CHAPTER VII

THE NUMERICAL RANGE AND POLARDECOMPOSITION OF AN OPERATOR

While giving a new proof of the result $\Sigma(T) = \mathrm{Cl}(W(T))$, for a bounded normal operator T, S. K. Berberian[1)] raised the following question in his paper:

If $T = UR$ is an invertible operator such that U is cramped, does it follow that $0 \notin \mathrm{Cl}(W(T))$?

Moreover, he also showed that the answer to the above question is in the affirmative for a particular type of operators namely normal operators. In this chapter, we show by constructing an example that the answer to the above question, in general, is in the negative*. However, we further show that for certain special types of operators and particularly, for hyponormal operators, the answer to the above question is in the affirmative. In fact, we prove the following three theorems:

THEOREM 7.1. *If* $T = UR$ *is an invertible hyponormal operator such that* U *is cramped, then* $0 \notin \mathrm{Cl}(W(T))$.

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1) S. K. Berberian [4]

* As this thesis was taking the final form for being presented my attention was drawn to a paper of E. Durszt [9] wherein he has also discussed the same question.

THEOREM 7.2. Let $T = UR$ be an invertible operator such that U is cramped. If $\Sigma(T)$ is a spectral set for T, then $0 \notin Cl(W(T))$.

THEOREM 7.3. If H is finite-dimensional, then there exists an invertible operator $T = UR$ such that U is cramped and $0 \in W(T)$.

PROOF OF THEOREM 7.1: Assume, to the contrary, that $0 \in Cl(W(T))$. Now $0 \notin s(T)$ and $0 \in Cl(W(T)) = \Sigma(T)$ by theorem 3.1 of chapter III implies that 0 lies in the convex hull of $s(T)$. Hence there exists at least one line through 0 such that it has non-empty intersection L with $\Sigma(T)$. If α and β are the end-points of L, then α and β are boundary points of $\Sigma(T)$ i.e. α, $\beta \in a(T)$.

Since $\alpha \in a(T)$, there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(T^* - \bar{\alpha}I)x_n\| \leq \|(T - \alpha I)x_n\| \to 0$$

i.e. $$\|Tx_n\| = \|Rx_n\| \to |\alpha| \ , \ (Tx_n \ , \ x_n) \to \alpha$$

and consequently $(Rx_n \ , \ x_n) \to |\alpha|$.

Hence

$$\|(R - |\alpha|I)x_n\|^2 = \|Rx_n\|^2 - |\alpha|(Rx_n \ , \ x_n) - |\alpha|(x_n \ , \ Rx_n) + |\alpha|^2 \to 0.$$

Now the relation

$$T - \alpha I = UR - \alpha I = U(R - |\alpha| I) - |\alpha|(-U + \frac{\alpha}{|\alpha|} I)$$

implies that

$$\big|\ \|(R - |\alpha| I)x_n\| - |\alpha|\,\|(U - \alpha/|\alpha| I)x_n\|\ \big| \leq \|(T - \alpha I)x_n\|.$$

In other words,

$$\|(U - \alpha/|\alpha| I)x_n\| \to 0 \quad \text{or} \quad \alpha/|\alpha| \in a(U).$$

Similarly $\beta/|\beta| \in a(U)$.

Since 0 lies on the line segment joining α and β, it follows that 0 also lies on the line segment joining $\alpha/|\alpha|$ and $\beta/|\beta|$ i.e. U is not cramped which leads to a contradiction. Hence $0 \notin Cl(W(T))$.

<u>PROOF OF THEOREM 7.2</u>: Assume, to the contrary, that $0 \in Cl(W(T))$. Since $\Sigma(T)$ is a spectral set for T, $\Sigma(T) = Cl(W(T))$ [24] and hence, as in the proof of theorem 7.1, 0 lies on the line segment joining two boundary points, say α and β of $\Sigma(T)$ and therefore α, $\beta \in a(T)$. Since $\Sigma(T)$ is a spectral set for T, we have, as in the proof of theorem 4.2 of chapter IV, $\|(T - \alpha I)x_n\| \to 0$ and $\|(T^* - \bar{\alpha} I)x_n\| \to 0$ for a sequence $\{x_n\}$ of unit vectors. Again, it can be shown as in the proof of theorem 7.1, that $\alpha/|\alpha|$ and $\beta/|\beta| \in s(U)$. This leads to a contradiction and the proof of theorem 7.2 is complete

PROOF OF THEOREM 7.3: Consider the operator

$$T = \begin{pmatrix} 1+i & 1-i \\ 2+i & 2-i \end{pmatrix}$$

in a unitary space of dimension two with basis $e_1 = (1\ ,\ 0)$ and $e_2 = (0\ ,\ 1)$.

Since $|T| = 2i$, T is invertible. Also $T\,T^* \neq T^*T$ i.e. T is not normal. If $T = UR$ be its polardecomposition, then

$$R = \begin{pmatrix} \sqrt{10}/2 & 3(1-2i)/\sqrt{10} \\ 3(1+2i)/\sqrt{10} & \sqrt{10}/2 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 2 - i/\sqrt{10} & 2 + i/\sqrt{10} \\ 1 + 2i/\sqrt{10} & 1 - 2i/\sqrt{10} \end{pmatrix}$$

The proper values α and β of U are

$$\alpha = (3+\sqrt{11}) + (-3+\sqrt{11})i/2\sqrt{10}$$

$$\beta = (3-\sqrt{11}) - (3+\sqrt{11})i/2\sqrt{10}$$

Hence, if θ is the angle such that

$$\cos\theta = 3 + \sqrt{11}\ /\ 2\sqrt{10} \quad \text{and} \quad \sin\theta = -3 + \sqrt{11}\ /\ 2\sqrt{10}\ ,$$

then $\tan\theta = -3 + \sqrt{11}\ /\ 3 + \sqrt{11} = 10 - 3\sqrt{11}$ i.e. $0 < \theta < \pi/4$.

Also $\alpha = e^{i\theta}$ and $\beta = e^{i(3\pi/2-\theta)}$ which shows that U is cramped.

Moreover, if we take the unit vector

$x = 1/\sqrt{2}\ e_1 - 1/\sqrt{2}\ e_2$, then $(Tx , x) = 0$ i.e. $0 \in W(T)$.

This completes the proof of the theorem.