, 1]$ and any $\epsilon > 0, \exists\ \delta(\epsilon, \alpha)$ such that $d(\tilde{f}(t, \tilde{x}), \tilde{f}(t_0, x_0)) < \epsilon$, where $|t - t_0| < \delta$ and $d(\tilde{x}, \tilde{x}_0) < \delta(\epsilon, \alpha)$.

2.8 Fuzzy derivatives

2.8.1 H-difference

The H-difference between two intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}] \in R$ is defined as,

$$A \ominus_h B = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$$

2.8.2 Hukuhara Derivative

For a fuzzy mapping $\tilde{f} : I \times E \rightarrow E$ and $t_0 \in I$, $\tilde{f}$ is said to be Hukuhara differentiable at $t_0 \in I$, if there exist an element $\dot{\tilde{f}}(t_0) \in E$ such that for all $h \geq 0$ sufficiently small $\exists \tilde{f}(t_0 + h) \ominus_h \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus_h \tilde{f}(t_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(t_0 + h) \ominus_h \tilde{f}(t_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

2.8.3 Strongly generalized differentiability

1. for all $h \geq 0$ sufficiently small $\exists \tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

2. for all $h \geq 0$ sufficiently small $\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)$, $\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{-h} = \dot{\tilde{f}}(t_0)$$

3. for all $h \geq 0$ sufficiently small $\tilde{f}(t_0+h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0-h) \ominus \tilde{f}(t_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(t_0+h) \ominus \tilde{f}(t_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(t_0-h) \ominus \tilde{f}(t_0)}{-h} = \dot{\tilde{f}}(t_0)$$

4. for all $h \geq 0$ sufficiently small $\tilde{f}(t_0) \ominus \tilde{f}(t_0+h)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0-h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0+h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0-h)}{h} = \dot{\tilde{f}}(t_0)$$

2.8.4 Generalized Hukuhara Difference

Generalized Hukuhara differences [59] is defined as follows, considering $K(X)$ as space of convex nonempty set of X, taking $A, B \in K(X)$ then generalized difference (gH) of A and B,

$$A \ominus_g B = C \iff \begin{cases} A = B + C \\ or\ B = A + (-1)C \end{cases}$$

The existence of gH-difference is either A contains translate of B or B contains translate of A. There are some properties of gH-difference are discussed as below,

1. If $C = A \ominus_g B$ exists, it is unique and if also $A \ominus B$ exists then $A \ominus_g B = A \ominus B$
2. If $A \ominus_g B$ exists and $B \ominus_g A$ then $B \ominus_g A = A \ominus_g B$.
3. $A \ominus_g A = \{0\}$
4. $(A + B \ominus_g B) = A$

2.8.5 Generalized Hukuhara Derivative

Let $\tilde{f} : I \times E \to E$ and $t_0 \in I$, $\tilde{f}$ is said to be generalized Hukuhara differentiable at $t_0 \in I$, if there exist element $\dot{\tilde{f}}(t_0)$ and

$$\dot{\tilde{f}}(t_0) = \lim_{h\to 0} \frac{\tilde{f}(t_0+h) \ominus_g \tilde{f}(t_0)}{h}$$

2.8.6 Seikkala derivative

$$SD(\overline{X}(t)) = \sup \max |(x_1(t,\alpha), z_1(t,\alpha)), x_2(t,\alpha), z_2(t,\alpha))|$$

where, $(\dot{x}_1(t,\alpha), \dot{x}_2(t,\alpha))$ are aSeikkala derivative is defined as in [7],lpha cuts.

We have carried out this work in direction of fully fuzzy dynamical system.

Since, our work is for the system of fully fuzzy differential equations, we have extended results like continuity, differentiability, integrability etc. in the generalized n vector space E^n as given in these research articles [53], [60].

3 Our work

Research in the area of fuzzy dynamical systems, till the start of our work majorly included the solution of scalar differential equation with fuzzy initial condition. To extend it, we initially started with solving Prey-Predator model with fuzzy initial condition. Before us, most of the people have solved Prey-Predator model by numerical techniques. We have solved Prey-Predator model by analytical technique and get the closed form solution. This solution clearly shows that model behaves like undamped oscillator because it contains periodic terms.
The analytical technique is given below briefly,

3.1 Analytical Method (Eigen vector and Eigen value)

In [61], we have considered following Prey-Predator model,

$$\begin{aligned} \dot{x}(t) &= ax - bxy \\ \dot{y}(t) &= -cy + dxy \end{aligned} \tag{1}$$

with $x(0) = \tilde{x}_0$ and $y(0) = \tilde{y}_0$.
The matrix form of equation (1) is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -bxy \\ dxy \end{bmatrix} ; \begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{bmatrix}$$

To solve the system (1), we linearize it arround the equilibrium point and we have obtained closed form solution by using eigen value and eigen vector method.
Now, the linearized form of equation (1) arround the equilibrium point is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \tag{2}$$

i.e

$$\dot{X}(t) = PX + Q \text{ with } \tilde{X}_0$$

where, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, $P = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix}$ and $\tilde{X}(0) = \begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{bmatrix}$.
For equation (2), fundamental matrix $\psi(t)$ is found to obtain state transition matrix $\phi(t, 0)$. Then the solution of (2) exists and is given as,

$$\tilde{X}(t) = \phi(t, 0)\tilde{X}_0 + \int_0^t \phi(t, \tau)Qdt$$

After this work, we solved system of differential equations involving fuzzy parameters as well as fuzzy initial condition. In the begining, we solved these systems using numerical techniques then using transform technique like Laplace Transform, and finally using semi-analytical technique Adomian decomposition method.

3.2 Numerical techniques

In [62-63], we have solved the system having fuzzy parameters with fuzzy initial conditions i.e fully fuzzy systems using two numerical techniques. First one numerical technique is based on descritization of Hukuhara derivative and the other one is Improved Euler Method. We have considered following system of differential equations,

$$\dot{\tilde{X}} = \tilde{f}(t, X) \tag{3}$$

with initial condition, $\tilde{X}(0) = \tilde{X}_0$

where, $\tilde{f} : I \times E^n \to E^n$ and,

$$\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} \text{ and, } \tilde{X}_0 = \begin{bmatrix} \tilde{x}_{10} \\ \tilde{x}_{20} \\ \vdots \\ \tilde{x}_{n0} \end{bmatrix}$$

Here, each $\tilde{f}_i$ is Hukuhara differentiable, $\forall i = 1, 2, 3, 4.....$ and $\tilde{f}(t, \tilde{X})$ can be nonlinear or linear. When we consider $\tilde{f}$ or

$$\tilde{f}(t, \tilde{X}) = \tilde{A} \otimes \tilde{X} \oplus \tilde{B} \tag{4}$$

with initial condition,

$$\tilde{X}(0) = \tilde{X}_0$$

where, $\tilde{A} = \begin{bmatrix} \tilde{a_{11}} & \cdots & \tilde{a_{1n}} \\ \vdots & \vdots & \vdots \\ \tilde{a_{n1}} & \cdots & \tilde{a_{nn}} \end{bmatrix}$ is $n \times n$ matrix and $\tilde{B} = \begin{bmatrix} \tilde{b_1} \\ \tilde{b_2} \\ \vdots \\ \tilde{b_n} \end{bmatrix}$ is column vector of $n \times 1$.

The parametric numerical scheme for equation (3), with $\tilde{f}$ as in equation (4) is given by,

$$\begin{bmatrix} \underline{X}_{k+1} \\ \overline{X}_{k+1} \end{bmatrix} = \begin{bmatrix} \underline{A} & 0 \\ 0 & \overline{A} \end{bmatrix} \cdot \begin{bmatrix} \underline{X}_k \\ \overline{X}_k \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \overline{B} \end{bmatrix}, for\ k = 0, 1, 2, 3, ... \tag{5}$$

$$\underline{X}(0) = \underline{X}_0, \overline{X}(0) = \overline{X}_0,$$

For equation (3), results for existence of solution and error analysis are proposed and proved. Further for a system such as equation (3) a numerical scheme on the line of Improved Euler Method is proved along with error analysis.

We have also, solved fully fuzzy dynamical system by Transformation technique i.e Fuzzy Laplace Transform.

3.3 Transformation technique (Fuzzy Laplace Transform)

Following numerical techniques, we worked with Laplace Transformation initially for linear homogeneous as well as non-homogeneous system as in [64]. For solution of both the systems, we used diagonalization concept then applied fuzzy Laplace transform.

We considered following model for homogeneous fuzzy dynamical system.

$$\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \tag{6}$$

$$\tilde{X}(0) = \tilde{X}_0$$

where, $\tilde{A}$ is as defined in equation (4).

The parametric form of equation (6) is,

$$[\dot{\underline{X}}, \dot{\overline{X}}] = [\min(\underline{A}\ \underline{X}, \underline{A}\ \overline{X}, \overline{A}\ \underline{X}, \overline{A}\ \overline{X}), \max(\underline{A}\ \underline{X}, \underline{A}\ \overline{X}, \overline{A}\ \underline{X}, \overline{A}\ \overline{X})] \tag{7}$$

$$[\underline{X}_0, \overline{X}_0]$$

The solution of equation (6), is given as,

$$\underline{X}(t) = \underline{X}_0 e^{(\underline{D})t}, \, \overline{X}(t) = \overline{X}_0 e^{(\overline{D})t}$$

where, $\underline{D}$ and $\overline{D}$ both are diagonal matrices, obtained from $\underline{A}$ and $\overline{A}$.

For equivalent non homogeneous linear systems i.e. $\dot{\tilde{X}} = \tilde{A} \otimes \tilde{X} \oplus \tilde{B}$ the solution of above system is given as ,

$$\underline{X} = \underline{P} L^{-1}(sI - \underline{D})^{-1}[\underline{P^{-1}} X(0) + \underline{P^{-1}} L(\underline{B})]$$

$$\overline{X} = \overline{P} L^{-1}(sI - \overline{D})^{-1}[\overline{P^{-1}} X(0) + \overline{P^{-1}} L(\overline{B})]$$

where, $\underline{P}$ and $\overline{P}$ are orthogonal matrices. The result for the existence of solution using fuzzy Laplace Transform also proved .

For the rigourous development of solution, to fully fuzzy dynamical system using Laplace Transform technique, we afresh defined Modified Hukuhara derivative for the first time. Although after some work in this area, we understand that it is more appropriate to call it as Modified Generalized Hukuhara derivative (mgh-derivative). Our proposed derivative has advantage that it gives unique solution of fuzzy dynamical system automatically without the author to select the suitable one as in case of Generalized Hukuhara derivative shown in [59]. The proposed Modified Generalized Hukuhare derivative is defined as follows.

3.3.1 Modified Generalized Hukuhare derivative (mgh-derivative)

Let $\tilde{f} : I \to E^n$ is said to be modified generalized Hukuhara differentiable at $t_0 \in I$, if there exist an element $\dot{\tilde{f}}(t_0) \in E^n$ such that for all $h \geq 0$ sufficiently small $\exists \tilde{f}(t_0 + h) - \tilde{f}(t_0)$, $\tilde{f}(t_0) - \tilde{f}(t_0 - h)$ should exist and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0)$$

The equivalent parametric form of the limits are given as,

$$\lim_{h \to 0} \frac{{}^{\alpha}\tilde{f}(t_0+h) \ominus {}^{\alpha}\tilde{f}(t_0)}{h}$$

$$= [\min(\lim_{h \to 0} \frac{\underline{f}(t_0+h) - \underline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\underline{f}(t_0+h) - \overline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\overline{f}(t_0+h) - \underline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\overline{f}(t_0+h) - \overline{f}(t_0)}{h}),$$
$$\max(\lim_{h \to 0} \frac{\underline{f}(t_0+h) - \underline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\underline{f}(t_0+h) - \lim_{h \to 0} \overline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\overline{f}(t_0+h) - \underline{f}(t_0)}{h}, \lim_{h \to 0} \frac{\overline{f}(t_0+h) - \overline{f}(t_0)}{h})]$$

$$\lim_{h\to 0}\frac{{}^{\alpha}\tilde{f}(t_0)\ominus{}^{\alpha}\tilde{f}(t_0-h)}{h}$$

$$= [\min(\lim_{h\to 0}\tfrac{\underline{f}(t_0)-\underline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\underline{f}(t_0)-\overline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\overline{f}(t_0)-\underline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\overline{f}(t_0)-\overline{f}(t_0-h)}{h}),$$
$$\max(\lim_{h\to 0}\tfrac{\underline{f}(t_0)-\underline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\underline{f}(t_0)-\overline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\overline{f}(t_0)-\underline{f}(t_0-h)}{h}, \lim_{h\to 0}\tfrac{\overline{f}(t_0)-\overline{f}(t_0-h)}{h})]$$

Alongwith the proposed mgh-derivative in [65], various results pertaining to the existence of fuzzy Laplace Transform for the function, derivative for function and fuzzy convolution theorem are given as listed below.

3.3.2 Fuzzy Laplace Transform under mgh-derivative

Consider fuzzy valued function $\tilde{f} = [\underline{f}, \overline{f}]$ in parametric form, which is bounded and piecewise continuous on the interval $[0, \infty)$ and suppose that $\tilde{f} \otimes e^{-st}$ is improper fuzzy Riemann integrable [48]. Then fuzzy Laplace transform of $\tilde{f}$ is defined as,

$$\tilde{F}(s) = L(\tilde{f}(t)) = \int_0^\infty e^{-st} \otimes \tilde{f}(t)dt$$

$$\tilde{F}(s) = L(\tilde{f}(t)) = \lim t \to \infty \int_0^t e^{-st} \otimes \tilde{f}(t)$$

Now parametric form of above equation,

$$L[\underline{f}(t), \overline{f}(t)] = \lim t \to \infty \int_0^t e^{-st} \otimes [\underline{f}(t), \overline{f}(t)]dt$$

$$L[\underline{f}(t), \overline{f}(t)] = \lim t \to \infty[\int_0^t e^{-st} \otimes \underline{f}(t)dt, \int_0^t e^{-st} \otimes \overline{f}(t)dt]$$

$$L(\underline{f}(t)) = \min[\lim t \to \infty[\int_0^t e^{-st} \otimes \underline{f}(t)dt, \int_0^t e^{-st} \otimes \overline{f}(t)dt]$$

$$L(\overline{f}(t)) = \max[\lim t \to \infty[\int_0^t e^{-st} \otimes \underline{f}(t)dt, \int_0^t e^{-st} \otimes \overline{f}(t)dt]$$

3.3.3 Fuzzy Laplace of Derivative

If $\tilde{f}(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ be continuous fuzzy valued function, $\lim_{t\to\infty} e^{-st}\underline{f}(t) \to 0$ and $\lim_{t\to\infty} e^{-st}\overline{f}(t) \to 0$ for large value of s and $\dot{\tilde{f}}(t)$ is piecewise continuous then $L(\dot{\tilde{f}}(t))$ exist, and is given by,

$$L(\dot{\tilde{f}}(t)) = sL(\tilde{f}(t)) \ominus \tilde{f}_0$$

3.3.4 Fuzzy convolution Theorem

Let $\tilde{f}(s)$ and $\tilde{g}(s)$ denote the fuzzy inverse Laplace transforms of $\tilde{f}(t)$ and $\tilde{g}(t)$ respectively. Then the product given by $\tilde{f}(s) \otimes \tilde{g}(s)$ is the fuzzy inverse Laplace transform of the convolution of $\tilde{f}(s)$ and $\tilde{g}(s)$ is given by,

$$L(\tilde{f}(t) \star \tilde{g}(t)) = \tilde{f}(s) \star \tilde{g}(s)$$

After that, we have solved fully fuzzy system by semi-analytical technique.

3.4 Semi-analytical Technique

Follwing semi-analytical techniques are used.

- Fuzzy Adomian Decomposition Method in parametric form (FADMP)
- Fuzzy Adomian Decomposition Method (FADM)

3.4.1 Fuzzy Adomian Decomposition Method in Parametric form (FADMP)

The article [66], investigates solution of fully fuzzy system by FADM in parametric form. In this method, we decompose the nonlinear part of fully fuzzy system into the series of fuzzy Adomian polynomials and obtain the solution of system in series form. We have also discussed the convergence of FADM in parametric form.

Consider nonlinear fuzzy differential equation given as, $L\tilde{u}+N\tilde{u}+R\tilde{u} = \tilde{g}$ where, L is linear (mgh-differentiable) operator, N is nonlinear operator, R is the operator of less order than that of L and $\tilde{g}$ is source term. Then applying, L^{-1} i.e. fuzzy integration operator on both sides we get,

$$\tilde{u} = L^{-1}\tilde{g} - L^{-1}(N\tilde{u}) - L^{-1}(R\tilde{u})$$

Parametric form of above equation is,as follows,

$$\begin{aligned}&[\underline{u}, \overline{u}] = L^{-1}[\min(\underline{g}, \overline{g}), \max(\underline{g}, \overline{g})] - L^{-1}[\min(N\underline{u}, N\overline{u}), \max(N\underline{u}N\overline{u})] - \\ &L^{-1}[\min(R\underline{u}, R\overline{u}), \max(R\underline{u}, R\overline{u})]\end{aligned} \tag{8}$$

Now, we decompose the nonlinear term, $[\min(N\underline{u}, N\overline{u}), \max(N\underline{u}, N\overline{u})]$ into series of Adomian polynomials $\underline{A}_1, \underline{A}_2, \underline{A}_3 ... \underline{A}_n$ and $\overline{A}_1, \overline{A}_2, \overline{A}_3 ... \overline{A}_n$, with the general polynomials as,

$$\underline{A}_n = \min(\frac{1}{n!}\frac{d^n}{d\lambda^n}N(\sum_k^{\infty}(\underline{X}_k)\lambda^n), \frac{1}{n!}\frac{d^n}{d\lambda^n}N(\sum_k^{\infty}(\overline{X}_k)\lambda^n))$$

$$\overline{A}_n = \max(\frac{1}{n!}\frac{d^n}{d\lambda^n}N(\sum_k^{\infty}(\underline{X}_k)\lambda^n), \frac{1}{n!}\frac{d^n}{d\lambda^n}N(\sum_k^{\infty}(\overline{X}_k)\lambda^n))$$

So, substituting these Adomian polynomials in equation (8), the desirable solution otained as,

$$[\underline{u}, \overline{u}] = [\underline{u_1}, \overline{u_1}] + [\underline{u_2}, \overline{u_2}] + [\underline{u_3}, \overline{u_3}] + ...$$

Using FADMP, we have solved following real life application in [67].

3.4.2 Application-Mathematical Modelling of Air Heating Solar Collectors with Fuzzy Parameters

Problem of solar collector to convert solar radiation into energy, which can be used for electricity generation, blow drying process, heating of water for many industrial applications, is considered.
Various physical factors play vital role in mathematical formulation [68]. The ambient temperature depends on weather and also varies from morning to evening, so taking range of such value is more realistic than the specific value of the same. Similarly, rate of mass of air flow also varies. Due to this reason, we have proposed the mathematical model of solar air collector involving fuzzy initial temprature and fuzzy rate of mass of air flow. Thus, the fuzzy model is proposed,

$$(\frac{\tilde{m}}{W})a(\frac{d\tilde{T}}{dx}) + (\frac{\tilde{m}}{W})b(\frac{d\tilde{T}}{dx})\tilde{T}(x) + F'U_L\tilde{T}(x) = F'(S + T_aU_L); \tilde{T}(0)$$

The values of the taken parameters are given below, $a = 980.54, b = 0.083$, $\mu = 2.05 \times 10^{-5}$, $W = 1.2, L = 4$, $U_l = 6.5$, $k = 0.029$, $\tau\alpha = 0.90$, $s = 0.015$, $G_L = 890$, $T_a = 293K$, $h_{r,p-b} = 7.395$, $\tilde{m} = (0.03, 0.06, 0.08)$ and $\tilde{T_0} = (320, 323, 325)$.
This problem is solved using the proposed FADMP technique.
Effect on temperature of fuzzy air mass flow and initial temperature, is given in the Fig.1, Fig.2 and Fig.3.

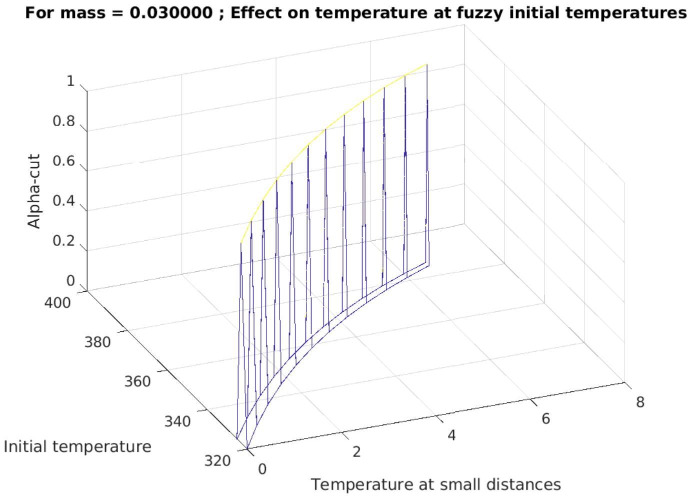

Figure 1: Effect on temperature at different fuzzy initial temperature; for mass (0.0300)

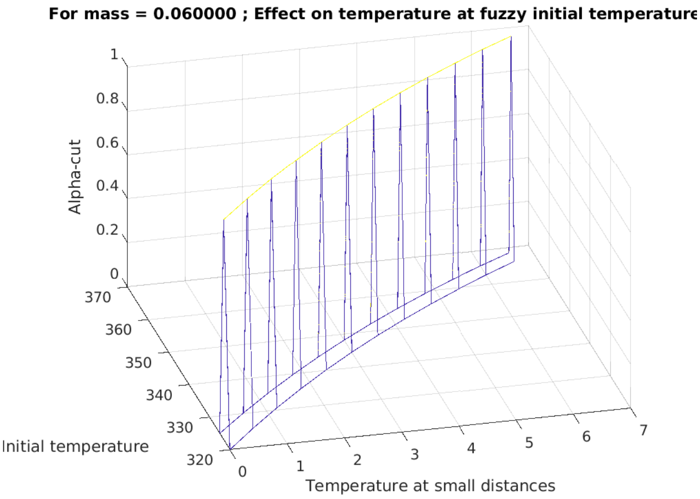

Figure 2: Effect on temperature at different fuzzy initial temperature; for mass (0.0600)

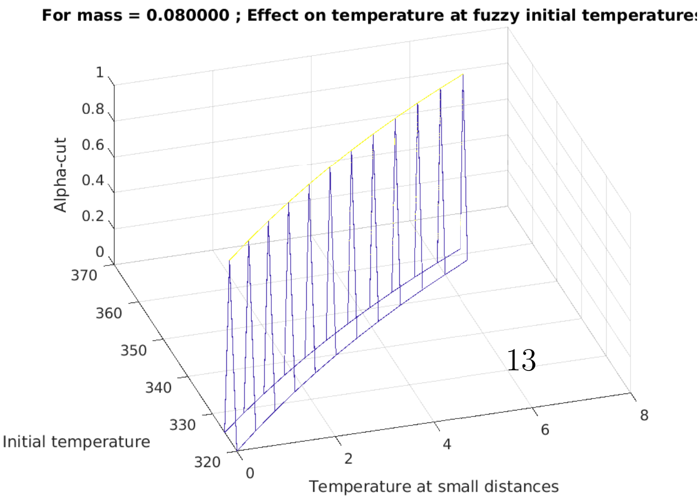

Figure 3: Effect on temperature at different fuzzy initial temperature; for mass (0.0800)

Till now the solution technique always required the fuzzy systems to be converted to the crisp by considering its equivalent parametric form. This motivated us to develop the results for fuzzy calculus, so that we can solve fuzzy systems directly.

3.4.3 Fuzzy Adomian Decomposition Method (FADM)

On the proposed new fuzzy derivative i.e. modified generalized Hukuhara derivative, we have developed few results required in fuzzy calculus like fuzzy power series, its convergence and Taylor's theorem. With the help of above mentioned results, we proposed and proved Fuzzy Adomian Decomposition method (FADM) [69]. The advantage of this technique is that we solve a problem completely in fuzzy environment instead of bringing it to the crisp for solving and going back in fuzzy as done in conventionally.
All the results are given below to support this fuzzy technique.

3.4.4 Fuzzy Taylor's theorem

If a fuzzy valued function $\tilde{f} : E^n \rightarrow E^n$ is n times modified generalized Hukuhara (mgh) differentiable, then fuzzy Taylor's expansion is given as,

$$\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x_0}) \oplus \dot{\tilde{f}}(\tilde{x_0}) \otimes (\tilde{x} \ominus \tilde{x_0}) \oplus \frac{(\tilde{x} \ominus \tilde{x_0})^2}{2!} \otimes \ddot{\tilde{f}}(\tilde{x_0}) \oplus \frac{(\tilde{x} \ominus \tilde{x_0})^3}{3!} \dddot{\tilde{f}}(\tilde{x_0}) \oplus ...$$

3.4.5 Fuzzy Power series and its radius of convergence:

A power series of fuzzy valued function arround the point $\tilde{x}_0$ can be given as $\sum_0^\infty \tilde{a}_n \otimes (\tilde{x} \ominus \tilde{x_0})^n = \tilde{a}_0 \oplus \tilde{a}_1 \otimes (\tilde{x} \ominus \tilde{x_0}) \oplus \tilde{a}_2 \otimes (\tilde{x} \ominus \tilde{x_0})^2 \oplus ...$ where $\tilde{a}_n$ is any fuzzy coefficients and n is positive integer. The radius of covergence for this power series is given by,

$$\tilde{R} = \lim_{n\rightarrow\infty} |\tfrac{\tilde{a}_n}{\tilde{a}_{n+1}}|$$

The above mentioned results are used for FADM technique as follows.

3.4.6 FADM in fuzzy environment

Consider a nonlinear fuzzy differential equation, $L(\tilde{u}) \oplus R(\tilde{u}) \oplus N(\tilde{u}) = \tilde{g}$ where, L is linear (modified generalized Hukuhara) differentiable operator, N is nonlinear operator, R is the operator of less order than that of L and $\tilde{g}$ is source term.

$$L^{-1}L(\tilde{u}) = L^{-1}(\tilde{g} \ominus R(\tilde{u}) \ominus N(\tilde{u}))$$

This method gives that unknown fuzzy function can be expressed as series of, $\tilde{u} = \sum \tilde{u}_n$ The method defines nonlinear fuzzy term by Adomian polynomials, and it is defined as, $N(\tilde{u}) = \sum_0^\infty \tilde{A}_n(\tilde{u}_0, \tilde{u}_1 ... \tilde{u}_n)$ where,

$$\tilde{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\sum_k^\infty (\tilde{X}_k)\lambda^n)$$

we have also proved convergence of fuzzy Adomian Decomposition method. We have solved one example by FADM, which is given as follows,

3.4.7 Example

$$\dot{\tilde{u}} = \tilde{u}^2, \tilde{u}(0) = \tilde{1} \tag{9}$$

$$L(\tilde{u}) = \tilde{u}^2$$

where, L is mgh-differentiable linear operator and L^{-1} i.e. fuzzy integrable operator on both side of equation (9).

$$L^{-1}L(\tilde{u}) = Ł^{-1}(\tilde{u}^2)$$

$$\tilde{u}(t) = \tilde{u}(0) \oplus L^{-1}(\tilde{u}^2)$$

Now, we decompose this term $L^{-1}(\tilde{u}^2)$ into Adomian polynomials as given in 3.4.6.

$$\tilde{A}_0 = \tilde{u}_0^2$$

$$\tilde{u}_1 = L^{-1}(\tilde{A}_0) = \tilde{1} \otimes \tilde{1}t$$

with $\tilde{1}$ represented by triangular fuzzy number $(0.5, 1, 1.5)$.

$$\tilde{u}_1 = \tilde{1}^2 t$$

$$\tilde{A}_1 = 2\tilde{u}_0 \otimes \tilde{u}_1$$

$$\tilde{A}_1 = 2\tilde{1}^3$$

$$\tilde{u}_2 = L^{-1}(\tilde{A}_1)$$

$$\tilde{u}_2 = \tilde{1}^3 t^2$$

Similarly,

$$\tilde{u}_3 = \tilde{1}^4 t^3$$

Continuing this process and we put these values $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$... in following expression,

$$\tilde{u} = \tilde{u}_1 \oplus \tilde{u}_2 \oplus \tilde{u}_3 ...$$

Thus, the obtained fuzzy solution is,

$$\tilde{u} = \tilde{1} \oplus \tilde{1}^2 t \oplus \tilde{1}^3 t^2 \oplus ....$$

For comparision with crisp solution, we put it in the parametric form as follows,

$$^{\alpha}\tilde{u} = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] + [(0.5 + 0.5\alpha)^2, (1.5 - 0.5\alpha)^2]t + [(0.5 + 0.5\alpha)^3, (1.5 - 0.5\alpha)^3]t^2 + ...$$

Evaluating at core gives us,

$$^{1}\tilde{u} = 1 + t + t^2 + t^3 + ... = (1 - t)^{-1}$$

Which is same as in case of crisp.
In this synopsis, fuzzy solution is constructed by using Decomposition theorem as in Klir [70].
In next section we have concluded this synopsis.

4 Conclusion

In this research work, we have applied different techniques on fully fuzzy linear as well as nonlinear dynamical system and compared result at core. We have proposed new fuzzy derivative i.e. Modified generalized Hukuhara derivative and used it for fuzzy Laplace transform. Further we have proposed and established results for solving nonlinear dynamical system namely Fuzzy Adomian Decomposition method. Advantage of this technique is that any fuzzy dynamical system can be solved completely in fuzzy environment. We have solved illustrative example and real life application i.e. air heating solar collectors with fuzzy parameters using proposed method.

References

[1] Zadeh, L.A. (1965), Fuzzy sets and systems, Proc. Syrup. on systems theory (Polytechnic Institute Press), Brooklyn NY.

[2] Chang, S. L. and Zadeh L. A. (1972), On Fuzzy Mapping and Control, IEEE Transactions on System, Man Cybernetics, Vol.2, 30-34.

[3] Dubois, D., and Prade, H. (1982), Towards Fuzzy Differential Calculus, Part 3, Differentiation, Fuzzy Sets and Systems, Vol. 8, 225-233.

[4] Puri, M. L and Ralescu, D.A (1983), Differential of Fuzzy Functions, J. Math. Anal. Appl. Vol. 91, 321-325.

[5] Hukuhara, M.(1967), Integration des applications measurables dont la valeur est un compact convexe. Fukc Ekvacioj Vol. 10, 205–223.

[6] Kaleva, O. (1987). Fuzzy Differential Equations, Fuzzy Sets and Systems, Vol. 24, 301-317.

[7] Seikala, S. (1987), On the fuzzy Initial value problem, Fuzzy Sets and Systems, Vol. 24(3) , 319-330.

[8] Kandel, A., Byatt, W.J. (1978), Fuzzy Differential equations, Mathematics.

[9] Bede, B., and Gal, S. G. (2005), Generalization of the differentiability of fuzzy number valued function with application to fuzzy differential equation, Fuzzy Sets and Systems, Vol. 151, 581-599.

[10] Ma, M., Friedman M. and Kandel, A. (1999), Numerical solutions of fuzzy differ-ential equations, Fuzzy Sets and Systems, Vol. 105, 133-138.

[11] Shokri J. (2007). Numerical Solutions of Fuzzy Differential Equations, Applied Mathematical Sciences, Vol. 1(45), 2231-2246.

[12] Abbasbandy, S. and Allahviranloo, T.(2002), Numerical solutions of fuzzy differential equations by taylor method, Computational Methods in Applied Mathematics, Vol.2, 113-124.

[13] Abbasbandy, S. and Allahviranloo, T.(2011),and Darabi P., Numerical solutions of N-order fuzzy differential equations by runge- kutta method, Mathematical and Computational Applications, Vol. 16, 935-946.

[14] Allahviranloo, T., Ehmady, N. and Ehmady, E.(2007), Numerical solutions of fuzzy differential equations by predictor- corrector method, Information Sciences, Vol. 177, 1633-1647

[15] Jayakumar T. Maheshkumar D. and Kanagrajan K. (2012) Numerical Solutions of Fuzzy Differential Equations By Runge- Kutta Method Of Order Five, Applied Mathematical Sciences, Vol. 6, 2089-3002.

[16] Parandin, N.(2013), Numerical solutions of fuzzy differential equations of 2nd order by runge- kutta method, Journal of Mathematical Extension Vol. 7, 47-62.

[17] Kanagrajan K. Suresh R. (2014) Numerical Solutions of Fuzzy Differential Equations Under Generalized Differentibility By Improved Euler Method, International Journal Of Applied Mathematics and Computation Vol. 6(1), 17-24.

[18] Allahviranloo, T., Ahmadi, M.B. (2010), fuzzy Laplace transforms, soft comput. Vol. 14, 235-243.

[19] Salahshour, S., Allahviranloo, T.(2013), Applications of Fuzzy Laplace Transforms, Soft Comput Vol. 17, 145-158.

[20] Eljaoui, E., Mellani, S., Saadia Chadli, L.(2015), Solving second order fuzzy differential equation by the fuzzy Laplace Transform method, Advance in Difference Equation, Vol. 66, 1-14.

[21] Hayder, A. K., Mohammad Ali Hawrra, F. (2015), Generalization of Fuzzy Laplace Transforms for Fuzzy derivatives,Journal of Kerbala University , Vol. 13(2), 120-137.

[22] Babolian, E., Vahidi, A.R. and Azimzadeh, Z. (2012), A Comparison Between Adomian's Decomposition Method and the Homotopy Perturbation Method for Solving Nonlinear Differential Equations. Journal of Applied Sciences, Vol.12, 793-797.

[23] Adomian, G. (1994), Solving Frontier problems of Physics: The decomposition method, Kluwer Academic Publishers.

[24] He, J.H. (2006), Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B, Vol. 20(10), 1141-1199.

[25] Charkrit, S. (2013), On the Solutions of First and Second Order Nonlinear Initial Value Problems, Proceedings of the World Congress on Engineering, WCE, July 3 - 5, London, U.K

[26] Jian, L.L. (2009), Adomian's decomposition method and homotopy perturbation method in solving nonlinear equations, Journal of computational and Applied Mathematics, Elsevier, Vol. 228(1), 168-173.

[27] He, J.H. (2000), Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput., Vol. 114, 115–123.

[28] Xu,Jinping, Zhigao, L., Neito, J.J (2010), A class of linear differential dynamical system with fuzzy matrices, J. Math. Anal. Appl., Vol. 368, 54-68.

[29] Ghazanfari, B., Niazi, S., Ghazanfari, A. G.(2012), Linear matrix differential dynamical system with fuzzy matrices, Applied Mathematical Modelling Vol. 36, 348-356.

[30] Pandit, P. (2012), Fully Fuzzy System of Linear Equations, International Journal of Soft Computing and Engineering (IJSCE), Vol. 2(5), 619-627.

[31] Biswas, S., Roy, T.K. (2019), A semianalytical method for fuzzy integro-differential equations under generalized Seikkala derivative, Soft Comput. Vol. 23, 7959–7975.

[32] Allahviranloo, T., Kiani, N.A., Barkhordari, M (2009), Toward the existence and uniqueness of solutions of second-order fuzzy differential equations. Inf Sci., Vol. 179, 1207–1215.

[33] Khastan, A., Nieto, J. J. and Rodríguez-López, R. (2011), Variation of constant formula for first order fuzzy differential equations, Fuzzy Sets and Systems, Vol. 177, 20–33.

[34] J. J. Nieto, Khastan, A. and Ivaz, K. (2009), Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis: Hybrid Systems, Vol. 3(4), 700–707.

[35] Tapaswini, S. and Chakraverty, S. (2014), "Euler based new solution method for fuzzy initial value problems," Int. J. Artific. Intell. Soft Comput., Vol. 4, 58–79.

[36] Xiaoping, X., Yongqiang, F. (2006), On the structure of solutions for fuzzy initial value problem, Fuzzy Sets and Systems, Vol. 157(2), 212-229.

[37] Prakash P, Kalaiselvi V (2009), Numerical solution of hybrid fuzzy differential equations by predictor corrector method. International Journal of Computer Mathematics, Vol. 86, 121–134.

[38] Prakash, P., Kalaiselvi, V.(2012), Numerical solutions of fuzzy differential equations by using hybrid methods, Fuzzy Inf. Eng., Vol. 4, 445–455.

[39] Tapaswini, S. and Chakraverty, S. (2014), Numerical Solution of Fuzzy Differential Equations and its Applications, Mathematics of Uncertainty Modeling in the Analysis of Engineering and Science Problems.

[40] Tapaswini, S. and Chakraverty, S. (2012), A new approach to fuzzy initial value problem by improved Euler method, Int. J. Fuzzy Inform. Eng., Vol. 4, 293–312.

[41] Fard, O. S. and Ghal Eh, N. (2011), Numerical solutions for linear system of first-order fuzzy differential equations with fuzzy constant coefficients Inform. Sci., Vol. 181, 4765–4779.

[42] Zimmermann, H. J. (2001), Fuzzy Set Theory and Its Application, Kluwer Academic Publishers, Boston/Dordrecht/London .

[43] Lan, H. Y. and Nieto, J. J. (2009), On initial value problems for first-order implicit impulsive fuzzy differential equations, Dynam. Syst. Appl., Vol. 18, 677–686 .

[44] Barros, L. C., Bassanezi, R. C. and Tonelli, P. A. (2000), Fuzzy modeling in population dynamics, Ecol. Model., Vol. 128, 27–33.

[45] Oberguggenberger, M. and Pittschmann, S. (1999), Differential Equations with Fuzzy Parameters, Mathematical and Computer Modelling of Dynamical Systems, Vol. 5(3), 181-202.

[46] Fard, O. S. (2009), A Numerical scheme for fuzzy calculus problem, J Uncertain Syst., Vol. 3(4), 307-314.

[47] Bede, B. (2013), Fuzzy Differential Equations in: Mathematics of Fuzzy Sets and Fuzzy Logic. Studies in Fuzziness and Soft Computing, Vol. 295, 171-191.

[48] Wu HC (1999), The improper fuzzy Riemann integral and its numerical integration, Infom Sci., Vol. 111, 109-137.

[49] Song , S. and Wu, C.(2000) Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations, Fuzzy sets and Systems, Vol. 110, 55-67.

[50] Bede, B.(2008), Note on Numerical solutions of fuzzy differential equations by predictor corrector method, Information Sciences, Vol. 178, 1917-1922.

[51] Abbasbandy, S., Allahviranloo, T., Lopez, O. and Nieto J.J. (2004), Numerical methods for fuzzy differential inclusions, Comput. Math. Appl., Vol. 48, 1633-1641.

[52] Jayakumar, T., Kanagarajan, K. and Indrakumar S.(2012), Numerical solution of Nth-order fuzzy differential equation by runge-kutta method of order five, Int. J. Math. Anal., Vol. 6(58), 2885-2896.

[53] Pandit, P. (2017). Exact Solution of Semi-linear Fuzzy System. The Journal Of The Indian Mathematical Society, Vol. 84(3-4), 225-238.

[54] Bede, B., Rudas, I.J. and Bencsik (2007), First order linear fuzzy differential equation under generalized differentiability, Inform. Sci, Vol. 177, 1648-1662.

[55] Georgion, D. N, Nieto, J. J. and Rodrigue-Lopez, R. (2005), Initial value problem for higher order fuzzy differential equations, Nonlinear Analysis, Vol. 63(4), 587-600.

[56] Buckley, J. J. and Feuring, T.(2001), Fuzzy initial value problem for nth-order linear differential equations, Fuzzy Sets and Systems, Vol. 121(2), 247-255.

[57] Mosleh, M.(2013), Fuzzy neural network for solving a system of fuzzy differential equations, Appl. Soft Comput., Vol. 13, 3597-3607

[58] Mosleh, M. and Otadi, M. (2012), Simulation and evaluation of fuzzy differential equations by fuzzy neural network, Appl. Soft Comput., Vol. 12, 2817-2827.

[59] Luciano, S. (2008), A generalization of Hukuhara difference for interval and fuzzy arithmetic, WP-EMS Working Papers Series in Economics, Mathematics and Statistics.

[60] Lupulescu, V. (2008) Initial value problem for fuzzy differential equations under dissipative conditions, Information Sciences, Vol. 178, 4523–4533.

[61] Pandit, P., Singh, P. (2014), Prey-Predator model and fuzzy initial condition, International Journal of Engineering and Innovative Technology (IJEIT) Vol. 3(12), 65-68.

[62] Pandit P., Singh, P. (2017), Numerical technique to solve dynamical system involving fuzzy parameters, International Journal of Emerging Trends and Technology in Computer Science (IJETTCS), Vol. 6(4), 51-57.

[63] Pandit, P., Singh, P. (2018), Improved Euler Method to solve Fully Fuzzy Dynamical System, presented in International conference on electrical, electricals, computers, communication, mechanical and computing EECCMC-2018, Vallore, Tamilnadu.

[64] Pandit, P. , Singh, P. (2017), Fuzzy Laplace Transform technique to solve linear dynamical system with fuzzy parameters, in proceeding International Conference on "Research and Innovations in Science, Engineering and Technology" ICRISET-2017.

[65] Pandit, P., Singh, P. (2019), Fully fuzzy Semi-Linear dynamical system solved by Fuzzy Laplace Transform under Modified Hukuhara derivative, SOCPROS, ebook ISBN: 978-981-15-0035-0, AISC, Springer.

[66] Pandit, P., Mistry, P, Singh, P. (2021), Population Dynamic Model of Two Species Solved by Fuzzy Adomian Decomposition Method, MMC-ITRE, ebook ISBN: 978-981-15-9953-8, AISC, Springer.

[67] Pandit, P., Mistry, P., Singh, P. (2020), Mathematical Modelling of Air Heating Solar Collectors with Fuzzy Parameters, Conference proceeding of ICET 2020, Energy Centre, Maulana Azad National Institute of Technology, Bhopal.

[68] Kalogirou, S. Solar energy (2009) engineering-processes and syatems, Kalo Elseviere, USA .

[69] Pandit, P. , Singh, P. (2021), Fuzzy Adomian Decomposition Method, Communicated to journal.

[70] George, Klir J. , Yuan, B. (1995) Fuzzy Sets and Fuzzy Logic: Theory and Applications , Prentice Hall PTR Upper Saddle River New Jersey.

Our Publications:

- Pandit, P., Singh, P. (2014), Prey-Predator model and fuzzy initial condition, International Journal of Engineering and Innovative Technology (IJEIT) Vol. 3(4), 65-68. (In UGC journal list till May, 2018)
- Pandit, P., Singh, P. (2017), Numerical technique to solve dynamical system involving fuzzy parameters, International Journal of Emerging Trends and Technology in Computer Science (IJETTCS), Vol. 6(12), 51-57. (In UGC journal list till May, 2018)
- Pandit, P., Singh, P. (2019), Fully fuzzy Semi-Linear dynamical system solved by Fuzzy Laplace Transform under Modified Hukuhara derivative, SOCPROS, ebook ISBN: 978-981-15-0035-0, AISC, Springer. (SCOPUS and UGC CARE listed)
- Pandit, P., Mistry, P, Singh, P. (2021), Population Dynamic Model of Two Species Solved by Fuzzy Adomian Decomposition Method, MMCITRE, ebook ISBN: 978-981-15-9953-8, AISC, Springer. (SCOPUS and UGC CARE listed)
- Pandit, P., Singh, P. (2021), Fuzzy Adomian Decomposition Method, Communicated to journal.

Conference Presentation:

- Pandit, P., Singh, P. (2015), Numerical Solution of fully fuzzy dynamical system, National Conference on Innovating for Development and Sustainability, Nuvrachna University, Vadodara, 30-31 october.
- Pandit, P., Singh, P. (2017), Fuzzy Laplace Transform technique to solve linear dynamical system with fuzzy parameters, in proceeding International Conference on "Research and Innovations in Science, Engineering and Technology" ICRISET-2017,17-19 February.
- Pandit, P., Singh, P., (2018), Improved Euler Method to solve Fully Fuzzy Dynamical System, presented in International conference on electrical, electricals, computers, communication, mechanical and computing (EECCMC-2018), Vallore, Tamilnadu, 28-29 January.
- Pandit, P., Singh, P. (2018) fully fuzzy Semi-Linear dynamical system solved by Fuzzy Laplace Transform under Modified Hukuhara derivative, Soft computing on Problem solving, presented in SOCPROS, VIT, Vellore, Chennai, 17-19, December.

- Pandit, P., Mistry, P., Singh, P. (2020), Mathematical Modelling of Air Heating Solar Collectors with Fuzzy Parameters, Conference proceeding of ICET 2020, Energy Centre, Maulana Azad National Institute of Technology, Bhopal, 27-28, August.

Payel Singh

Candidate

P K Pandit
22/4/2021

Guide

22/04/2021

Head of Department

Head
Applied Mathematics Department
Faculty of Technology & Engineering
The Maharaja Sayajirao University of Baroda
Vadodara

Dean

Dean
Faculty of Tech. & Engg.
M. S. University of Baroda,
Baroda